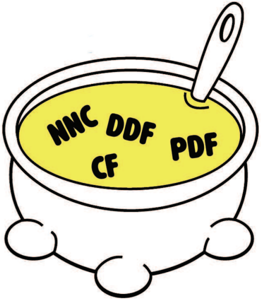
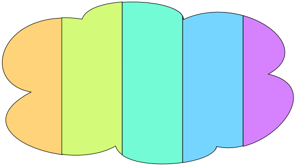
Coding for Wireless Relay Networks
Alphabet Soups and the Network Cutlet Bound

Young-Han Kim

University of California, San Diego



Australian School of Information Theory
Adelaide, Australia
November 11, 2014



Theorem 1

history of communication = history of wireless

Japanese scientists dug 50 meters underground and discovered small pieces of copper. After studying these pieces for a long time, Japan announced that the ancient Japanese 20,000 years ago had a nation-wide telephone network.

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(Blank) scientists were outraged. They dug 50, 100 and 200 meters underground, but found nothing. They concluded that the ancient (blank) 40,000 years ago had cellular telephones.

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Theorem 2

history of wireless = history of relaying


海東八道烽火山岳地圖

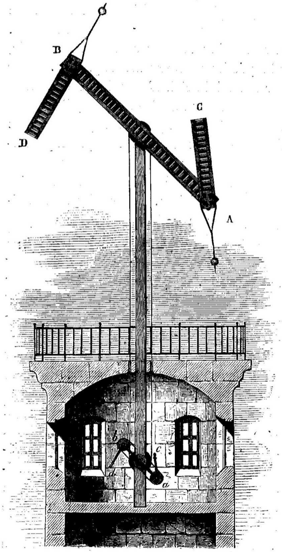
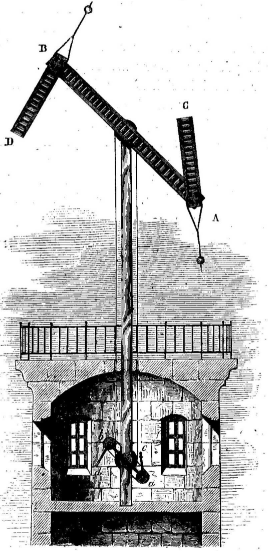


Fig. 19. — Télégraphe de Chappe.



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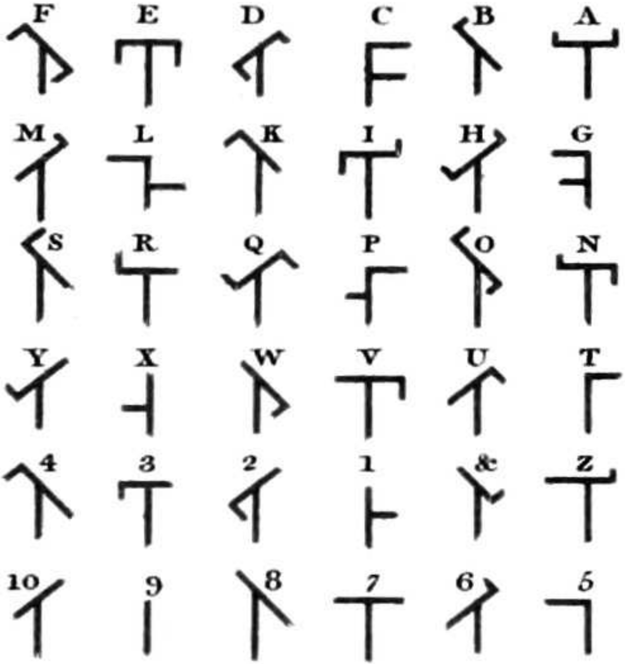

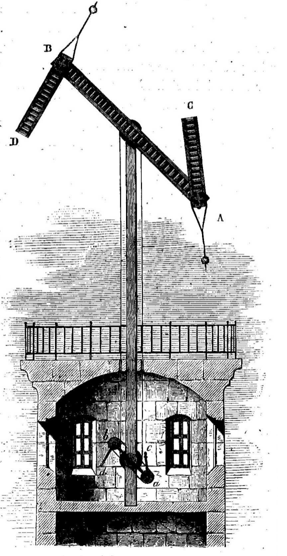

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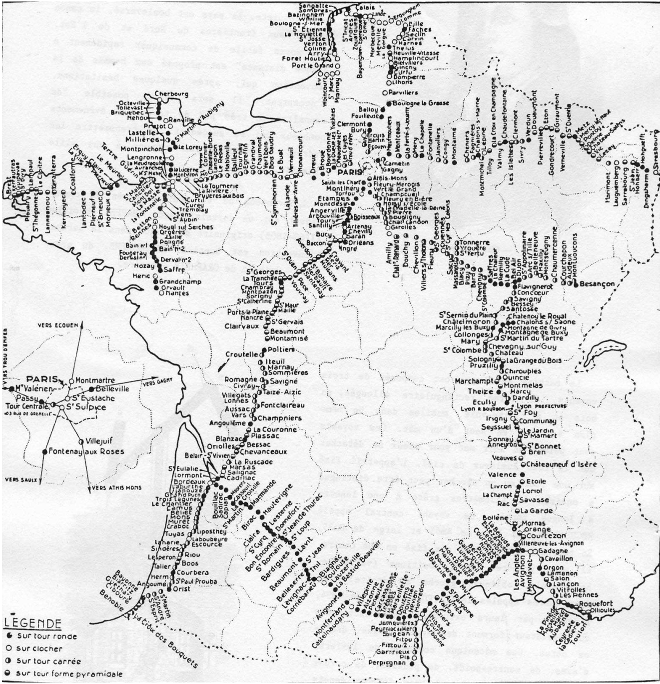

HOW *Radio-Relay* WORKS

The microwaves used for telephone transmission travel in a straight line. So relay towers, like those shown, are usually built on hilltops, averaging about 30 miles apart. Each tower picks up microwaves from its neighbor, and with complex electronic equipment amplifies and focuses them like a searchlight, then beams them accurately at the next tower. And hundreds of Long Distance telephone calls ride the beam at the same time.

New skyway spans nation with words and pictures

BELL SYSTEM *Radio-Relay* BUILT FOR LONG DISTANCE CALLS AND TELEVISION

There's something new on the national horizon! Bell Telephone construction crews have completed the last link in a coast-to-coast *Radio-Relay* system that is unique in all the world. Today, communications ride on radio microwaves, flashed through the air from tower to tower.

It was an historic event in 1915, when wires first carried the human voice across three thousand miles of mountains and prairie. By 1942, telephone messages were carried across the United States by another means — cable, both underground and overhead. And now comes *Radio-Relay* to supplement wire and cable!

The new system is already in use for Long Distance telephone service and coast-to-coast television. This new skyway helps make America's vast communications network even stronger and more flexible. And it could hardly happen at a better time. The demands of defense are heavy and urgent.

BELL TELEPHONE SYSTEM

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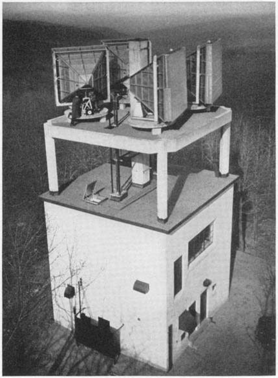
Seven towers on seven hilltops



One of seven relay stations — to test use of radio "microwaves" for Long Distance calls.

Built by the Bell System, they will provide a new kind of Long Distance communication.

Each hilltop tower is a relay station between New York and Boston* for very short radio waves. These "microwaves" are free from static and most man-made interference. But they shoot off into space instead of following the earth's curve. So they have to be gathered into a beam and aimed at the next tower, about 30 miles away. That's the job of the four big, square, metal lenses on each tower. They focus microwaves very much as a magnifying glass focuses the sun's rays.

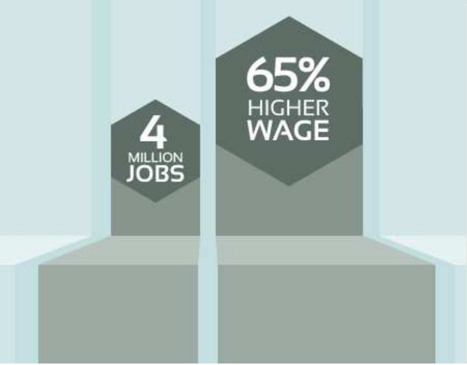
These radio relay systems may be used for Long Distance telephone calls and to transmit pictures, radio broadcasts and television programs.

This is another example of the Bell System's effort to provide more and better Long Distance service.

BELL TELEPHONE SYSTEM

**We have applied to the Federal Communications Commission for authority to start a similar link later between New York and Chicago*

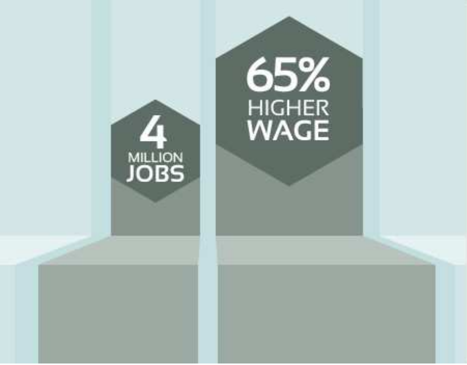

VICTIM OF WIRELESS TECHNOLOGY



Wireless Jobs, Good Pay

Nearly four million U.S. jobs are directly or indirectly associated with wireless, and they pay 65% higher wages than the national average.

Source: Larry Summers, "Technological Opportunities, Job Creation, and Economic Growth," Remarks at the New America Foundation, June 28, 2010, at http://www.whitehouse.gov/administration/eop/nec/speeches/technological-opportunities-job-creation-economic-growth.



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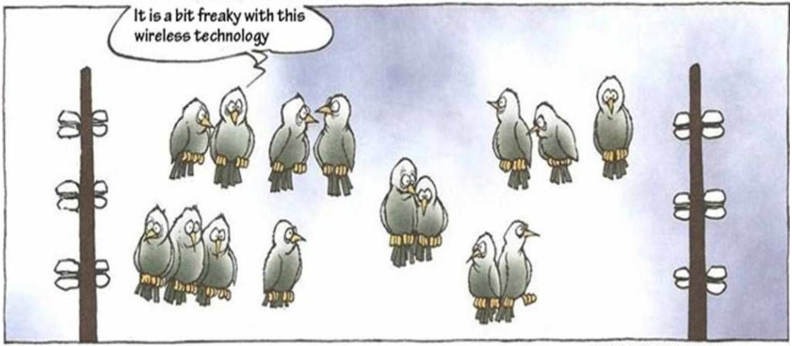
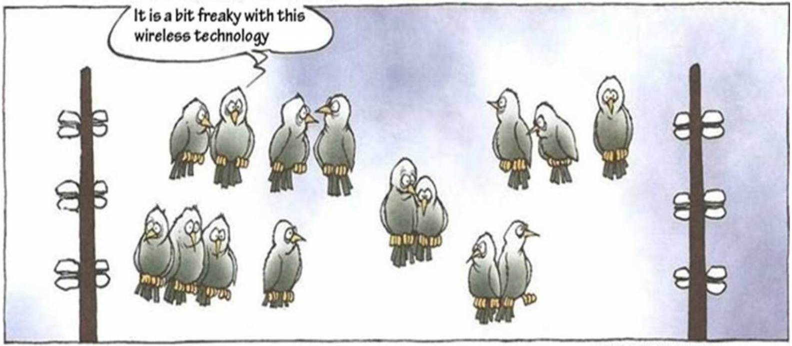
U.S. Wireless Major Industry

The U.S. wireless industry is valued at nearly $196 billion, which is larger than publishing, agriculture, hotels and lodging, air transportation, and motor vehicle manufacturing segments.

Source: Roger Entner, *The Wireless Industry: The Essential Engine of US Economic Growth*, Recon Analytics, April 2012, *available at* http://reconanalytics.com/wp-content/uploads/2012/04/Wireless-The-Ubiquitous-Engine-by-Recon-Analytics-1.pdf. [at p.1]


It is a bit freaky with this wireless technology



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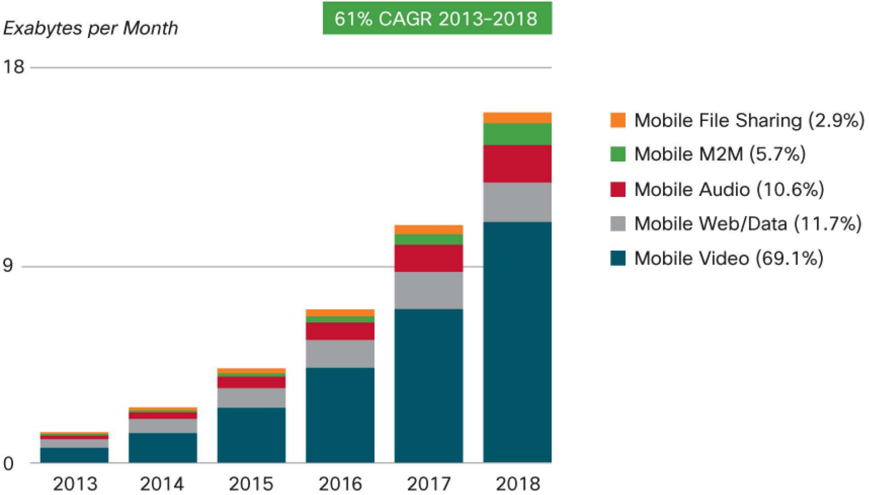
SPECIAL Report

WORDS **BENNETT RING** ILLUSTRATION **NIGEL BUCHANAN**

CONNECTIVITY -IS KING-

From smart watches that synchronise with smartphones, to portable high-definition cameras that can be remotely monitored from anywhere on Earth, 2014 has been a year in which to be linked in is everything.

Where is wireless going?



Figures in parentheses refer to traffic share in 2018.

Source: Cisco VNI Mobile, 2014

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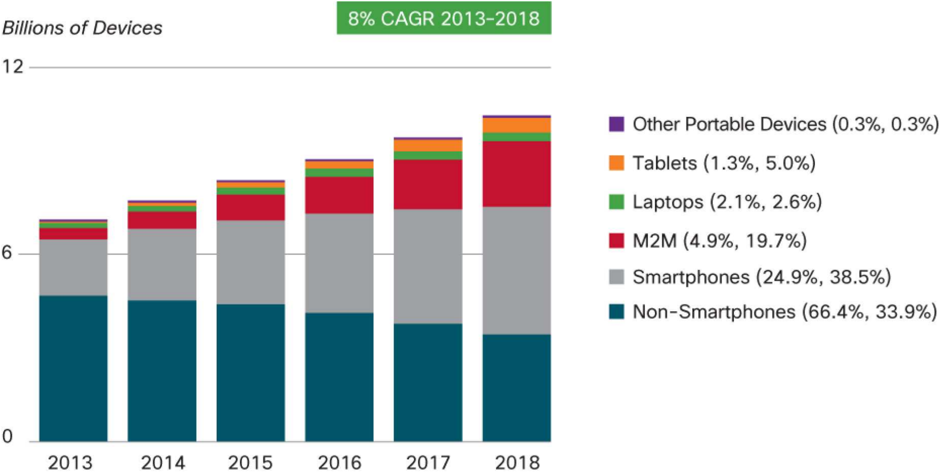

Figures in parentheses refer to device or connections share in 2013, 2018.

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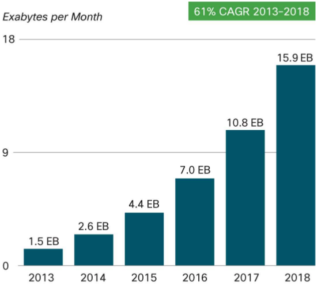
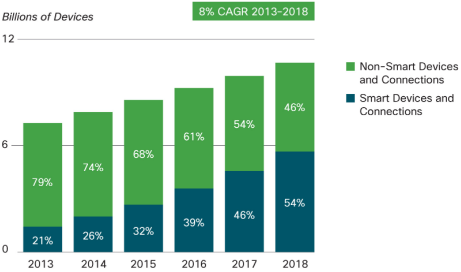
Where is wireless going?

Mobile data per month

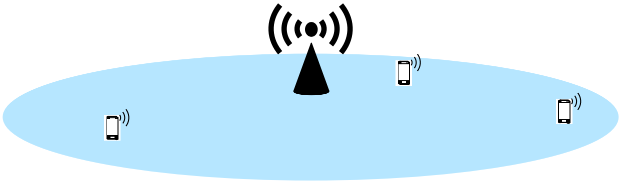


Source: Cisco VNI Mobile, 2014

Number of devices



Percentages refer to device or connections share.
Source: Cisco VNI Mobile, 2014



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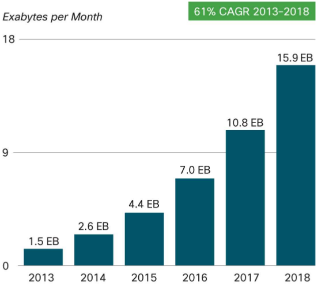

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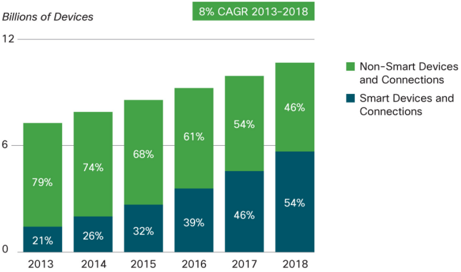

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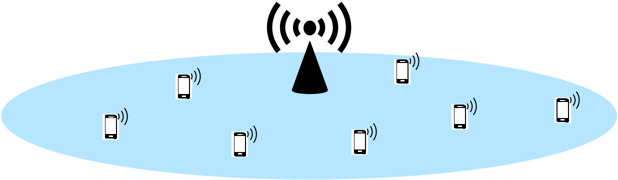

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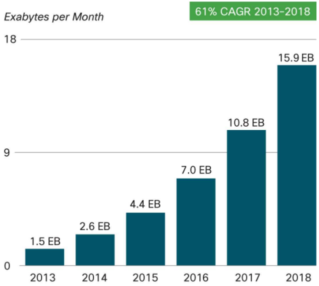

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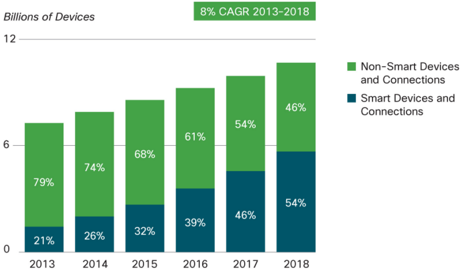

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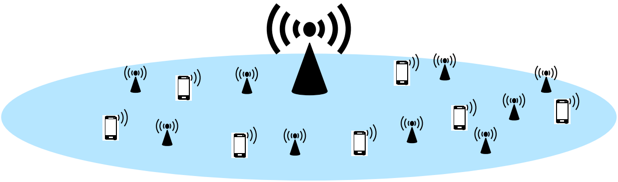
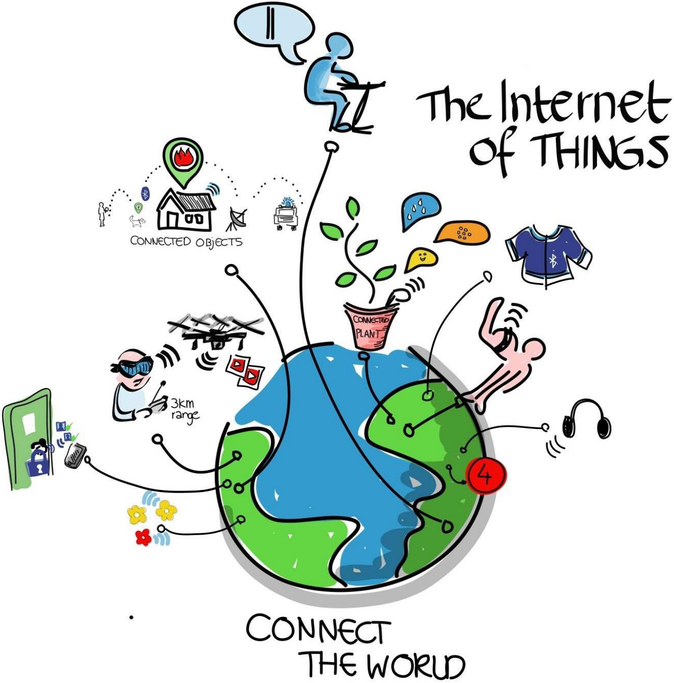
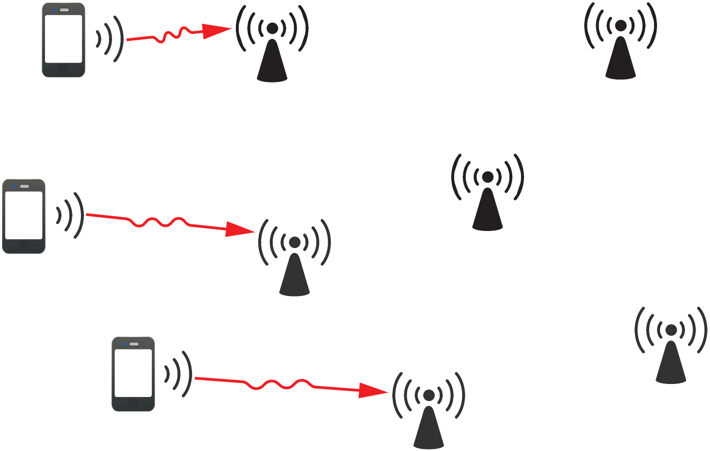



The Internet of THINGS
CONNECTED OBJECTS
CONNECTED PLANT
3km range
4
CONNECT THE WORLD

Cloud radio access network (C-RAN)



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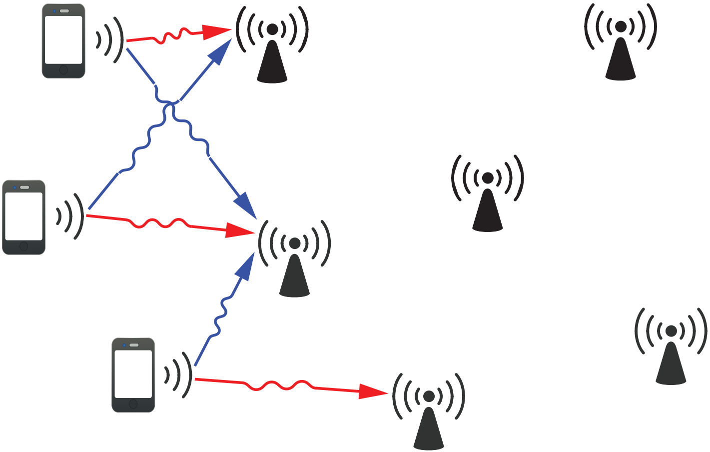

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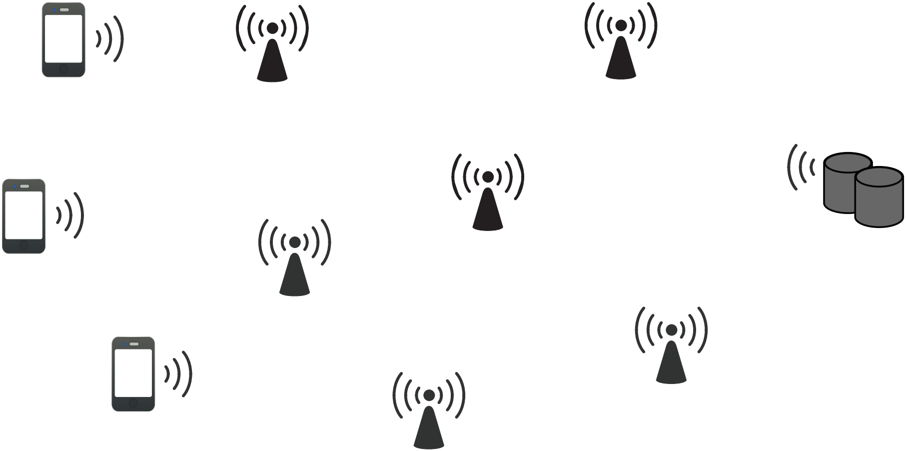

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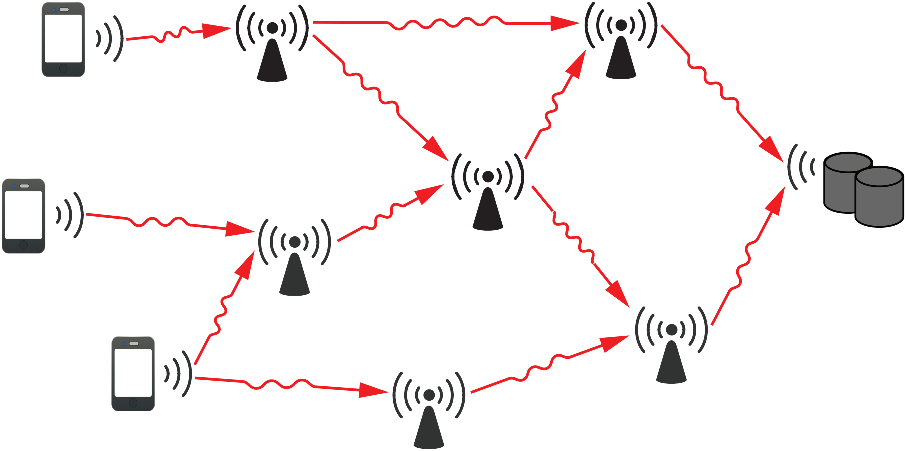

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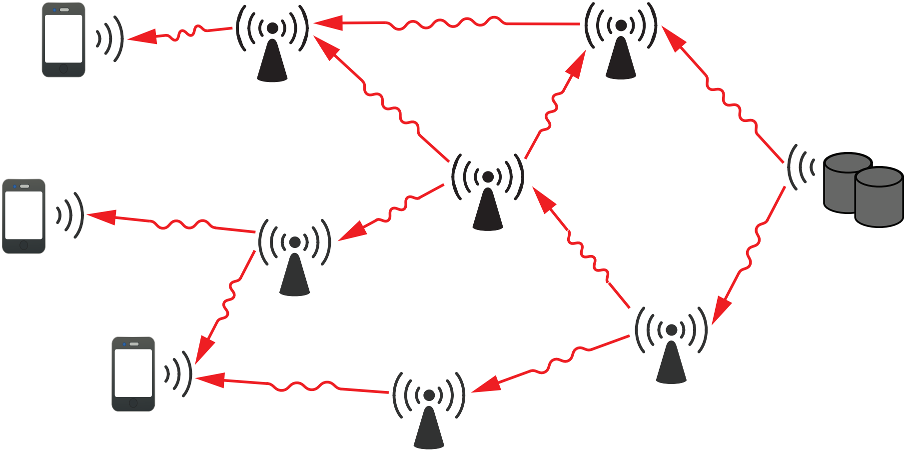
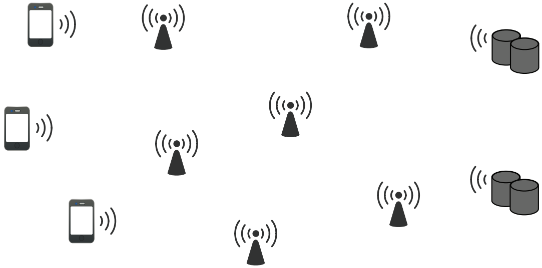


About this tutorial



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  - What is the limit on the amount of reliable communication?
  - What are the coding schemes that achieve this limit?

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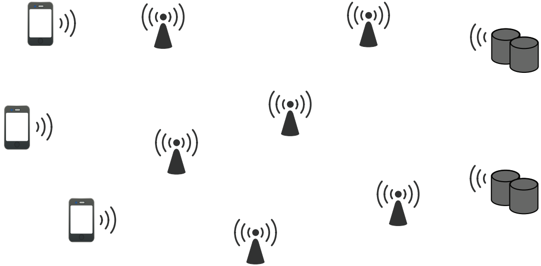

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- Broad coverage of recent results

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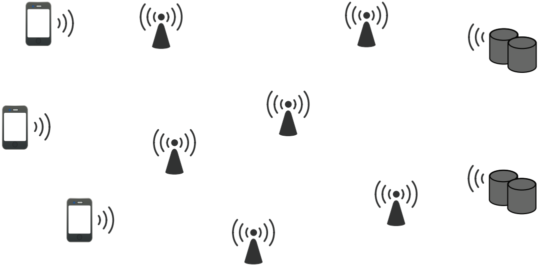

- Information theory for relay networks
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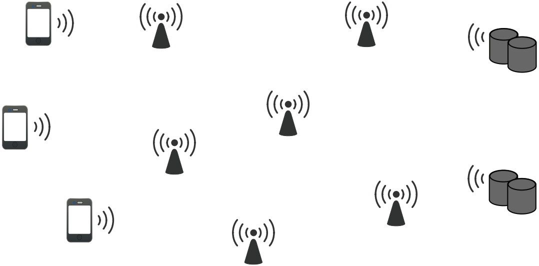
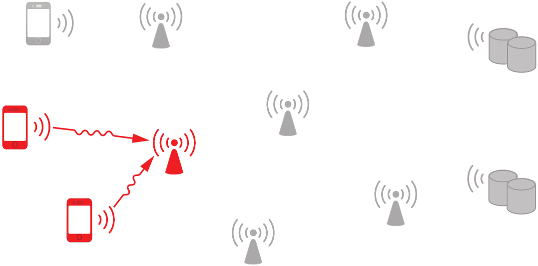
  - What are the coding schemes that achieve this limit?
- Broad coverage of recent results
- Personal angle on the topic (not a comprehensive survey of the literature)

Organization



Organization

- Single-hop communication
  - Multiple access (many-to-one)



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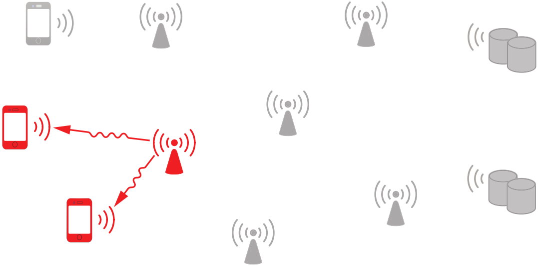

Organization

- Single-hop communication
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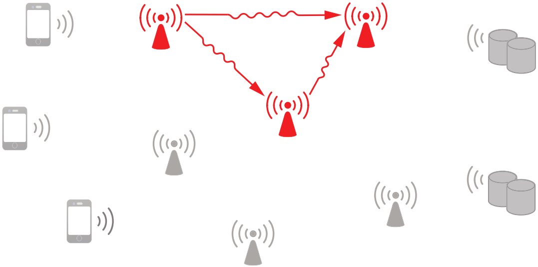
- Basic relaying techniques
  - Partial decode–forward (PDF)
  - Compress–forward (CF)



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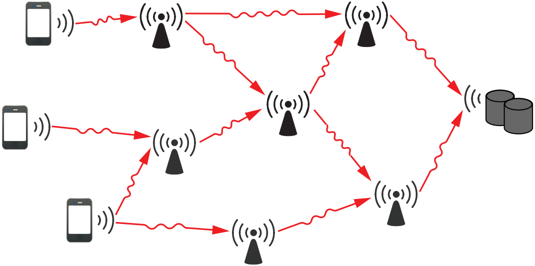

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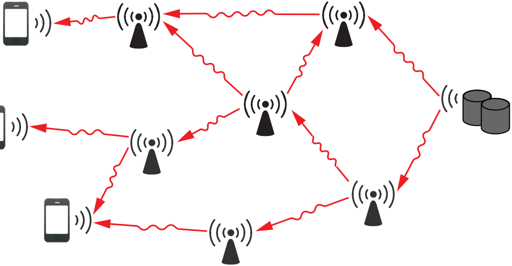
    - Distributed decode–forward (DDF)



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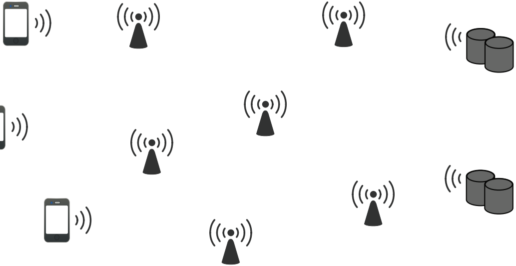
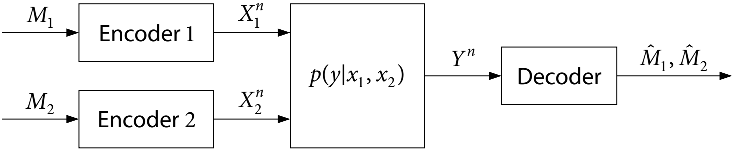
- Limit on communication
  - Cutset bound
  - Cutlet bound



$$C \le \ldots I(X;Y) \ldots$$

Multiple access channel



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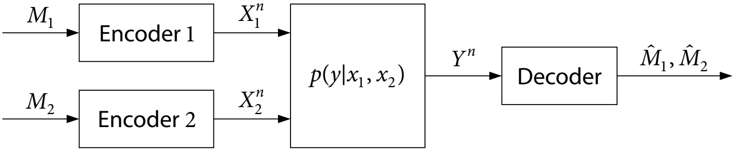

- A $(2^{nR_1}, 2^{nR_2}, n)$ code:
  - Message sets: $[1:2^{nR_1}]$ and $[1:2^{nR_2}]$
  - Encoder $j = 1, 2$: $x_j^n(m_j)$
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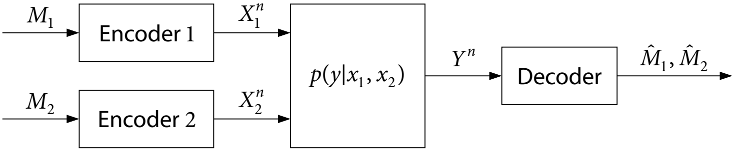
Multiple access channel



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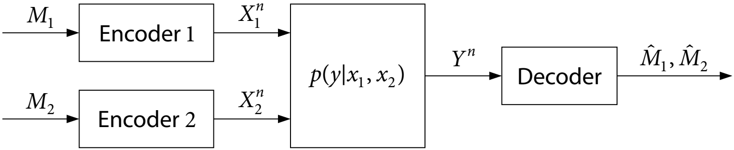

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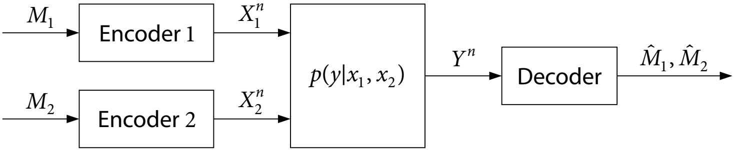

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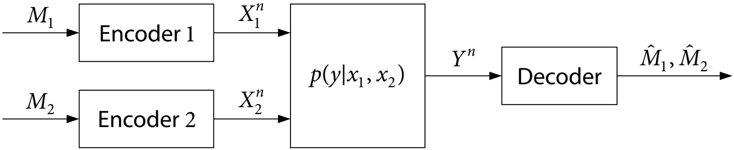
- (R_1, R_2) achievable if $\exists$ $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$

Multiple access channel



- A $(2^{nR_1}, 2^{nR_2}, n)$ code:
  - Message sets: $[1:2^{nR_1}]$ and $[1:2^{nR_2}]$
  - Encoder $j = 1, 2$: $x_j^n(m_j)$
  - Decoder: $(\hat{m}_1(y^n), \hat{m}_2(y^n))$
- $(M_1, M_2) \sim \mathrm{Unif}([1:2^{nR_1}] \times [1:2^{nR_2}])$
- Average probability of error: $P_e^{(n)} = \mathrm{P}\{(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\}$
- (R_1, R_2) achievable if $\exists$ $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$
- Capacity region $\mathscr{C}$: Closure of the set of achievable rate pairs (R_1, R_2)

Random coding and simultaneous decoding

- Codebook generation:
  - Independently generate 2^{nR_1} sequences $x_1^n(m_1) \sim \prod_{i=1}^n p_{X_1}(x_{1i})$, $m_1 \in [1:2^{nR_1}]$
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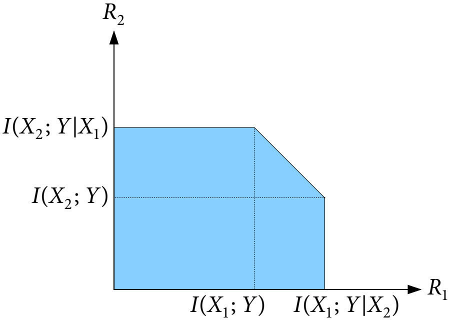

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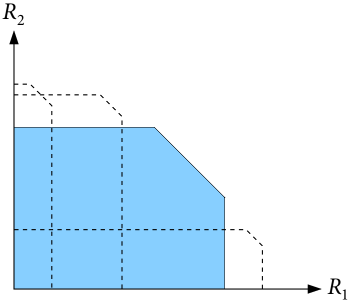

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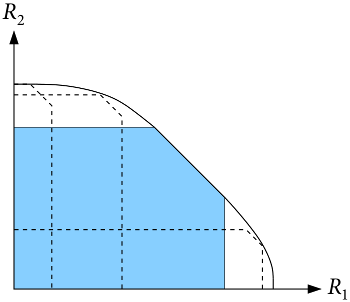

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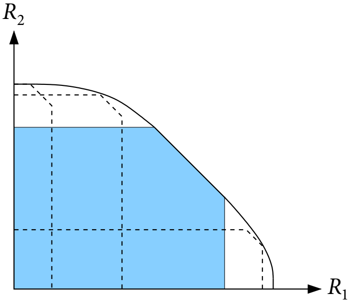
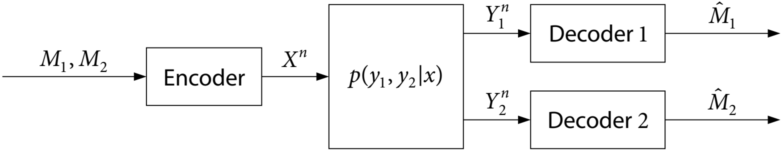


- Capacity region (Ahlswede 1971, Liao 1972)

Broadcast channel



- $(2^{nR_1}, 2^{nR_2}, n)$ code, $P_e^{(n)}$, achievability, $\mathscr{C}$: Same as before

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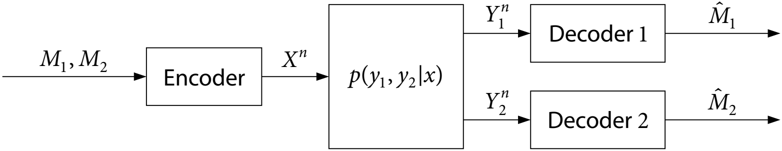
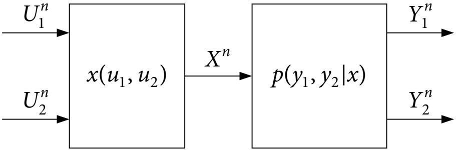


- $(2^{nR_1}, 2^{nR_2}, n)$ code, $P_e^{(n)}$, achievability, $\mathscr{C}$: Same as before
- Capacity region is not known in general

Superposition coding



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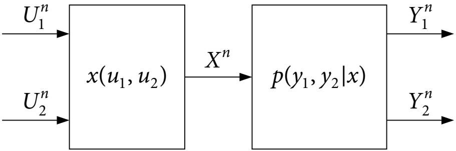

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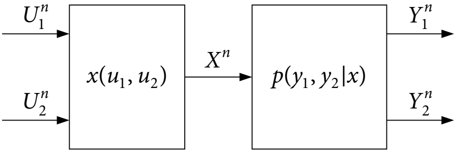


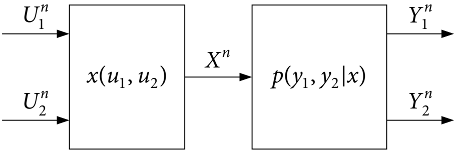
- Independent $U_1^n(m_1)$ and $U_2^n(m_2)$ (as in MAC)

- A simple inner bound: (R_1, R_2) is achievable if

$$\begin{aligned} R_1 &< I(U_1; Y_1), \\ R_2 &< I(U_2; Y_2) \end{aligned}$$

for some $p(u_1)p(u_2)$ and function $x(u_1, u_2)$

Marton coding



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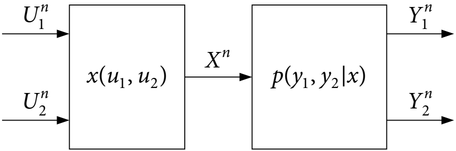

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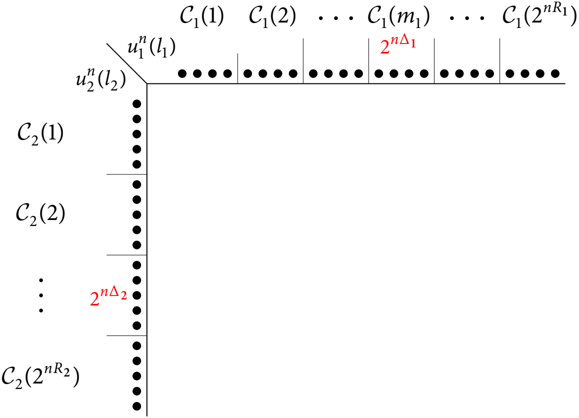
Multicoding and joint typicality encoding

- For each m_j, generate a subcodebook $\mathcal{C}_j(m_j)$ consisting of $u_j^n(l_j) \sim \prod_{i=1}^n p_{U_j}(u_{ji})$
- For each (m_1, m_2), find jointly typical $(u_1^n(l_1), u_2^n(l_2)) \in \mathcal{C}_1(m_1) \times \mathcal{C}_2(m_2)$



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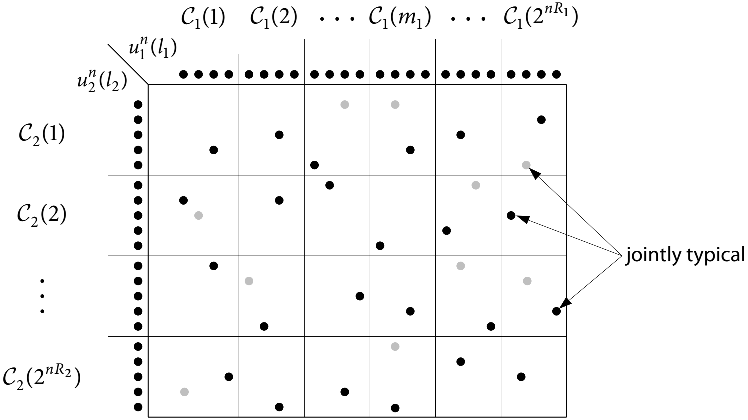

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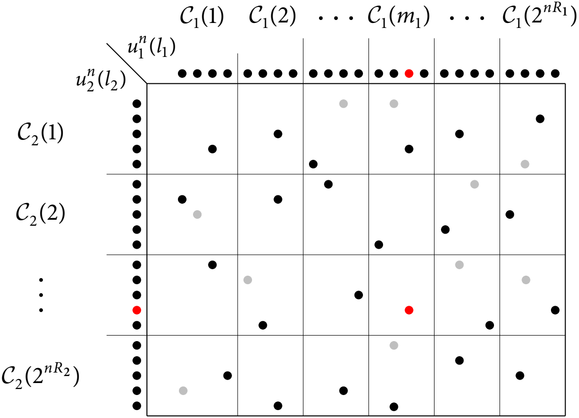

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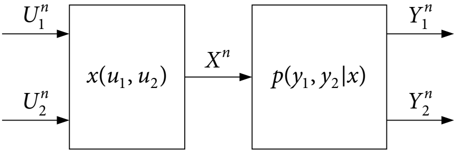

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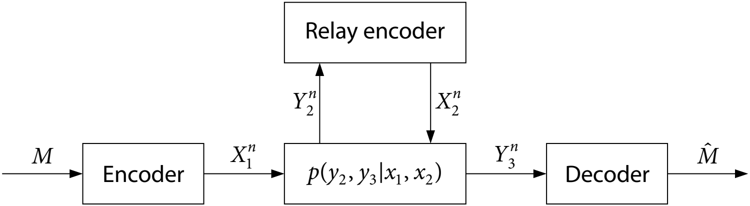
- Essentially the best known scheme for broadcast

Noosa, Sunshine Coast

Relay channel



- A $(2^{nR}, n)$ code:
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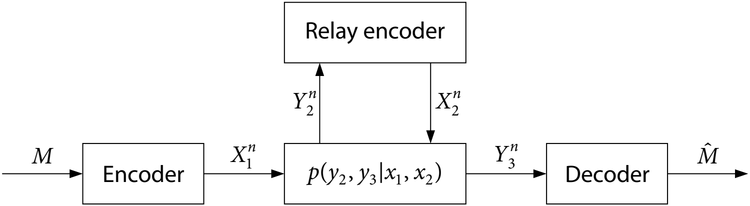

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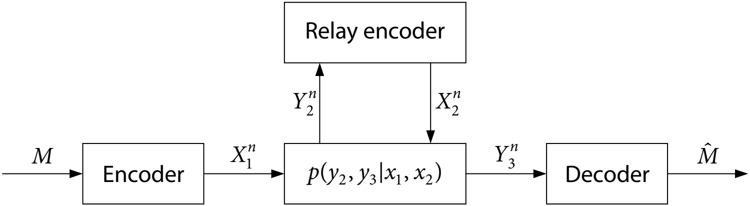

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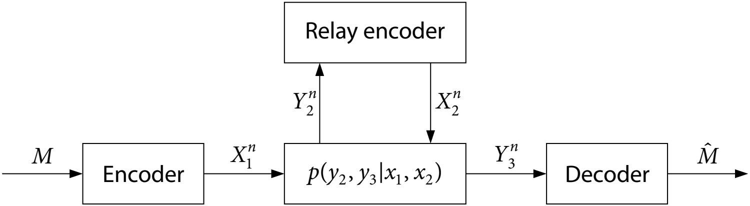

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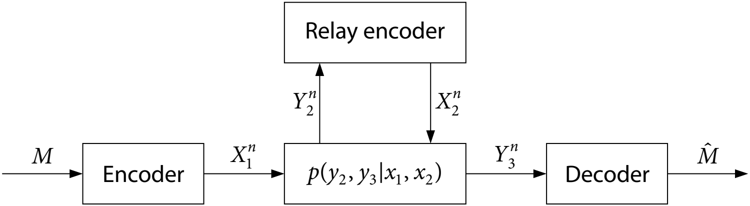

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Dictionary of relaying schemes

- Standard parlance: decode–forward, amplify–forward, compress–forward

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Cutset upper bound

Cover–El Gamal (1979)

$$C \leq R_{\mathrm{CS}} = \max_{p(x_1,x_2)} \min\bigl\{ I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3 | X_2) \bigr\}$$

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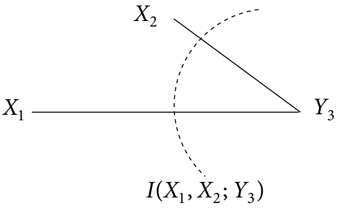
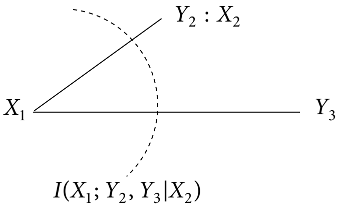
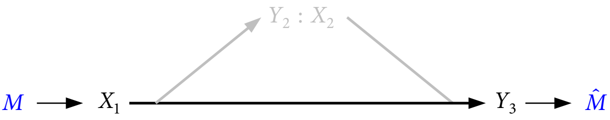


Cooperative MAC bound



Cooperative BC bound

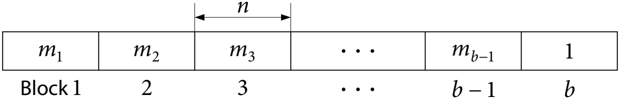
Direct transmission



$$C \geq \max_{p(x_1),\, x_2} I(X_1; Y_3 | X_2 = x_2)$$

Multihop

- Block Markov coding: Send $b-1$ messages over b n-transmission blocks



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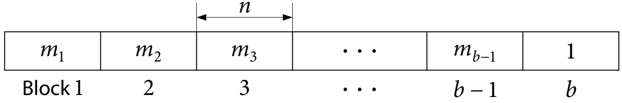
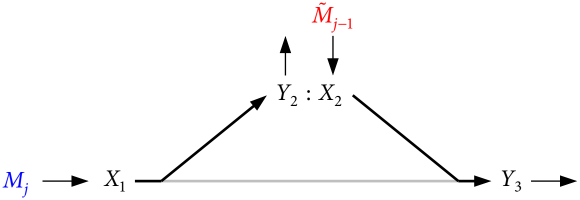


m_1	m_2	m_3	$\cdots$	m_{b-1}	1
Block 1	2	3	$\cdots$	$b-1$	b



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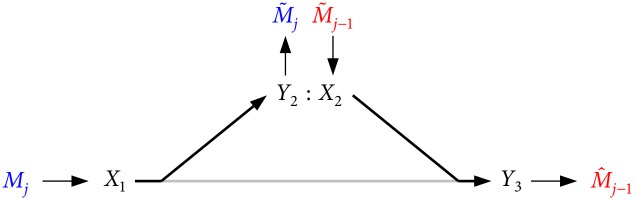

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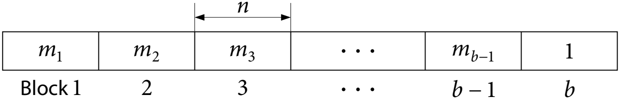

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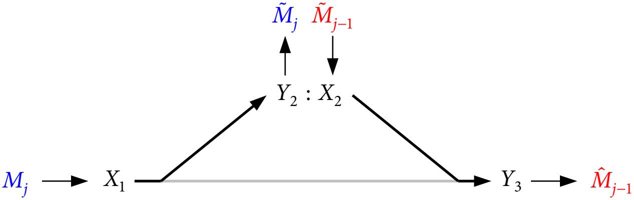

$$C \geq \max_{p(x_1)p(x_2)} \min\{I(X_2;Y_3),\ I(X_1;Y_2|X_2)\}$$

Coherent multihop

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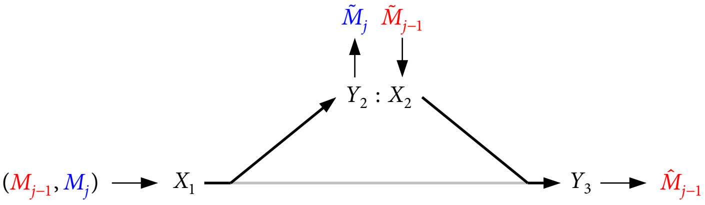

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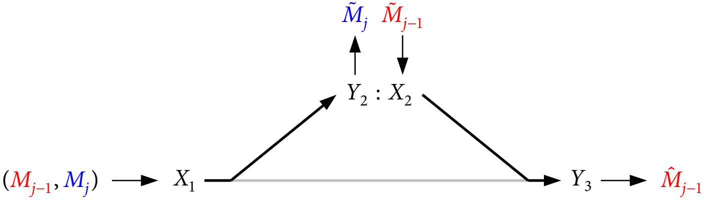

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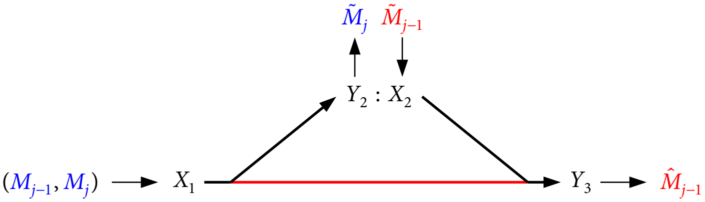
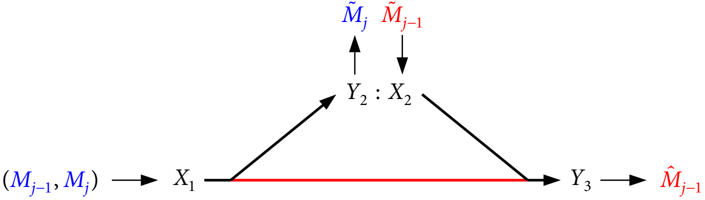
Decode–forward

- Also utilize the direct path via backward decoding



Decode–forward

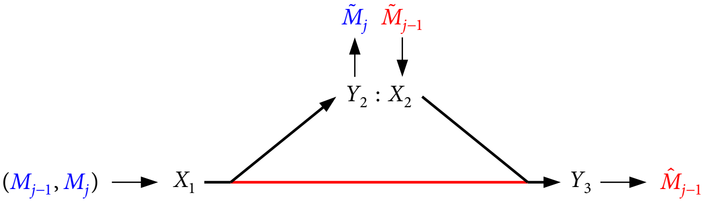
- Also utilize the direct path via backward decoding



Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1 \mid 1)$	$x_1^n(m_2 \mid m_1)$	$x_1^n(m_3 \mid m_2)$	…	$x_1^n(m_{b-1} \mid m_{b-2})$	$x_1^n(1 \mid m_{b-1})$
Y_2						
X_2						
Y_3						

Decode–forward

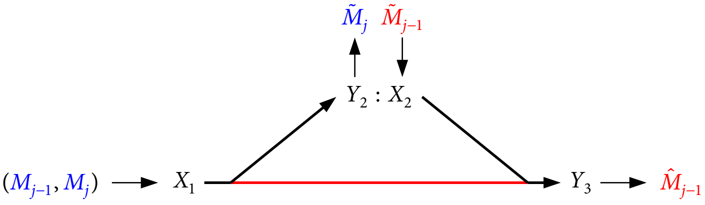
- Also utilize the direct path via backward decoding



Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1 \vert 1)$	$x_1^n(m_2 \vert m_1)$	$x_1^n(m_3 \vert m_2)$	…	$x_1^n(m_{b-1} \vert m_{b-2})$	$x_1^n(1 \vert m_{b-1})$
Y_2	$\tilde{m}_1 \rightarrow$	$\tilde{m}_2 \rightarrow$	$\tilde{m}_3 \rightarrow$	…	$\tilde{m}_{b-1}$	$\emptyset$
X_2						
Y_3						

Decode–forward

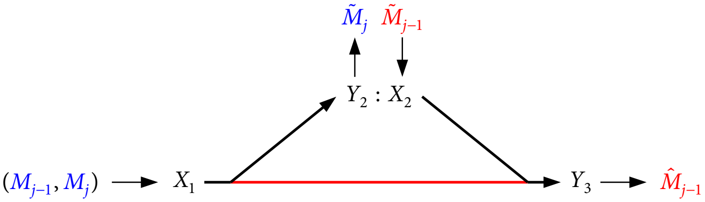
- Also utilize the direct path via backward decoding



Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1 \vert 1)$	$x_1^n(m_2 \vert m_1)$	$x_1^n(m_3 \vert m_2)$	…	$x_1^n(m_{b-1} \vert m_{b-2})$	$x_1^n(1 \vert m_{b-1})$
Y_2	$\tilde{m}_1 \rightarrow$	$\tilde{m}_2 \rightarrow$	$\tilde{m}_3 \rightarrow$	…	$\tilde{m}_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(\tilde{m}_1)$	$x_2^n(\tilde{m}_2)$	…	$x_2^n(\tilde{m}_{b-2})$	$x_2^n(\tilde{m}_{b-1})$
Y_3						

Decode–forward

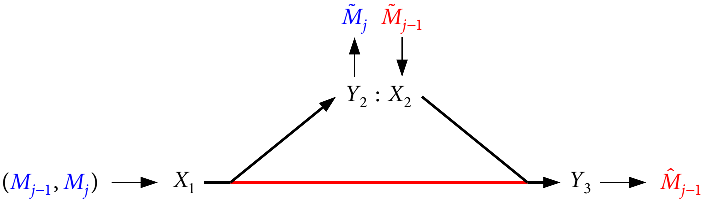
- Also utilize the direct path via backward decoding



Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1\|1)$	$x_1^n(m_2\|m_1)$	$x_1^n(m_3\|m_2)$	…	$x_1^n(m_{b-1}\|m_{b-2})$	$x_1^n(1\|m_{b-1})$
Y_2	$\tilde{m}_1 \rightarrow$	$\tilde{m}_2 \rightarrow$	$\tilde{m}_3 \rightarrow$	…	$\tilde{m}_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(\tilde{m}_1)$	$x_2^n(\tilde{m}_2)$	…	$x_2^n(\tilde{m}_{b-2})$	$x_2^n(\tilde{m}_{b-1})$
Y_3	$\emptyset$	$\hat{m}_1$	$\leftarrow \hat{m}_2$	…	$\leftarrow \hat{m}_{b-2}$	$\leftarrow \hat{m}_{b-1}$

Decode–forward

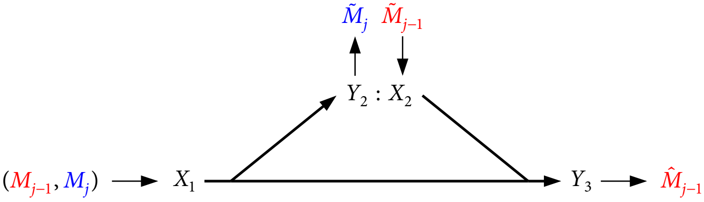
- Also utilize the direct path via backward decoding



Cover–El-Gamal (1979)

$$C \geq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2|X_2)\}$$

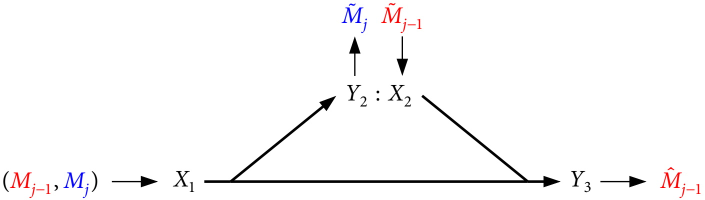
Decode–forward



- Decode–forward: $C \geq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2|X_2)\}$

  Cutset bound: $C \leq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3|X_2)\}$

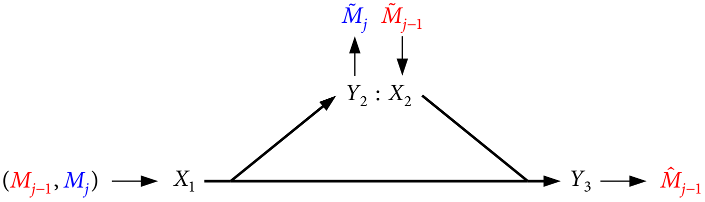
Decode–forward



- Decode–forward: $C \geq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2|X_2)\}$

  Cutset bound: $C \leq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3|X_2)\}$
- Performs well when the relay is stronger than the receiver

Decode–forward



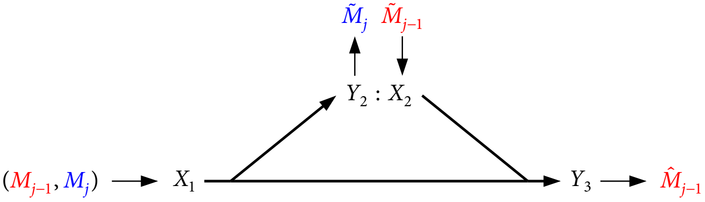
- Decode–forward: $C \geq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2|X_2)\}$

  Cutset bound: $C \leq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3|X_2)\}$

- Performs well when the relay is stronger than the receiver

- But worse than even direct transmission if the relay is weaker

Decode–forward



- Decode–forward: $C \geq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2|X_2)\}$

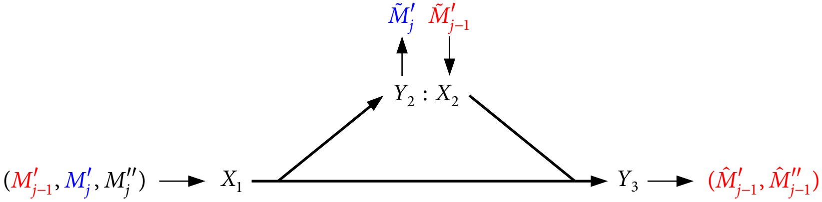
  Cutset bound: $C \leq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3|X_2)\}$

- Performs well when the relay is stronger than the receiver

- But worse than even direct transmission if the relay is weaker

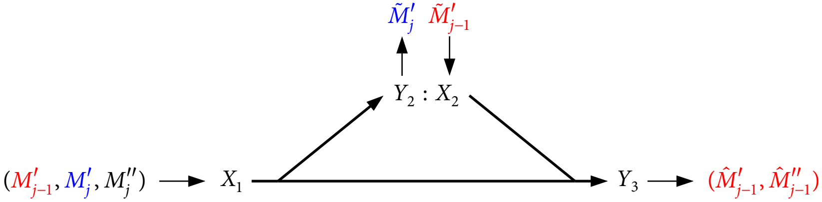
- Solutions:
  - Partial decode–forward: Recover only part of the message
  - Compress–forward: Do not recover the message at all

Partial decode–forward



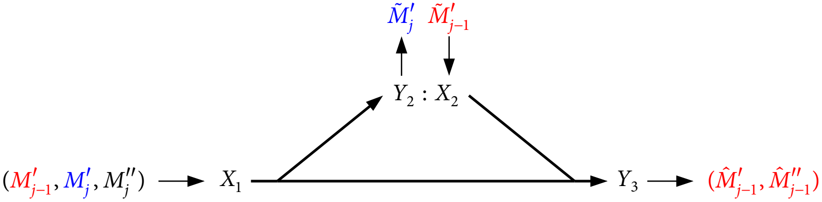
- Codebook structure: $(u^n(m'_j | m'_{j-1}), x_1^n(m'_j, m''_j | m'_{j-1}), x_2^n(m'_{j-1}))$

Partial decode–forward



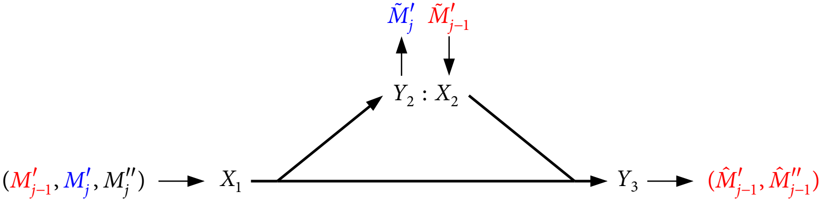
- Codebook structure: $(u^n(m'_j|m'_{j-1}), x_1^n(m'_j, m''_j|m'_{j-1}), x_2^n(m'_{j-1}))$
- Decode–forward of M'_j over $p(y_2, y_3|u, x_2)$

Partial decode–forward



- Codebook structure: $(u^n(m'_j|m'_{j-1}), x_1^n(m'_j, m''_j|m'_{j-1}), x_2^n(m'_{j-1}))$
- Decode–forward of M'_j over $p(y_2, y_3|u, x_2)$
- Direct transmission of M''_j over $p(y_3|x_1, u, x_2)$

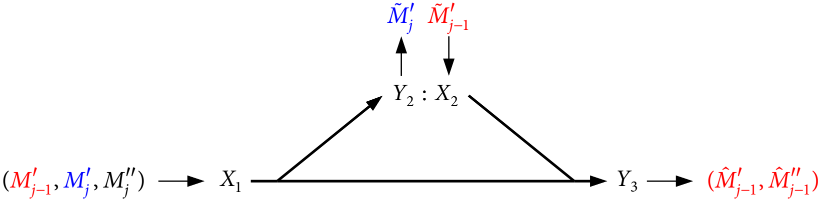
Partial decode–forward



Cover–El Gamal (1979)

$$C \geq R_{\text{PDF}} = \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(U; Y_2 | X_2) + I(X_1; Y_3 | X_2, U)\}$$

Partial decode–forward



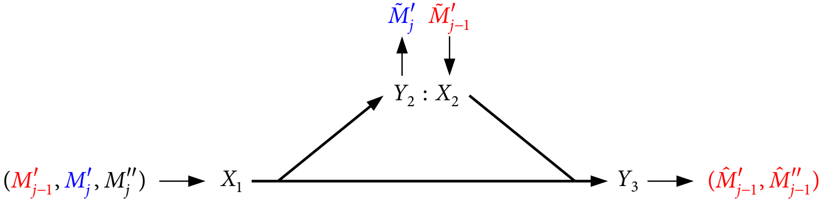
Cover–El Gamal (1979)

$$C \geq R_{\text{PDF}} = \max_{p(u,x_1,x_2)} \min\bigl\{I(X_1, X_2; Y_3),\ I(U; Y_2|X_2) + I(X_1; Y_3|X_2, U)\bigr\}$$

- Direct transmission ($U = \emptyset$, $M' = \emptyset$):

$$R_{\text{DT}} = \max_{p(x_1)} I(X_1; Y_3)$$

Partial decode–forward



Cover–El Gamal (1979)

$$C \geq R_{\mathrm{PDF}} = \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(U; Y_2 | X_2) + I(X_1; Y_3 | X_2, U)\}$$

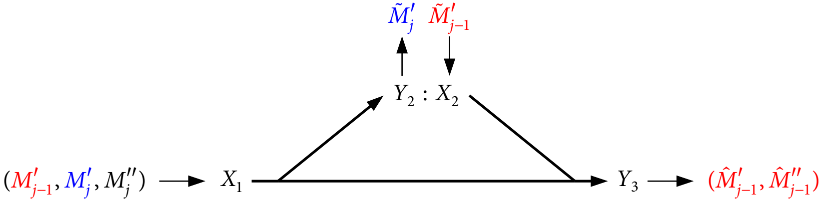
- **Direct transmission** ($U = \emptyset$, $M' = \emptyset$):

$$R_{\mathrm{DT}} = \max_{p(x_1)} I(X_1; Y_3)$$

- **Decode–forward** ($U = X_1$, $M'' = \emptyset$):

$$R_{\mathrm{DF}} = \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2 | X_2)\}$$

Partial decode–forward



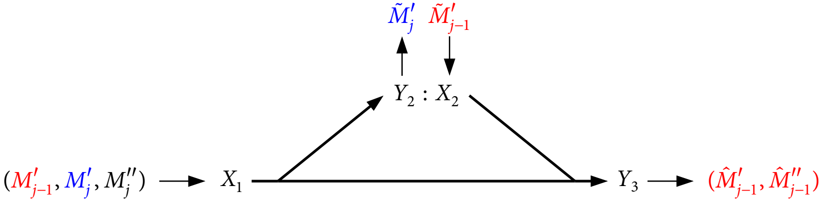
Cover–El Gamal (1979)

$$C \geq R_{\mathrm{PDF}} = \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(U; Y_2 | X_2) + I(X_1; Y_3 | X_2, U)\}$$

- Alternative representation (El Gamal–Aref 1982):

$$R_{\mathrm{PDF}} = \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; U, Y_3 | X_2) - I(U; X_1 | X_2, Y_2)\}$$

Partial decode–forward



Cover–El Gamal (1979)

$$C \geq R_{\text{PDF}} = \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(U; Y_2|X_2) + I(X_1; Y_3|X_2, U)\}$$

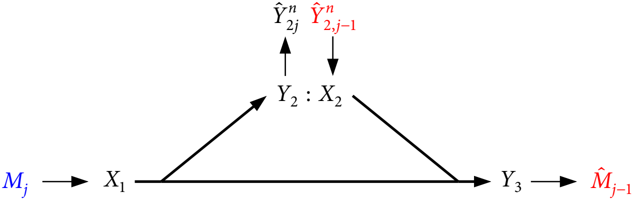
- Alternative representation (El Gamal–Aref 1982):

$$R_{\text{PDF}} = \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; U, Y_3|X_2) - I(U; X_1|X_2, Y_2)\}$$

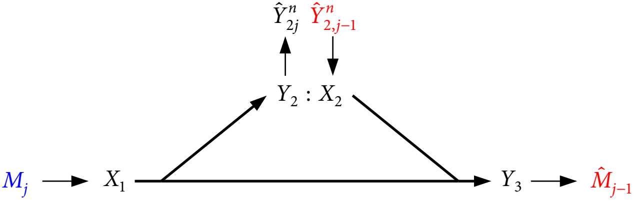
- Comparison to the cutset bound:

$$R_{\text{CS}} = \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3|X_2)\}$$

Compress–forward

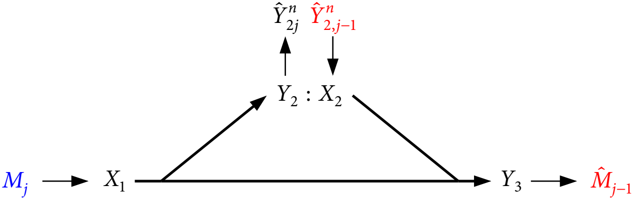


Compress–forward



- Codebook structure: $(x_1^n(m_j), x_2^n(l_{j-1}), \hat{y}_2^n(k_j|l_{j-1}))$

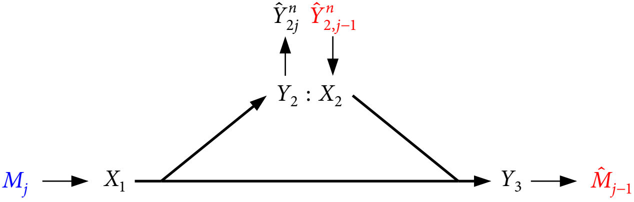
Compress–forward



- Codebook structure: $(x_1^n(m_j), x_2^n(l_{j-1}), \hat{y}_2^n(k_j|l_{j-1}))$

Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1)$	$x_1^n(m_2)$	$x_1^n(m_3)$	…	$x_1^n(m_{b-1})$	$x_1^n(1)$
Y_2						
X_2						
Y_3						

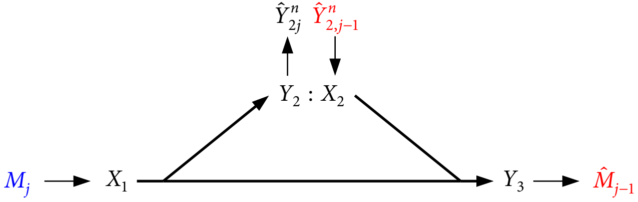
Compress–forward



- Codebook structure: $(x_1^n(m_j), x_2^n(l_{j-1}), \hat{y}_2^n(k_j|l_{j-1}))$

Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1)$	$x_1^n(m_2)$	$x_1^n(m_3)$	…	$x_1^n(m_{b-1})$	$x_1^n(1)$
Y_2	$\hat{y}_2^n(k_1\|1), l_1$	$\hat{y}_2^n(k_2\|l_1), l_2$	$\hat{y}_2^n(k_3\|l_2), l_3$	…	$\hat{y}_2^n(k_{b-1}\|l_{b-2}), l_{b-1}$	$\emptyset$
X_2						
Y_3						

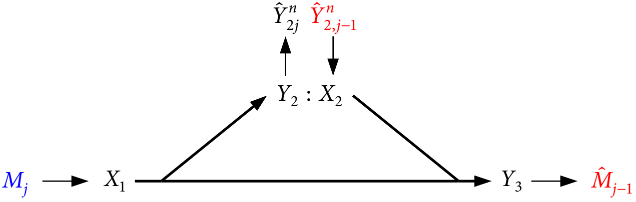
Compress–forward



- Codebook structure: $(x_1^n(m_j), x_2^n(l_{j-1}), \hat{y}_2^n(k_j|l_{j-1}))$

Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1)$	$x_1^n(m_2)$	$x_1^n(m_3)$	…	$x_1^n(m_{b-1})$	$x_1^n(1)$
Y_2	$\hat{y}_2^n(k_1\|1), l_1$	$\hat{y}_2^n(k_2\|l_1), l_2$	$\hat{y}_2^n(k_3\|l_2), l_3$	…	$\hat{y}_2^n(k_{b-1}\|l_{b-2}), l_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(l_1)$	$x_2^n(l_2)$	…	$x_2^n(l_{b-2})$	$x_2^n(l_{b-1})$
Y_3						

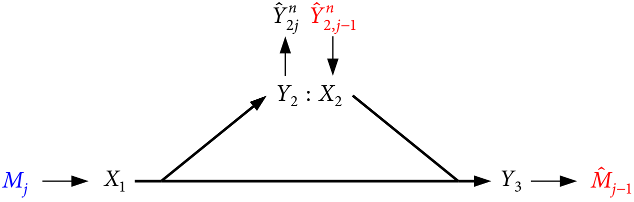
Compress–forward



- Codebook structure: $(x_1^n(m_j), x_2^n(l_{j-1}), \hat{y}_2^n(k_j | l_{j-1}))$

Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1)$	$x_1^n(m_2)$	$x_1^n(m_3)$	…	$x_1^n(m_{b-1})$	$x_1^n(1)$
Y_2	$\hat{y}_2^n(k_1\|1), l_1$	$\hat{y}_2^n(k_2\|l_1), l_2$	$\hat{y}_2^n(k_3\|l_2), l_3$	…	$\hat{y}_2^n(k_{b-1}\|l_{b-2}), l_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(l_1)$	$x_2^n(l_2)$	…	$x_2^n(l_{b-2})$	$x_2^n(l_{b-1})$
Y_3	$\emptyset$	$\hat{l}_1, \hat{k}_1, \hat{m}_1$	$\hat{l}_2, \hat{k}_2, \hat{m}_2$	…	$\hat{l}_{b-2}, \hat{k}_{b-2}, \hat{m}_{b-2}$	$\hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}$

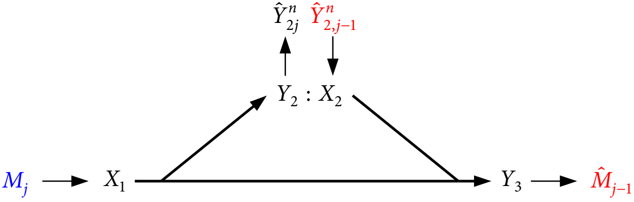
Compress–forward



Cover–El Gamal (1979)

$$C \geq R_{\text{CF}} = \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2):I(X_2;Y_3)\geq I(Y_2;\hat{Y}_2|X_2,Y_3)} I(X_1;\hat{Y}_2,Y_3|X_2)$$

Compress–forward



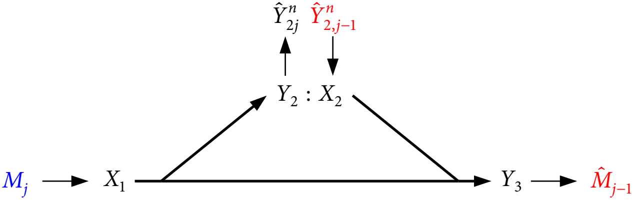
Cover–El Gamal (1979)

$$C \geq R_{\text{CF}} = \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2): I(X_2;Y_3) \geq I(Y_2;\hat{Y}_2|X_2,Y_3)} I(X_1; \hat{Y}_2, Y_3 | X_2)$$

- Alternative representation (El-Gamal–Mohseni–Zahedi 2006):

$$R_{\text{CF}} = \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2)} \min\{I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2 | X_1, X_2, Y_3),\ I(X_1; \hat{Y}_2, Y_3 | X_2)\}$$

Compress–forward



Cover–El Gamal (1979)

$$C \geq R_{\text{CF}} = \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2): I(X_2;Y_3) \geq I(Y_2;\hat{Y}_2|X_2,Y_3)} I(X_1; \hat{Y}_2, Y_3 | X_2)$$

- Alternative representation (El-Gamal–Mohseni–Zahedi 2006):

$$R_{\text{CF}} = \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2)} \min\{I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2 | X_1, X_2, Y_3),\ I(X_1; \hat{Y}_2, Y_3 | X_2)\}$$

- Comparison to the cutset bound:

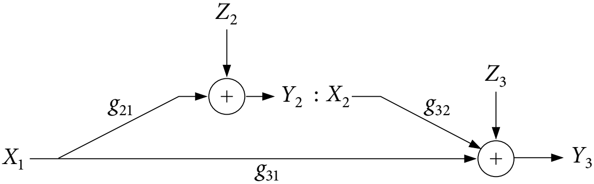
$$R_{\text{CS}} = \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3 | X_2)\}$$

Partial decode–forward vs. compress–forward

$$\begin{aligned}
R_{\text{PDF}} &= \max\min\{I(X_1, X_2; Y_3), \quad -I(U; X_1 | X_2, Y_2) + I(X_1; U, \ Y_3 | X_2)\}, \\
R_{\text{CF}} &= \max\min\{I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2 | X_1, X_2, Y_3), \quad I(X_1; \hat{Y}_2, Y_3 | X_2)\}, \\
R_{\text{CS}} &= \max\min\{I(X_1, X_2; Y_3), \quad I(X_1; Y_2, Y_3 | X_2)\}
\end{aligned}$$

Partial decode–forward vs. compress–forward

$$\begin{aligned}
R_{\mathrm{PDF}} &= \max\min\{I(X_1, X_2; Y_3), \quad -I(U; X_1 | X_2, Y_2) + I(X_1; U,\ Y_3 | X_2)\}, \\
R_{\mathrm{CF}} &= \max\min\{I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2 | X_1, X_2, Y_3), \quad I(X_1; \hat{Y}_2, Y_3 | X_2)\}, \\
R_{\mathrm{CS}} &= \max\min\{I(X_1, X_2; Y_3), \quad I(X_1; Y_2, Y_3 | X_2)\}
\end{aligned}$$



Partial decode–forward vs. compress–forward

$$
\begin{aligned}
R_{\mathrm{PDF}} &= \max\min\{I(X_1, X_2; Y_3), \quad -I(U; X_1 | X_2, Y_2) + I(X_1; U, Y_3 | X_2)\},\\
R_{\mathrm{CF}} &= \max\min\{I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2 | X_1, X_2, Y_3), \quad I(X_1; \hat{Y}_2, Y_3 | X_2)\},\\
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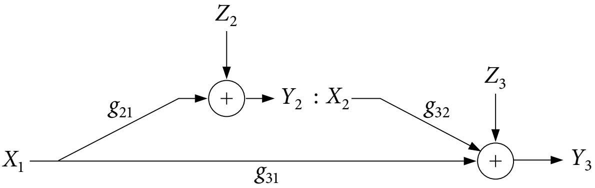
\end{aligned}
$$



- $\Delta_{\mathrm{PDF}} = R_{\mathrm{CS}} - R_{\mathrm{PDF}} \le 1/2$ and $\Delta_{\mathrm{CF}} = R_{\mathrm{CS}} - R_{\mathrm{CF}} \le 1/2$ (cf. Jin–Kim 2014)

Partial decode–forward vs. compress–forward

$$\begin{aligned}
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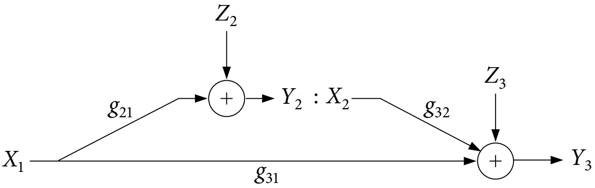

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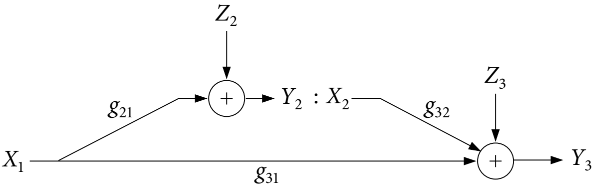

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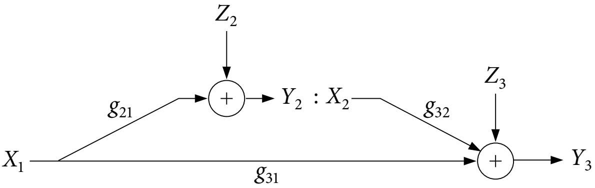

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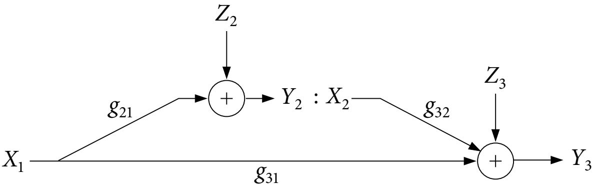

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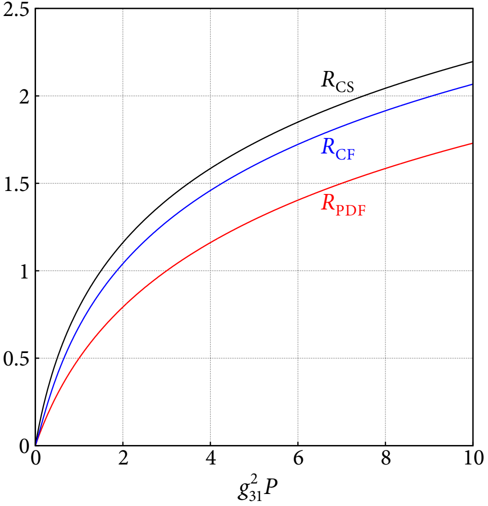
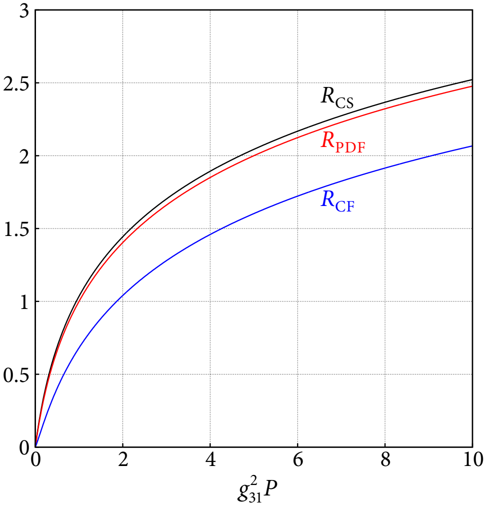
Comparison of coding schemes



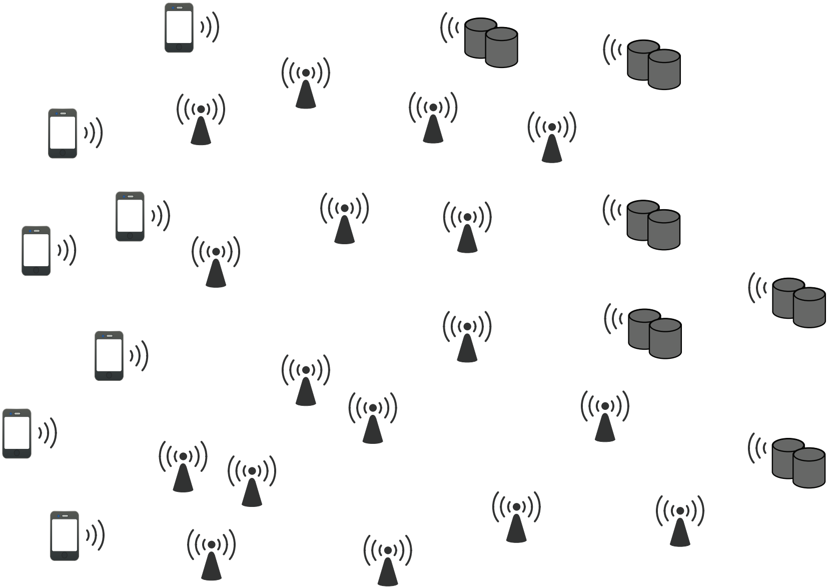
$$g_{21} = g_{31}, g_{32} = 2g_{31}$$

Comparison of coding schemes

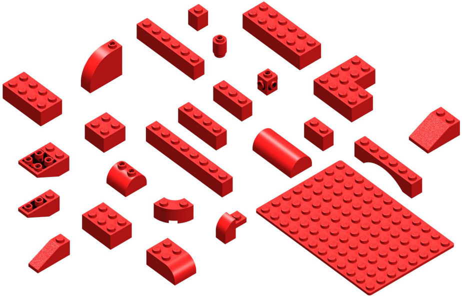


$$g_{21} = 2g_{31}, g_{32} = g_{31}$$

Snapper Rock, Gold Coast

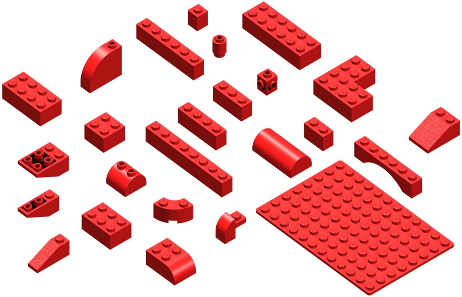


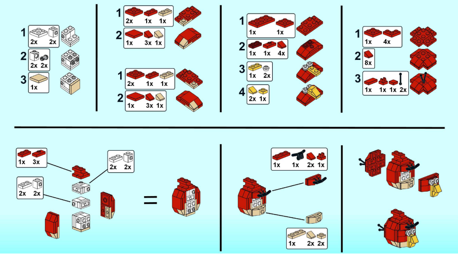
Relaying for networks



- Simultaneous decoding
- Superposition coding
- Multicoding
- Block Markov coding
- Decode–forward
- Compress–forward

Relaying for networks





- Simultaneous decoding
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- Decode–forward
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- Noisy network coding
- Distributed decode–forward

Putting things together

- Often vanilla extensions do not work

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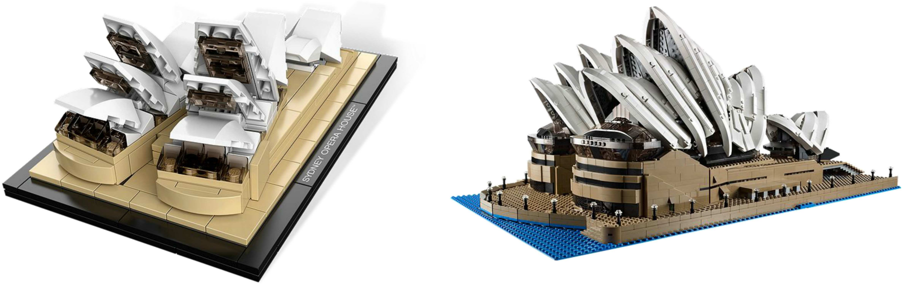
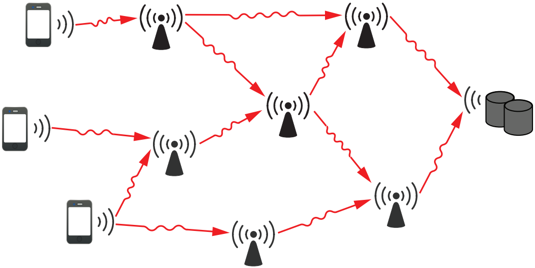
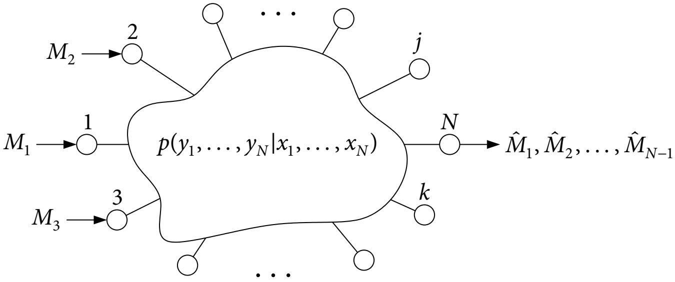
- Often vanilla extensions do not work
- Building blocks should be refined



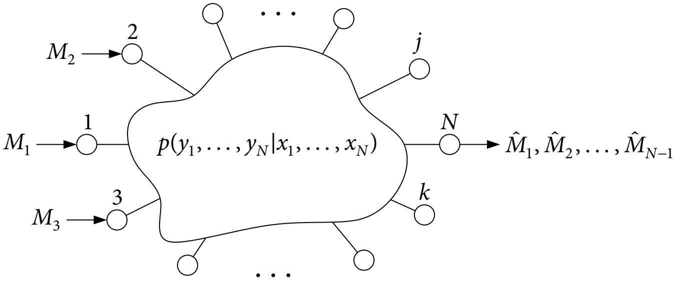
Multiple access relay network



Multiple access relay network

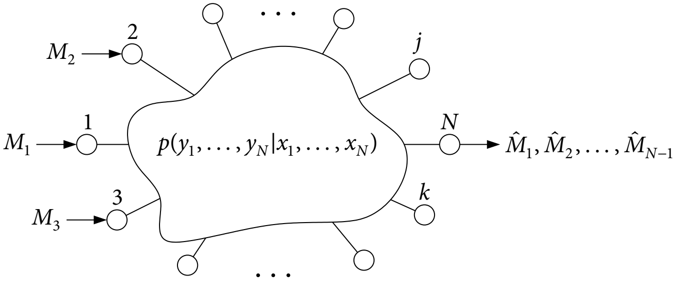


Multiple access relay network



- Multihop uplink communication

Multiple access relay network



- Multihop uplink communication
- A $(2^{nR_1},\ldots,2^{nR_{N-1}},n)$ code:
  - Message set: $[1:2^{nR_1}]\times\cdots\times[1:2^{nR_{N-1}}]$
  - Encoder: $x_{ji}(m_j, y_j^{i-1}), j\in[1:N-1], i\in[1:n]$
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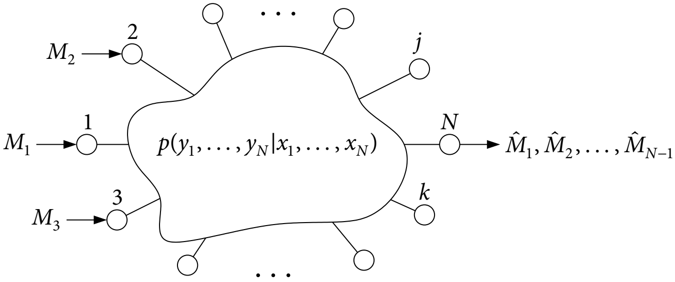
Multiple access relay network



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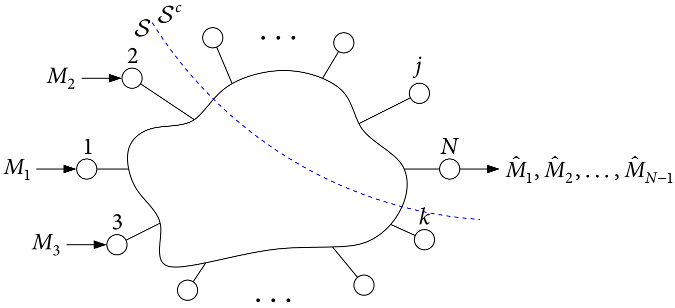
- Capacity region? Optimal coding scheme?

Cutset outer bound

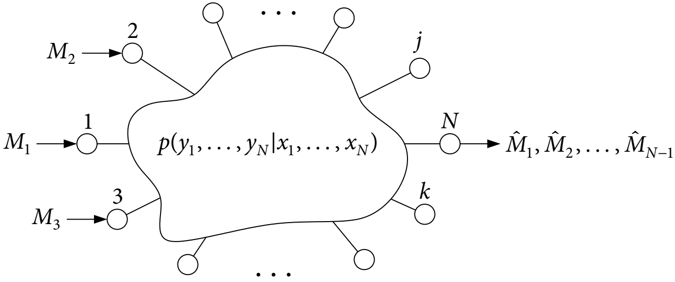


El Gamal (1981)

$$R(\mathcal{S}) := \sum_{j \in \mathcal{S}} R_j \leq I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c)), \quad \forall \mathcal{S}$$

for some $p(x^N)$

Noisy network coding



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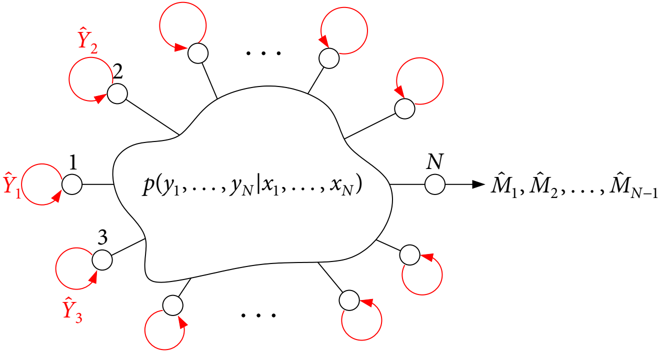
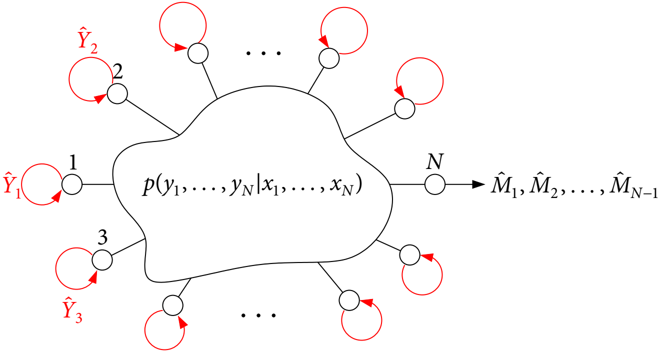
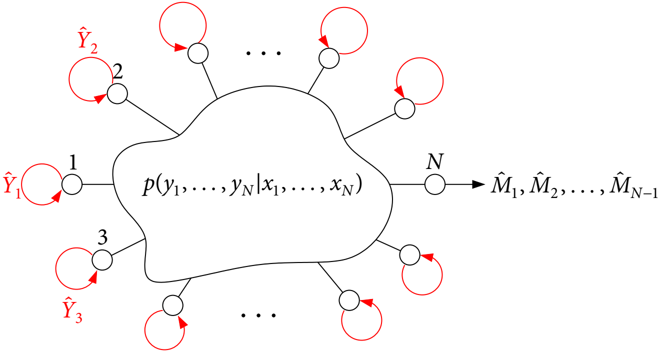


Noisy network coding



- Block Markov coding:
- Relay compression:
- Auxiliary index:
- Simultaneous decoding:

Noisy network coding



- Block Markov coding: Splitting of M_* over b blocks of n transmissions
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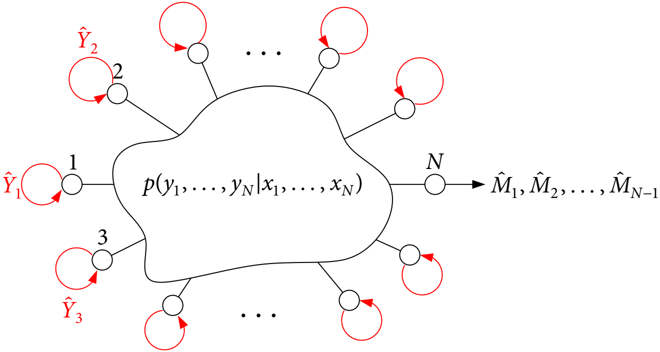
Noisy network coding



- **Block Markov coding:** Splitting of M_* over b blocks of n transmissions
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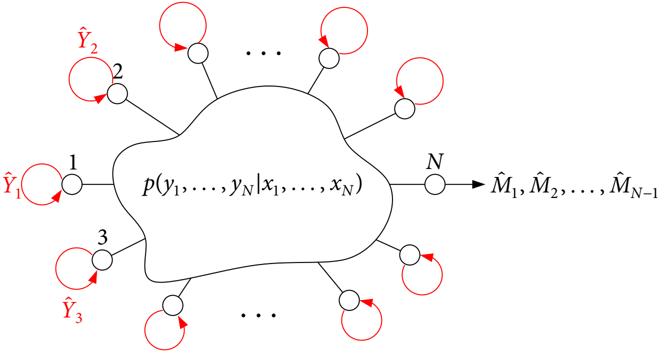
Noisy network coding



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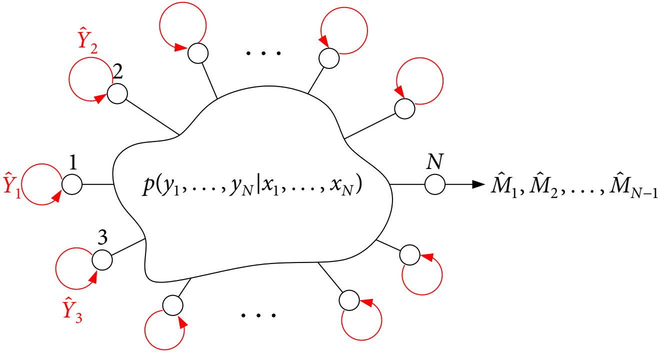
Noisy network coding



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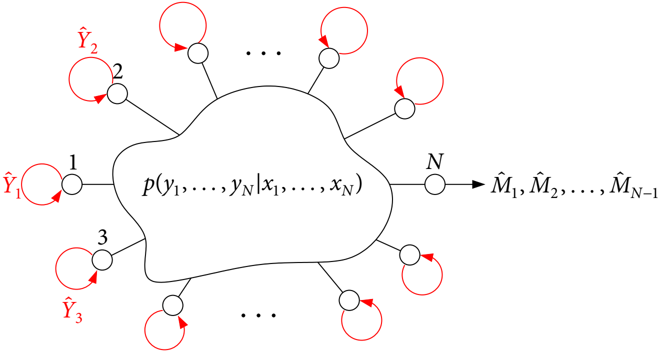
- Simultaneous decoding: Y_N recovers (M_*, L_*)

Noisy network coding

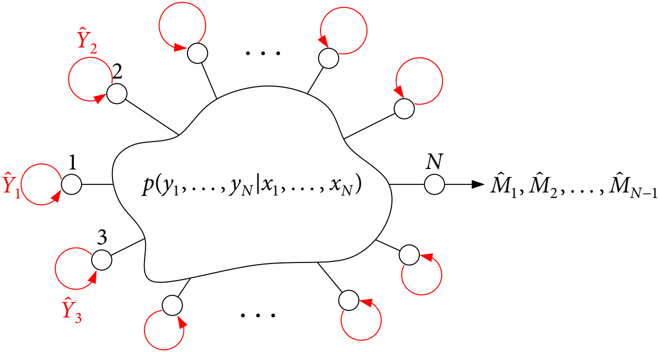


Lim–Kim–El Gamal–Chung (2011)

$$R(\mathcal{S}) \le I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c) | X(\mathcal{S}^c)) - I(Y(\mathcal{S}); \hat{Y}(\mathcal{S}) | X^N, \hat{Y}(\mathcal{S}^c)), \quad \forall \mathcal{S}$$

for some $\prod_{k=1}^{N} p(x_k) p(\hat{y}_k | y_k, x_k)$

Noisy network coding



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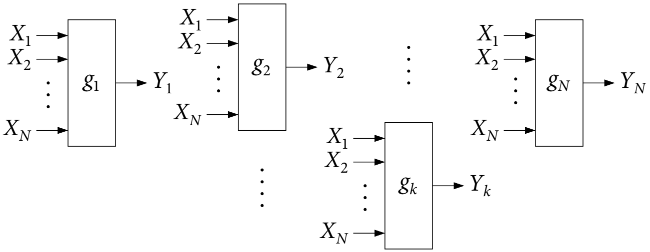
Deterministic network (Avestimehr–Diggavi–Tse 2011)



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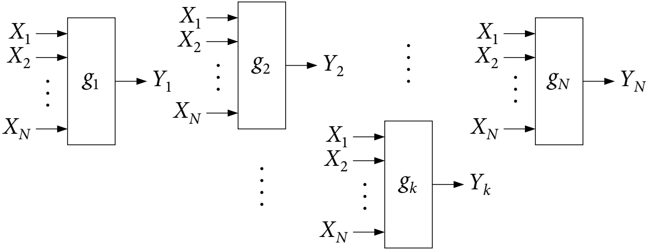

Cutset

$$R(\mathcal{S}) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

Noisy network coding

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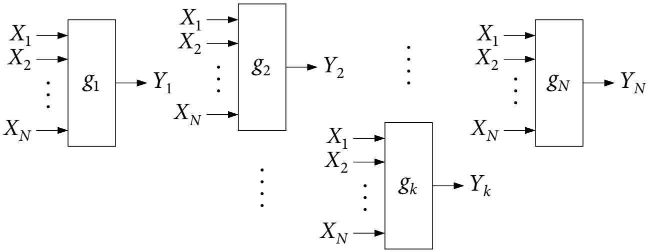

Cutset

$$\begin{aligned} R(\mathcal{S}) &\le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c)) \\ &= H(Y(\mathcal{S}^c) | X(\mathcal{S}^c)), \quad \forall \mathcal{S} \end{aligned}$$

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Noisy network coding

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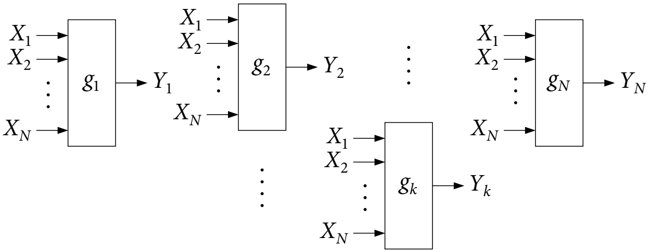

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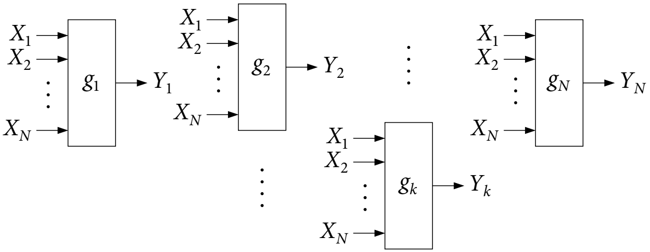

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for some $\prod_{k=1}^N p(x_k)$

- Tight for graphical networks and deterministic networks with no interference

Gaussian network

- Channel model:

$$Y_k = \sum_j g_{kj} X_j + Z_k, \quad k \in [1:N]$$

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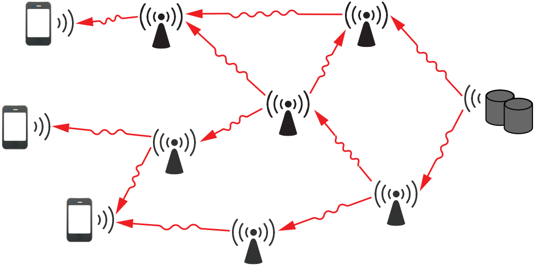
Lim–Kim–El Gamal–Chung (2011)

If $(R_1, \ldots, R_N) \in \mathscr{R}_{\mathrm{CS}}$, then $(R_1 - 0.63N, \ldots, R_N - 0.63N) \in \mathscr{R}_{\mathrm{NNC}}$

Broadcast relay network



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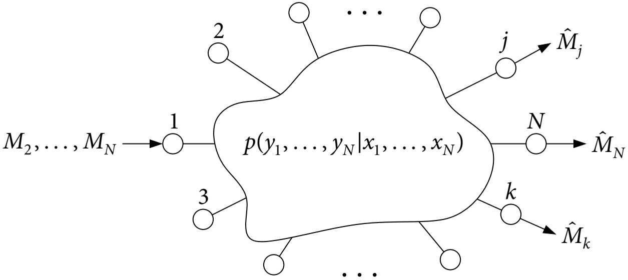
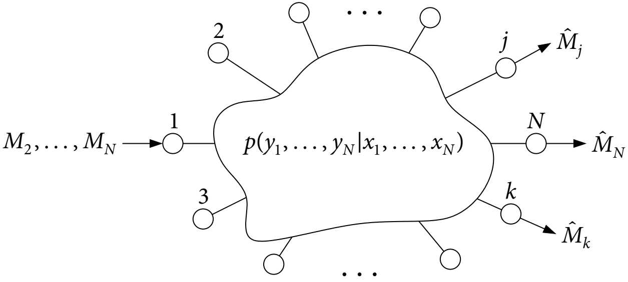
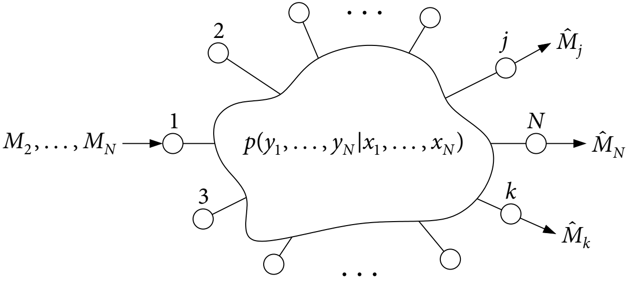


Broadcast relay network



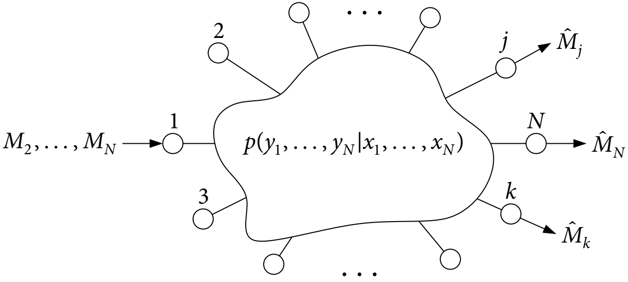
- Multihop downlink communication

Broadcast relay network



- Multihop downlink communication
- A $(2^{nR_2}, \ldots, 2^{nR_N}, n)$ code:
  - Message set: $[1:2^{nR_2}] \times \cdots \times [1:2^{nR_N}]$
  - Encoder: $x_{1i}(m_2, \ldots, m_N, y_1^{i-1})$, $i \in [1:n]$
  - Relay encoder: $x_{ji}(y_j^{i-1})$, $j \in [2:N]$, $i \in [1:n]$
  - Decoder: $\hat{m}_j(y_j^n)$, $j \in [2:N]$

Broadcast relay network



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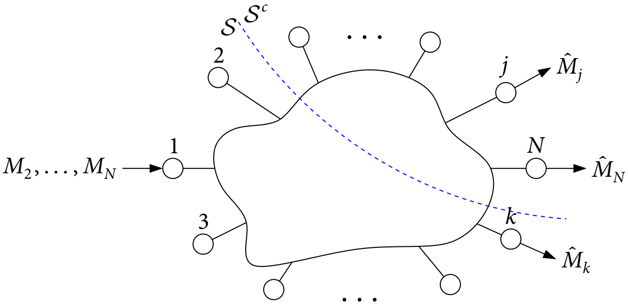
  - Decoder: $\hat{m}_j(y_j^n)$, $j \in [2:N]$
- Capacity region? Optimal coding scheme?

Cutset outer bound

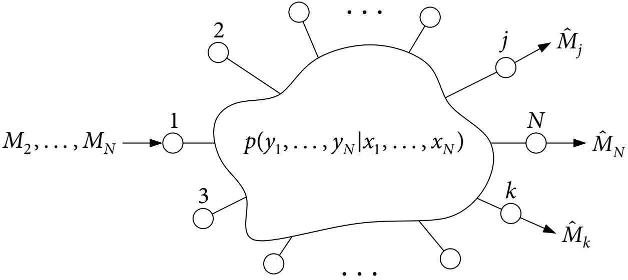


El Gamal (1981)

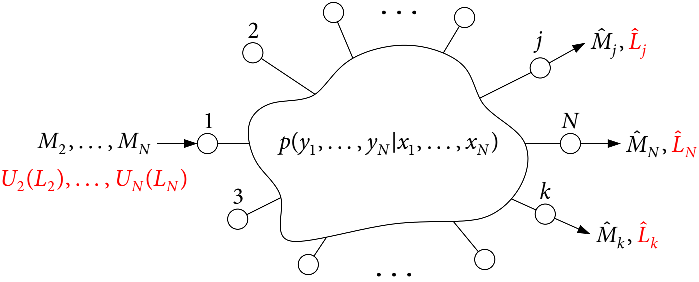
$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c)), \quad \forall \mathcal{S}$$

for some $p(x^N)$

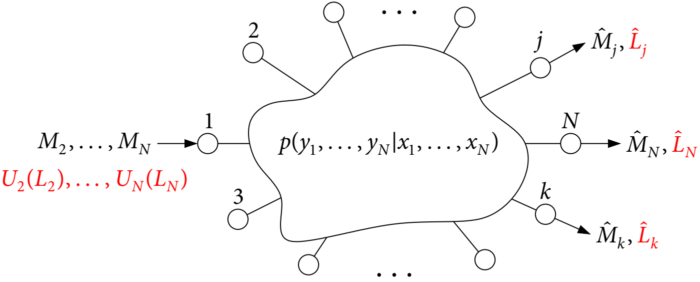
Distributed decode–forward



Distributed decode–forward

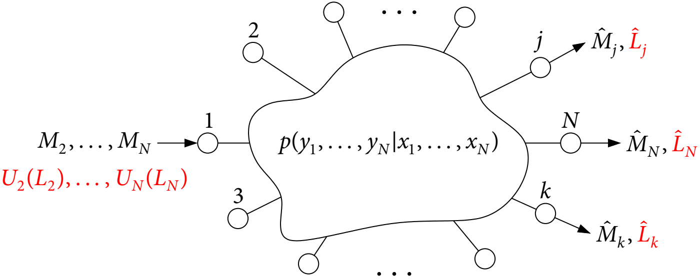


Distributed decode–forward



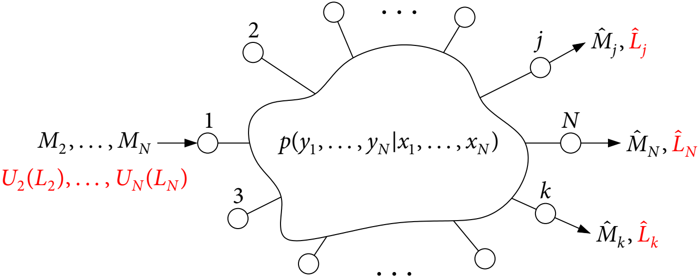
- Block Markov coding:
- Backward encoding:
- Partial decoding:
- Auxiliary index:

Distributed decode–forward



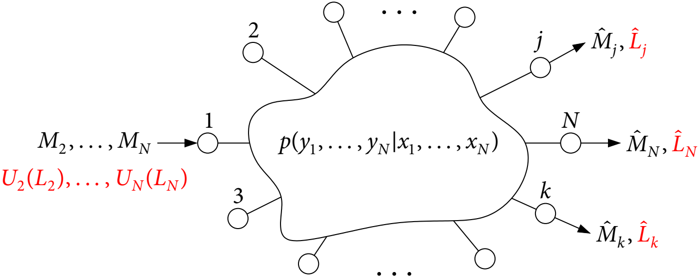
- Block Markov coding: Splitting of M_* over b blocks of n transmissions
- Backward encoding:
- Partial decoding:
- Auxiliary index:

Distributed decode–forward



- Block Markov coding: Splitting of M_* over b blocks of n transmissions
- Backward encoding: Multicoding of $X_*(M_*)$ and $U_2(L_2), \dots, U_N(L_N)$
- Partial decoding:
- Auxiliary index:

Distributed decode–forward



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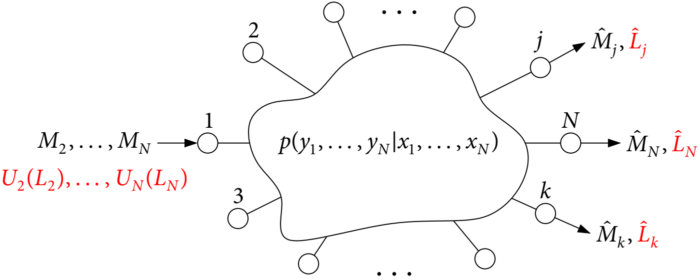
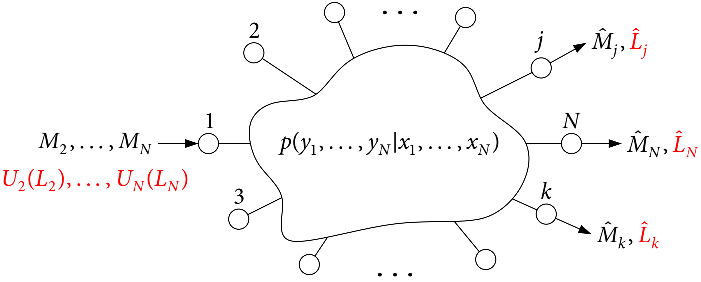
Distributed decode–forward



- Block Markov coding: Splitting of M_* over b blocks of n transmissions
- Backward encoding: Multicoding of $X_*(M_*)$ and $U_2(L_2), \ldots, U_N(L_N)$
- Partial decoding: Relay k decodes Y_k to recover $U_k(L_k)$
- Auxiliary index: $(U_2(L_2), \ldots, U_N(L_N))$ conveys the messages implicitly

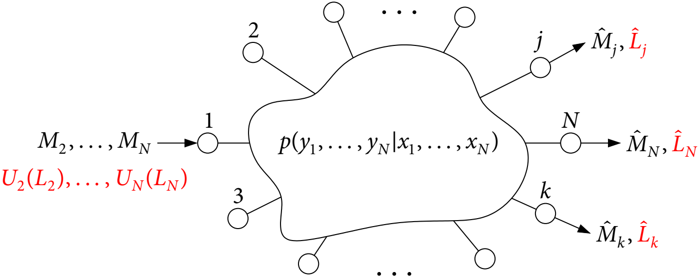
Distributed decode–forward



- Codebook structure: $x_1^n(\mathbf{m}_j|\mathbf{l}_j, \mathbf{l}_{j-1}), x_k^n(l_{k,j-1}), u_k^n(m_{kj}, l_{kj}|l_{k,j-1})$

Block	1	2	3	$\cdots$	b
X_1					
X_k					
Y_k					

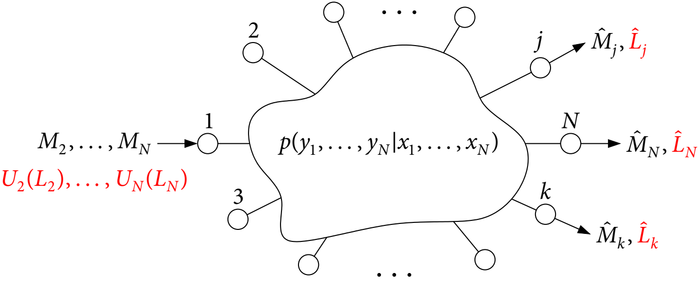
Distributed decode–forward



- Codebook structure: $x_1^n(\mathbf{m}_j | \mathbf{l}_j, \mathbf{l}_{j-1}), x_k^n(l_{k,j-1}), u_k^n(m_{kj}, l_{kj} | l_{k,j-1})$

Block	1	2	3	$\cdots$	b
X_1	$\mathbf{l}_0$	$\leftarrow \mathbf{l}_1$	$\leftarrow \mathbf{l}_2$	$\cdots$	$\leftarrow \mathbf{l}_{b-1}$
X_k					
Y_k					

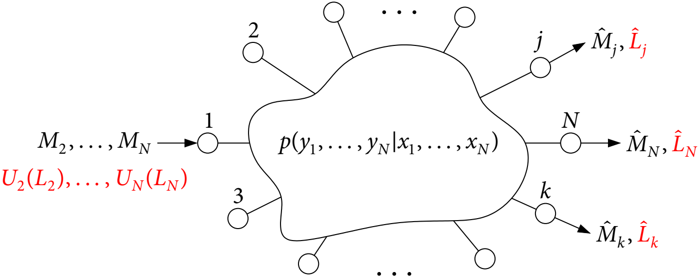
Distributed decode–forward



- Codebook structure: $x_1^n(\mathbf{m}_j | \mathbf{l}_j, \mathbf{l}_{j-1}), x_k^n(l_{k,j-1}), u_k^n(m_{kj}, l_{kj} | l_{k,j-1})$

Block	1	2	3	$\cdots$	b
X_1	$\mathbf{l}_0$	$\leftarrow \mathbf{l}_1$	$\leftarrow \mathbf{l}_2$	$\cdots$	$\leftarrow \mathbf{l}_{b-1}$
	$x_1^n(\mathbf{m}_1 \| \mathbf{l}_1, \mathbf{l}_0)$	$x_1^n(\mathbf{m}_2 \| \mathbf{l}_2, \mathbf{l}_1)$	$x_1^n(\mathbf{m}_3 \| \mathbf{l}_3, \mathbf{l}_2)$	$\cdots$	$x_1^n(\mathbf{m}_b \| \mathbf{l}_b, \mathbf{l}_{b-1})$
X_k					
Y_k					

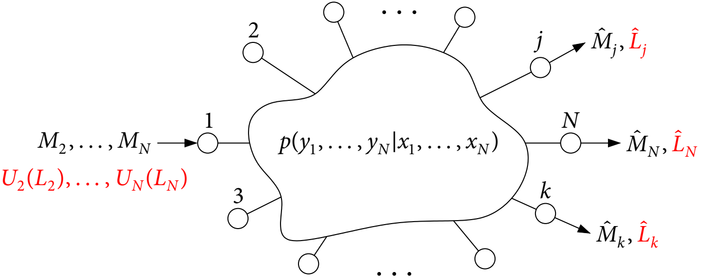
Distributed decode–forward



- Codebook structure: $x_1^n(\mathbf{m}_j | \mathbf{l}_j, \mathbf{l}_{j-1}), x_k^n(l_{k,j-1}), u_k^n(m_{kj}, l_{kj} | l_{k,j-1})$

Block	1	2	3	$\cdots$	b
X_1	$\mathbf{l}_0$	$\leftarrow \mathbf{l}_1$	$\leftarrow \mathbf{l}_2$	$\cdots$	$\leftarrow \mathbf{l}_{b-1}$
	$x_1^n(\mathbf{m}_1 \| \mathbf{l}_1, \mathbf{l}_0)$	$x_1^n(\mathbf{m}_2 \| \mathbf{l}_2, \mathbf{l}_1)$	$x_1^n(\mathbf{m}_3 \| \mathbf{l}_3, \mathbf{l}_2)$	$\cdots$	$x_1^n(\mathbf{m}_b \| \mathbf{l}_b, \mathbf{l}_{b-1})$
X_k					
Y_k	$\hat{m}_{k1}, \hat{l}_{k1}$	$\hat{m}_{k2}, \hat{l}_{k2}$	$\hat{m}_{k3}, \hat{l}_{k3}$	$\cdots$	$\hat{m}_{kb}, \hat{l}_{kb}$

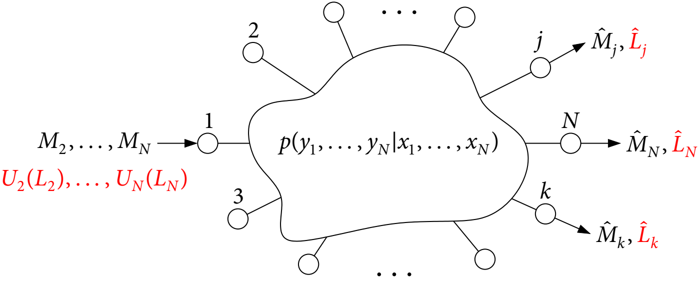
Distributed decode–forward



- Codebook structure: $x_1^n(\mathbf{m}_j|\mathbf{l}_j, \mathbf{l}_{j-1}), x_k^n(l_{k,j-1}), u_k^n(m_{kj}, l_{kj}|l_{k,j-1})$

Block	1	2	3	$\cdots$	b
X_1	$\mathbf{l}_0$	$\leftarrow \mathbf{l}_1$	$\leftarrow \mathbf{l}_2$	$\cdots$	$\leftarrow \mathbf{l}_{b-1}$
	$x_1^n(\mathbf{m}_1|\mathbf{l}_1, \mathbf{l}_0)$	$x_1^n(\mathbf{m}_2|\mathbf{l}_2, \mathbf{l}_1)$	$x_1^n(\mathbf{m}_3|\mathbf{l}_3, \mathbf{l}_2)$	$\cdots$	$x_1^n(\mathbf{m}_b|\mathbf{l}_b, \mathbf{l}_{b-1})$
X_k	$x_k^n(\hat{l}_{k0})$				
Y_k	$\hat{m}_{k1}, \hat{l}_{k1}$	$\hat{m}_{k2}, \hat{l}_{k2}$	$\hat{m}_{k3}, \hat{l}_{k3}$	$\cdots$	$\hat{m}_{kb}, \hat{l}_{kb}$

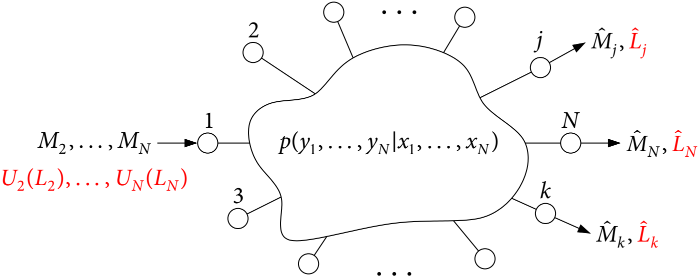
Distributed decode–forward



- Codebook structure: $x_1^n(\mathbf{m}_j | \mathbf{l}_j, \mathbf{l}_{j-1}), x_k^n(l_{k,j-1}), u_k^n(m_{kj}, l_{kj} | l_{k,j-1})$

Block	1	2	3	$\cdots$	b
X_1	$\mathbf{l}_0$	$\leftarrow \mathbf{l}_1$	$\leftarrow \mathbf{l}_2$	$\cdots$	$\leftarrow \mathbf{l}_{b-1}$
	$x_1^n(\mathbf{m}_1 | \mathbf{l}_1, \mathbf{l}_0)$	$x_1^n(\mathbf{m}_2 | \mathbf{l}_2, \mathbf{l}_1)$	$x_1^n(\mathbf{m}_3 | \mathbf{l}_3, \mathbf{l}_2)$	$\cdots$	$x_1^n(\mathbf{m}_b | \mathbf{l}_b, \mathbf{l}_{b-1})$
X_k	$x_k^n(\hat{l}_{k0})$	$x_k^n(\hat{l}_{k1})$	$x_k^n(\hat{l}_{k2})$	$\cdots$	$x_k^n(\hat{l}_{k,b-1})$
Y_k	$\hat{m}_{k1}, \hat{l}_{k1}$	$\hat{m}_{k2}, \hat{l}_{k2}$	$\hat{m}_{k3}, \hat{l}_{k3}$	$\cdots$	$\hat{m}_{kb}, \hat{l}_{kb}$

Distributed decode–forward

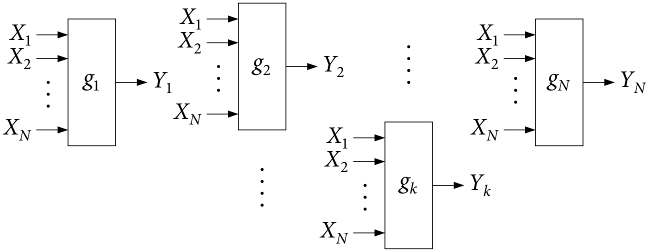


Lim–Kim–Kim (2014)

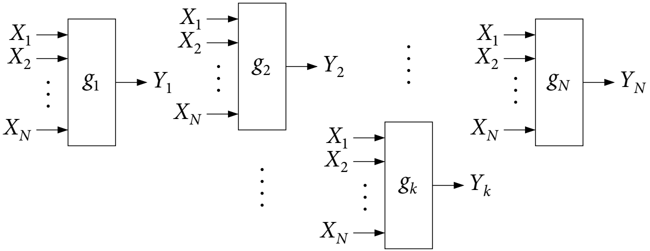
$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) - \sum_{k \in \mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k), \quad \forall \mathcal{S}$$

for some $(\prod_{k=2}^N p(x_k)) p(x_1, u_2^N | x_2^N)$

Deterministic network (Kannan–Raja–Viswanath 2012)



Deterministic network (Kannan–Raja–Viswanath 2012)



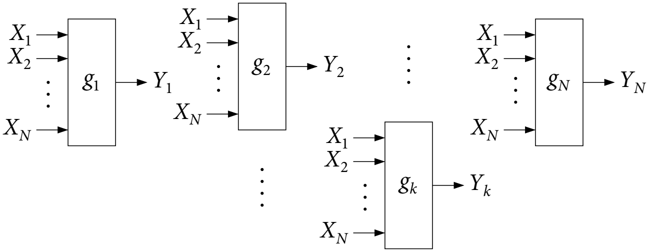
Cutset

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

Distributed decode–forward

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) - \sum_{k \in \mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k)$$

Deterministic network (Kannan–Raja–Viswanath 2012)



Cutset

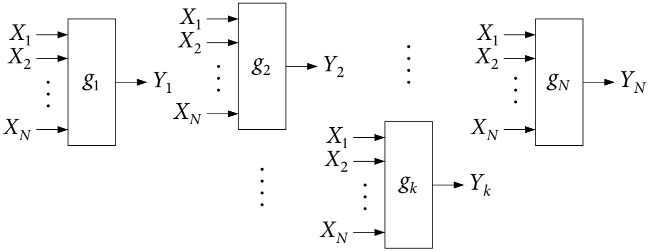
$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c)) \\ = H(Y(\mathcal{S}^c) | X(\mathcal{S}^c)), \quad \forall \mathcal{S}$$

for some $p(x^N)$

Distributed decode–forward

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) \\ - \sum_{k \in \mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k)$$

Deterministic network (Kannan–Raja–Viswanath 2012)



Cutset

$$\begin{aligned} R(\mathcal{S}^c) &\le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c)) \\ &= H(Y(\mathcal{S}^c) | X(\mathcal{S}^c)), \quad \forall \mathcal{S} \end{aligned}$$

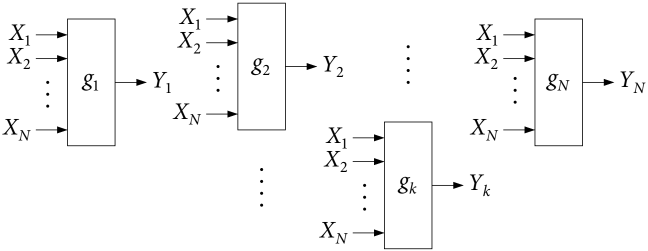
for some $p(x^N)$

Distributed decode–forward

$$\begin{aligned} R(\mathcal{S}^c) &\le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) \\ &\quad - \sum_{k \in \mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k) \\ &= H(Y(\mathcal{S}^c) | X(\mathcal{S}^c)), \quad \forall \mathcal{S} \end{aligned}$$

for some $(\prod_{k=2}^N p(x_k))p(x_1|x_2^N)$

Deterministic network (Kannan–Raja–Viswanath 2012)



Cutset

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c)) = H(Y(\mathcal{S}^c)|X(\mathcal{S}^c)), \quad \forall \mathcal{S}$$

for some $p(x^N)$

Distributed decode–forward

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) - \sum_{k\in\mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k) = H(Y(\mathcal{S}^c)|X(\mathcal{S}^c)), \quad \forall \mathcal{S}$$

for some $(\prod_{k=2}^N p(x_k))p(x_1|x_2^N)$

- Tight for graphical networks and deterministic networks with no interference

Gaussian network

- Channel model:

$$Y_k = \sum_j g_{kj} X_j + Z_k, \quad k \in [1:N]$$

Gaussian network

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- Cutset bound:

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

Gaussian network

- Channel model:

$$Y_k = \sum_j g_{kj} X_j + Z_k, \quad k \in [1:N]$$

- Cutset bound:

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

- Distributed decode–forward bound:

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) - \sum_{k \in \mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k)$$

Gaussian network

- Channel model:

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$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) - \sum_{k \in \mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k)$$

- Set $U_k = \sum_j g_{kj} X_j + \hat{Z}_k \sim Y_k$, where $\hat{Z}_k \sim \mathrm{N}(0,1)$

Gaussian network

- Channel model:

$$Y_k = \sum_j g_{kj} X_j + Z_k, \quad k \in [1:N]$$

- Cutset bound:

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

- Distributed decode–forward bound:

$$R(\mathcal{S}^c) \le I(X(\mathcal{S}); U(\mathcal{S}^c) | X(\mathcal{S}^c)) - \sum_{k \in \mathcal{S}^c} I(U_k; U(\mathcal{S}_k^c), X^N | X_k, Y_k)$$

- Set $U_k = \sum_j g_{kj} X_j + \hat{Z}_k \sim Y_k$, where $\hat{Z}_k \sim \mathrm{N}(0,1)$

Lim–Kim–Kim (2014)

If $(R_1, \ldots, R_N) \in \mathscr{R}_{\mathrm{CS}}$, then $(R_1 - 0.5N, \ldots, R_N - 0.5N) \in \mathscr{R}_{\mathrm{DDF}}$

Distributed decode–forward (DDF)

Broadcast

Noisy network coding (NNC)

Multiple access

Distributed decode–forward (DDF)

Broadcast

Simple decoder

Noisy network coding (NNC)

Multiple access

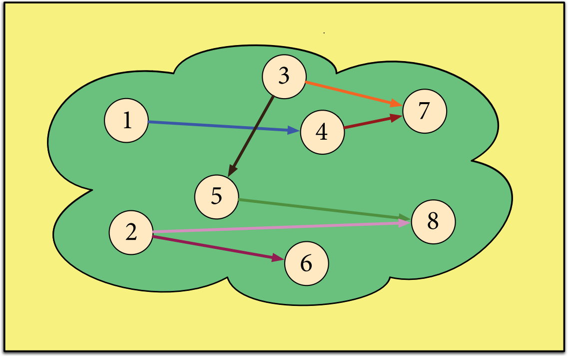
Simple encoder

Distributed decode–forward (DDF)	Noisy network coding (NNC)
Broadcast	Multiple access
Simple decoder	Simple encoder
Marton’s inner bound	MAC capacity region

Distributed decode–forward (DDF)	Noisy network coding (NNC)
Broadcast	Multiple access
Simple decoder	Simple encoder
Marton’s inner bound	MAC capacity region
Partial decode–forward	Compress–forward

Distributed decode–forward (DDF)	Noisy network coding (NNC)
Broadcast	Multiple access
Simple decoder	Simple encoder
Marton’s inner bound	MAC capacity region
Partial decode–forward	Compress–forward
$0.5N$	$0.63N$

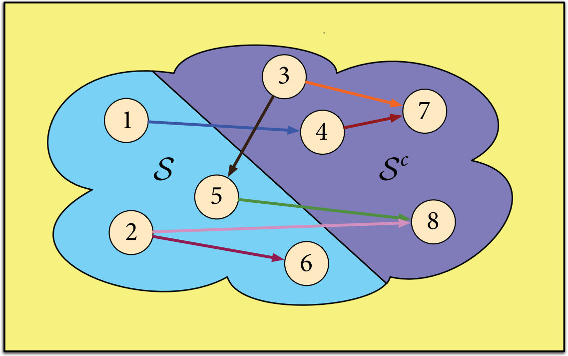
Multiple unicast network



- Capacity region $\mathscr{C}$: Closure of the set of achievable rate tuples

$$(R_{1\to4}, R_{2\to6}, R_{2\to8}, R_{3\to5}, R_{3\to7}, R_{4\to7}, R_{5\to8})$$

Cutset outer bound

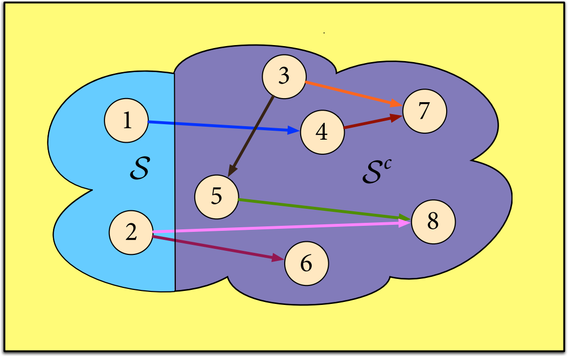


El Gamal (1981), Cover–Thomas (2006)

$$\sum_{j,k:j\in\mathcal{S},k\in\mathcal{S}^c} R_{j\to k} \le I(X(\mathcal{S});Y(\mathcal{S}^c)|X(\mathcal{S}^c)), \quad \forall\mathcal{S}\subseteq[1:8]$$

for some $p(x^N)$

Cutset outer bound

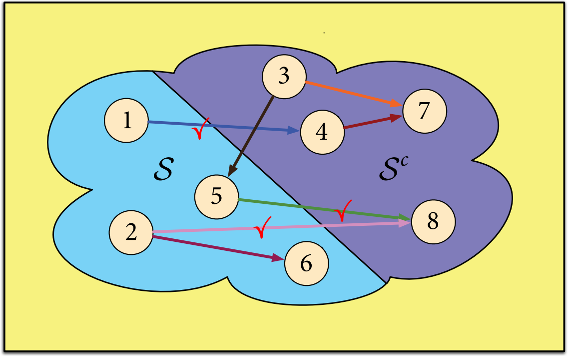


El Gamal (1981), Cover–Thomas (2006)

$$\sum_{j,k:j\in\mathcal{S},k\in\mathcal{S}^c} R_{j\to k} \le I(X(\mathcal{S});Y(\mathcal{S}^c)|X(\mathcal{S}^c)), \quad \forall\mathcal{S}\subseteq[1:8]$$

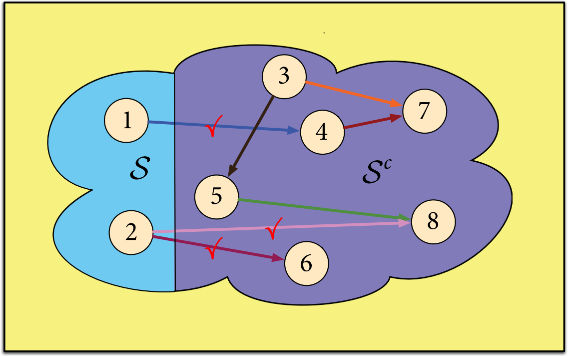
for some $p(x^N)$

Cutset outer bound



$$R_{1\to 4} + R_{2\to 8} + R_{5\to 8} \le I(X_1, X_2, X_5, X_6; Y_3, Y_4, Y_7, Y_8 | X_3, X_4, X_7, X_8)$$

Cutset outer bound



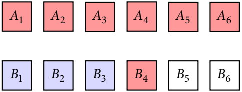
$$R_{1\to4} + R_{2\to8} + R_{5\to8} \le I(X_1, X_2, X_5, X_6; Y_3, Y_4, Y_7, Y_8 | X_3, X_4, X_7, X_8)$$

$$R_{1\to4} + R_{2\to6} + R_{2\to8} \le I(X_1, X_2; Y_3, Y_4, Y_5, Y_6, Y_7, Y_8 | X_3, X_4, X_5, X_6, X_7, X_8)$$

Directed information

- Mutual information

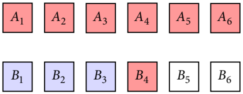
$$I(A_1, \ldots, A_N; B_1, \ldots, B_N) = \sum_{j=1}^{N} I(A^N; B_j | B^{j-1})$$



Directed information

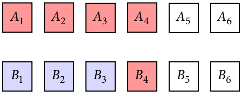
- Mutual information

$$I(A_1, \ldots, A_N; B_1, \ldots, B_N) = \sum_{j=1}^{N} I(A^N; B_j | B^{j-1})$$



- Directed information (Massey 1990)

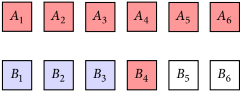
$$I(A_1, \ldots, A_N \to B_1, \ldots, B_N) = \sum_{j=1}^{N} I(A^j; B_j | B^{j-1})$$



Directed information

- Mutual information

$$I(A_1,\ldots,A_N;B_1,\ldots,B_N)$$
$$=\sum_{j=1}^{N} I(A^N;B_j|B^{j-1})$$



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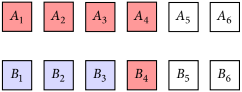
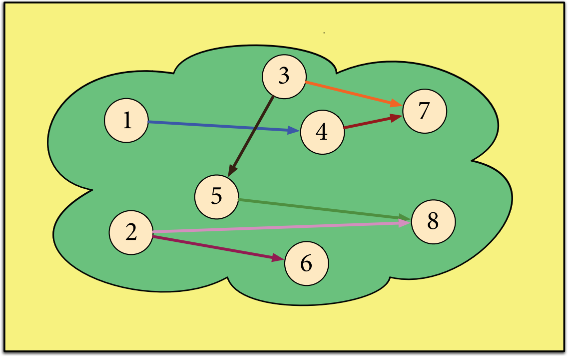


- Amount of information A^N causally provides about B^N (Permuter et al. 2011)

Cutlet outer bound

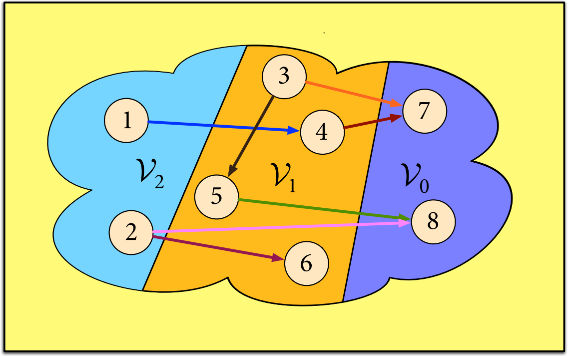


Kamath–Kim (2014)

$$\sum_{j,k:j\in\mathcal{V}_+,k\in\mathcal{V}_-} R_{j\to k} \le I(Y(\mathcal{V}_0),\ldots,Y(\mathcal{V}_{L-1}) \to X(\mathcal{V}_1),\ldots,X(\mathcal{V}_L)|X(\mathcal{V}_0)), \quad \forall \mathcal{V}_0,\ldots,\mathcal{V}_L$$

for some $p(x^N)$

Cutlet outer bound

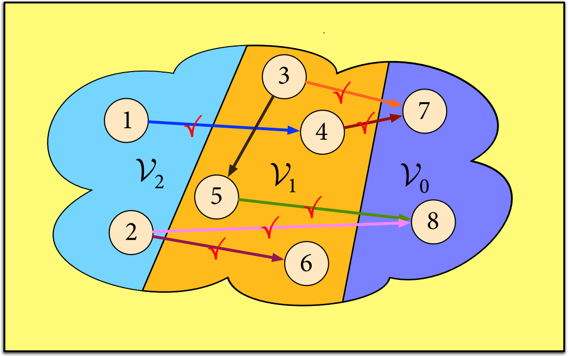


Kamath–Kim (2014)

$$\sum_{j,k: j\in\mathcal{V}_+, k\in\mathcal{V}_-} R_{j\to k} \le I(Y(\mathcal{V}_0),\dots,Y(\mathcal{V}_{L-1}) \to X(\mathcal{V}_1),\dots,X(\mathcal{V}_L) | X(\mathcal{V}_0)), \quad \forall \mathcal{V}_0,\dots,\mathcal{V}_L$$

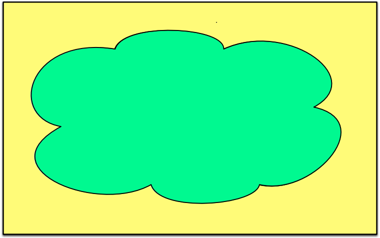
for some $p(x^N)$

Cutlet outer bound

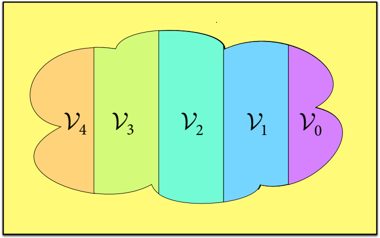


$$R_{1\to4}+R_{2\to6}+R_{2\to8}+R_{3\to7}+R_{4\to7}+R_{5\to8}$$
$$\le I\big((Y_7,Y_8),(Y_3,Y_4,Y_5,Y_6)\to(X_3,X_4,X_5,X_6),(X_1,X_2)\,|\,(X_7,X_8)\big)$$
$$=I(Y(7:8);X(3:6)|X(7:8))+I(Y(3:6),Y(7:8);X(1:2)|X(3:6),X(7:8))$$

Cutset vs. cutlet

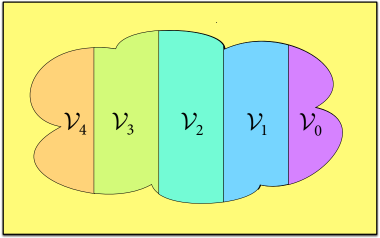


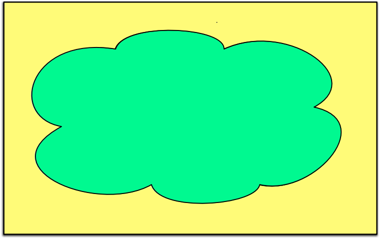
Cutset vs. cutlet



$$
\begin{aligned}
R_{\text{sum}} \le I\big(&Y(\mathcal{V}_0), Y(\mathcal{V}_1), Y(\mathcal{V}_2), Y(\mathcal{V}_3) \\
&\to X(\mathcal{V}_1), X(\mathcal{V}_2), X(\mathcal{V}_3), X(\mathcal{V}_4) | X(\mathcal{V}_0)\big)
\end{aligned}
$$

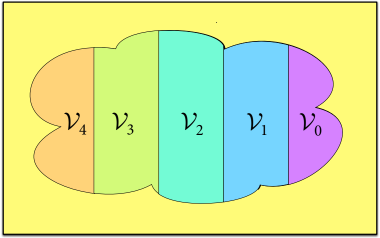
Cutset vs. cutlet

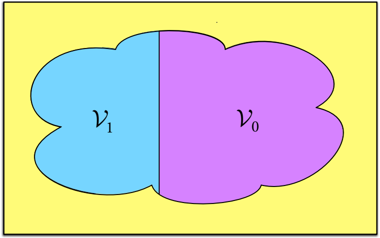




$$R_{\text{sum}} \le I\big(Y(\mathcal{V}_0), Y(\mathcal{V}_1), Y(\mathcal{V}_2), Y(\mathcal{V}_3) \\ \to X(\mathcal{V}_1), X(\mathcal{V}_2), X(\mathcal{V}_3), X(\mathcal{V}_4) | X(\mathcal{V}_0)\big)$$

Cutset vs. cutlet

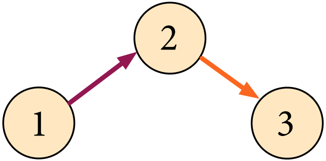




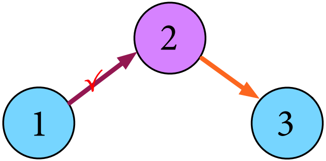
$$\begin{aligned} R_{\text{sum}} &\le I\big(Y(\mathcal{V}_0), Y(\mathcal{V}_1), Y(\mathcal{V}_2), Y(\mathcal{V}_3) \\ &\qquad \to X(\mathcal{V}_1), X(\mathcal{V}_2), X(\mathcal{V}_3), X(\mathcal{V}_4) \,|\, X(\mathcal{V}_0)\big) \end{aligned}$$

$$\begin{aligned} R_{\text{sum}} &\le I(Y(\mathcal{V}_0) \to X(\mathcal{V}_1) \,|\, X(\mathcal{V}_0)) \\ &= I(X(\mathcal{V}_1); Y(\mathcal{V}_0) \,|\, X(\mathcal{V}_0)) \end{aligned}$$

Example: $p(y_2, y_3|x_1, x_2)$



Example: $p(y_2, y_3 | x_1, x_2)$

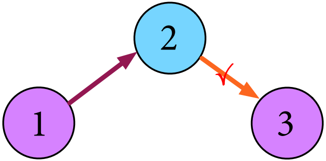


- Cutset bound

$$R_{1\to 2} \le I(X_1; Y_2 | X_2)$$

for some $p(x_1, x_2)$

Example: $p(y_2, y_3|x_1, x_2)$

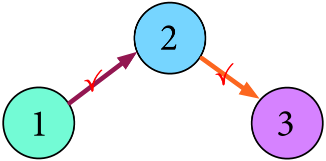


- Cutset bound

$$R_{1\to 2} \le I(X_1; Y_2|X_2)$$
$$R_{2\to 3} \le I(X_2; Y_3|X_1)$$

for some $p(x_1, x_2)$

Example: $p(y_2, y_3|x_1, x_2)$



- Cutset bound

$$\begin{aligned} R_{1\to 2} &\le I(X_1; Y_2|X_2) \\ R_{2\to 3} &\le I(X_2; Y_3|X_1) \end{aligned}$$

for some $p(x_1, x_2)$

- Cutlet bound

$$\begin{aligned} R_{1\to 2} &\le I(X_1; Y_2|X_2) \\ R_{2\to 3} &\le I(X_2; Y_3|X_1) \end{aligned}$$

$$\begin{aligned} R_{1\to 2} + R_{2\to 3} &\le I(Y_3, Y_2 \to X_2, X_1) \\ &= I(X_2; Y_3) + I(X_1; Y_2, Y_3|X_2) \end{aligned}$$

for some $p(x_1, x_2)$

To learn more

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