Capacity Achieving Codes: There and Back Again

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Outline

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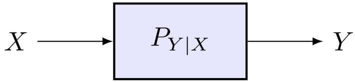
Capacity of Point-to-Point Communication



- Coding for Discrete-Time Memoryless Channels
  - Transition probability: $P_{Y|X}(y|x)$ for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$
  - Transmit a length-n codeword $\underline{x} \in \mathcal{C} \subset \mathcal{X}^n$
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$$X \longrightarrow \boxed{P_{Y|X}} \longrightarrow Y$$

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- Channel Capacity introduced by Shannon in 1948
  - Random code of rate $R \triangleq \frac{1}{n} \log_2 |\mathcal{C}|$ (bits per channel use)
  - As $n \to \infty$, reliable transmission possible if $R < C$ with

$$C \triangleq \max_{p(x)} I(X;Y)$$

The Binary Erasure Channel (BEC)

- Denoted BEC(ε) when erasure probability is ε
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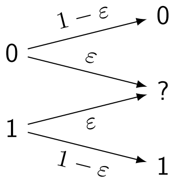

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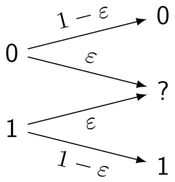

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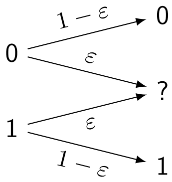

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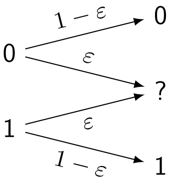

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  - One can achieve capacity by drawing H uniformly at random!

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- 1999-2011: Understanding LDPC convolutional codes and coupling

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- EXtrinsic Information Transfer (EXIT) Curves

Applications of These Tools

- Error-Correcting Codes
  - Random code defined by random factor graph
  - Low-complexity decoding via belief propagation
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- Compressed Sensing
  - Random measurement matrix defined by random factor graph
  - Low-complexity reconstruction via message passing
  - Schemes provably achieve the information-theoretic limit!

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- The solution of the simpler problem often provides insight that allows one to crack the harder problem.
- To achieve channel capacity in practice, we now know that a good "easy" problem would have been:
  - "Design a code that achieves capacity on the BEC and is encodable and decodable in quasi-linear time"

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Factor Graphs

- A factor graph provides a graphical representation of the local dependence structure for a set of random variables
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\begin{aligned}
P(x_1, x_2, x_3, x_4) &\triangleq \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_4 = x_4 | Y = y) \\
&\propto f(x_1, x_2, x_3, x_4) \\
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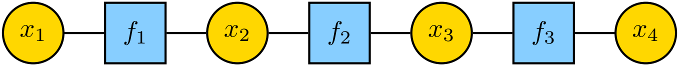
- Given $Y = y$, this describes a Markov chain whose factor graph is



Conditional Independence for Factor Graphs

- Let $A, B, S \subset [n]$ be disjoint subsets of VNs in factor graph G
  - If S separates A from B (i.e., there is no path in G from A to B that avoids S), then we have $X_A \perp\!\!\!\perp X_B \mid X_S$

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- Sketch of Proof:
  - Fixing $X_S = x_S$ separates the FG into disjoint components
  - Groups of VNs in different components are independent
  - $X_A \perp\!\!\!\perp X_B$ because A and B are in different components

Inference via Marginalization

- Marginalizing out all variables except X_1 gives

$$\mathbb{P}(X_1 = x_1 | Y = y) \propto g_1(x_1) \triangleq \sum_{(x_2,\ldots,x_4)\in\mathcal{X}^3} f(x_1, x_2, x_3, x_4)$$

- Thus, the maximum a posteriori decision for X_1 given $Y = y$ is

$$\hat{x}_1 = \arg\max_{x_1\in\mathcal{X}} \sum_{(x_2,\ldots,x_4)\in\mathcal{X}^3} f(x_1, x_2, x_3, x_4)$$

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- Marginalization is efficient for tree-structured factor graphs
  - For the Markov chain, roughly $5\,|\mathcal{X}|^2$ operations required

$$g_1(x_1) = \sum_{x_2\in\mathcal{X}} f_1(x_1, x_2) \sum_{x_3\in\mathcal{X}} f_2(x_2, x_3) \sum_{x_4\in\mathcal{X}} f_3(x_3, x_4)$$

The Importance of Factorization (1)

- Consider a random vector $(X_1, X_2, \ldots, X_6) \in \mathcal{X}^6$ where

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- If f factors as follows, then the marginalization can be simplified:

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = f_1(x_1, x_2, x_3) f_2(x_1, x_4, x_6) f_3(x_4) f_4(x_4, x_5)$$

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For example, we can write $g_1(x_1)$ as:

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&= \sum_{x_2^3} f_1(x_1, x_2, x_3) \left[\sum_{x_4} f_3(x_4) \left[\sum_{x_5} f_4(x_4, x_5) \right] \left[\sum_{x_6} f_2(x_1, x_4, x_6) \right] \right]
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The Importance of Factorization (2)

For example, we can write $g_1(x_1)$ as:

$$=\sum_{x_2^6} f_1(x_1,x_2,x_3)f_2(x_1,x_4,x_6)f_3(x_4)f_4(x_4,x_5)$$

$$=\sum_{x_2^5} f_1(x_1,x_2,x_3)f_3(x_4)f_4(x_4,x_5)\left[\sum_{x_6} f_2(x_1,x_4,x_6)\right]$$

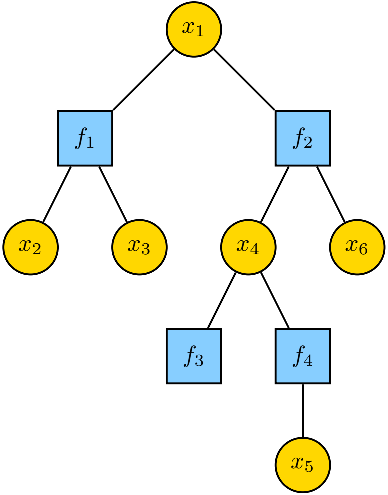
$$=\sum_{x_2^4} f_1(x_1,x_2,x_3)f_3(x_4)\left[\sum_{x_5} f_4(x_4,x_5)\right]\left[\sum_{x_6} f_2(x_1,x_4,x_6)\right]$$

$$=\sum_{x_2^3} f_1(x_1,x_2,x_3)\left[\sum_{x_4} f_3(x_4)\left[\sum_{x_5} f_4(x_4,x_5)\right]\left[\sum_{x_6} f_2(x_1,x_4,x_6)\right]\right]$$

$$=\sum_{x_2}\left[\sum_{x_3} f_1(x_1,x_2,x_3)\right]\left[\sum_{x_4} f_3(x_4)\left[\sum_{x_5} f_4(x_4,x_5)\right]\left[\sum_{x_6} f_2(x_1,x_4,x_6)\right]\right]$$

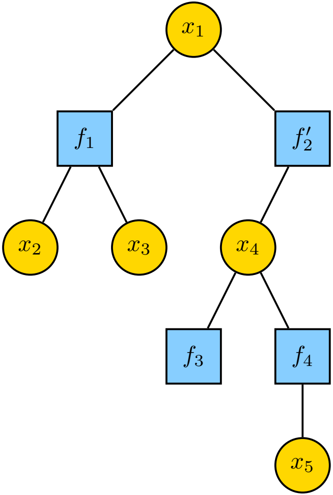
This implementation requires roughly $2\,|\mathcal{X}|^3+5\,|\mathcal{X}|^2$ operations

The Factor Graph and Leaf Removal



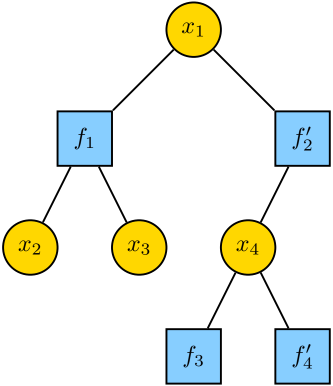
$$g_1(x_1) = \sum_{x_2^5} f_1(x_1, x_2, x_3) f_3(x_4) f_4(x_4, x_5) \sum_{x_6} f_2(x_1, x_4, x_6)$$

The Factor Graph and Leaf Removal



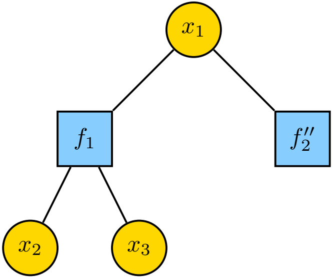
$$g_1(x_1) = \sum_{x_2^4} f_1(x_1, x_2, x_3) f_3(x_4) \left[\sum_{x_5} f_4(x_4, x_5) \right] f_2'(x_1, x_4)$$

The Factor Graph and Leaf Removal



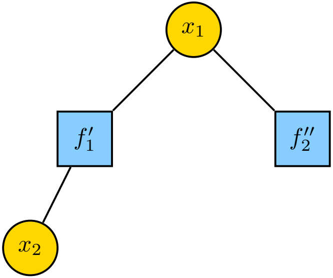
$$g_1(x_1) = \sum_{x_2^3} f_1(x_1, x_2, x_3) \left[\sum_{x_4} f_3(x_4) f_4'(x_4) f_2'(x_1, x_4) \right]$$

The Factor Graph and Leaf Removal



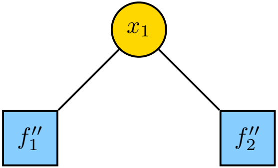
$$g_1(x_1) = \sum_{x_2} \left[\sum_{x_3} f_1(x_1, x_2, x_3) \right] f_2''(x_1)$$

The Factor Graph and Leaf Removal



$$g_1(x_1) = \left[\sum_{x_2} f_1'(x_1, x_2)\right] f_2''(x_1)$$

The Factor Graph and Leaf Removal



$$g_1(x_1) = f_1''(x_1) f_2''(x_1)$$

Constraint Satisfaction and Zero-One Factors

- A non-negative function $f\colon \mathcal{X}^n \to \mathbb{R}$ defines a distribution on $\mathcal{X}^n$:

$$
\begin{aligned}
P(\underline{x}) &\triangleq \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) \\
&= \frac{1}{Z} f(\underline{x}) \triangleq \frac{1}{Z} \prod_{a=1}^{m} f_a(\underline{x}_{\partial a}),
\end{aligned}
$$

  - where $\underline{x}_{\partial a}$ is the subvector of variables involved in factor a
  - and $Z \triangleq \sum_{\underline{x}} f(\underline{x})$ is called the partition function

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  - where $\underline{x}_{\partial a}$ is the subvector of variables involved in factor a
  - and $Z \triangleq \sum_{\underline{x}} f(\underline{x})$ is called the partition function

- For Constraint Satisfaction Problems (CSPs)
  - All factors $f_a(\underline{x}_{\partial a})$ take values in $\{0, 1\}$
  - The set of valid configurations is $\{\underline{x} \in \mathcal{X}^n | f(\underline{x}) = 1\}$
  - Thus, Z equals the number of valid configurations
  - $P(\underline{x})$ is uniform over the set of valid configurations

Outline

Marginalization via Belief Propagation

- Factor Graph $G = (V \cup F, E)$
  - Variable nodes V, Factor nodes F
  - Edges: $(i, a) \in E \subseteq V \times F$
  - $F(i)/V(a) =$ set of neighbors for node-i/a
  - Messages: $\mu_{i \to a}^{(t)}(x_i)$ and $\hat{\mu}_{a \to i}^{(t)}(x_i)$

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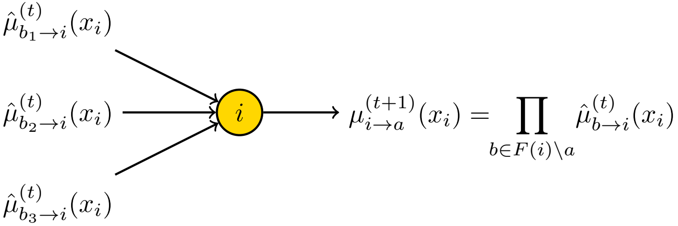

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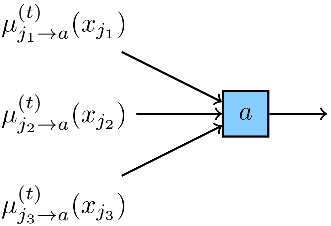

$$\hat{\mu}^{(t)}_{a \to i}(x_i) = \sum_{\underline{x}_{V(a)\setminus i}} f_a(\underline{x}_{V(a)}) \prod_{j \in V(a)\setminus i} \mu^{(t)}_{j \to a}(x_j)$$

Marginalization via Belief Propagation

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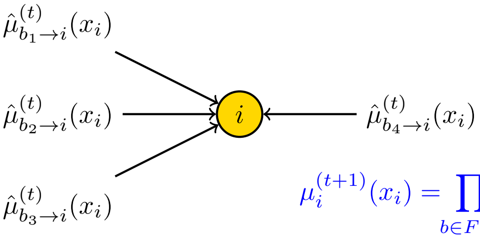
  - Messages: $\mu^{(t)}_{i \to a}(x_i)$ and $\hat{\mu}^{(t)}_{a \to i}(x_i)$
- variable-i marginal

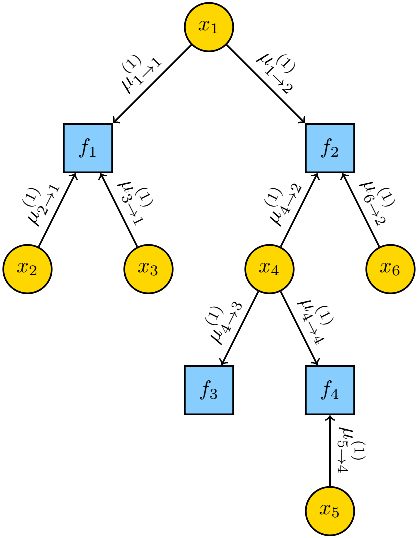


$$\mu^{(t+1)}_{i}(x_i) = \prod_{b \in F(i)} \hat{\mu}^{(t)}_{b \to i}(x_i)$$

Marginalization via Belief Propagation: Example

iteration 1: variable to factor

$$\mu_{i\to a}^{(1)}(x_i) = 1$$



Marginalization via Belief Propagation: Example

iteration 1: variable to factor

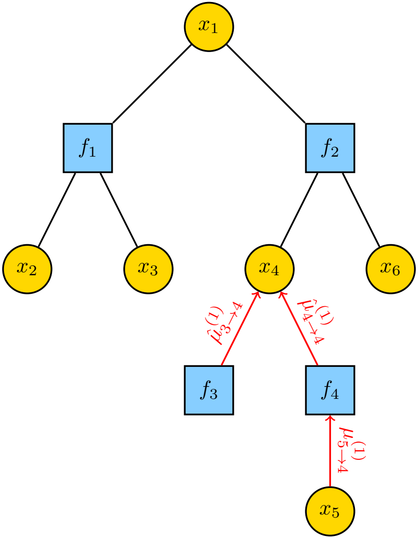
$$\mu_{i\to a}^{(1)}(x_i) = 1$$

iteration 1: factor to variable

$$\hat{\mu}_{4\to 4}^{(1)}(x_4) = \sum_{x_5} f_4(x_4, x_5)\mu_{5\to 4}^{(1)}(x_i)$$

$$= \sum_{x_5} f_4(x_4, x_5)$$

$$\hat{\mu}_{3\to 4}^{(1)}(x_4) = f_3(x_4)$$



Marginalization via Belief Propagation: Example

iteration 1: factor to variable

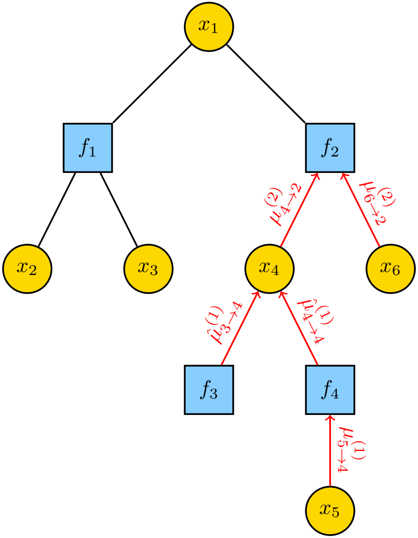
$$\hat{\mu}^{(1)}_{4\to 4}(x_4) = \sum_{x_5} f_4(x_4, x_5)\mu^{(1)}_{5\to 4}(x_i)$$
$$= \sum_{x_5} f_4(x_4, x_5)$$

$$\hat{\mu}^{(1)}_{3\to 4}(x_4) = f_3(x_4)$$

iteration 2: variable to factor

$$\mu^{(2)}_{4\to 2}(x_4) = \hat{\mu}^{(1)}_{4\to 4}(x_4)\hat{\mu}^{(1)}_{3\to 4}(x_4)$$
$$= f_3(x_4)\sum_{x_5} f_4(x_4, x_5)$$

$$\mu^{(2)}_{6\to 2}(x_6) = 1$$



Marginalization via Belief Propagation: Example

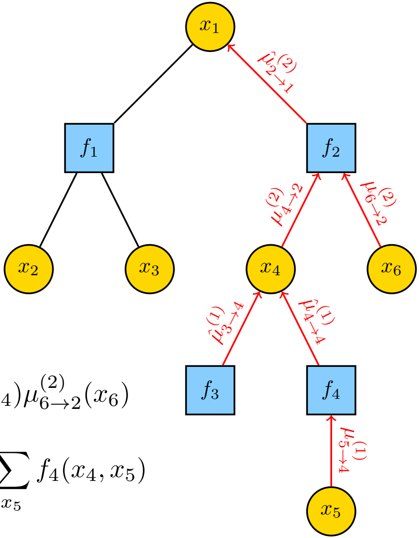
iteration 2: variable to factor

$$\mu^{(2)}_{4\to 2}(x_4) = \hat{\mu}^{(1)}_{4\to 4}(x_4)\hat{\mu}^{(1)}_{3\to 4}(x_4)$$
$$= f_3(x_4)\sum_{x_5} f_4(x_4, x_5)$$

$$\mu^{(2)}_{6\to 2}(x_6) = 1$$

iteration 2: factor to variable

$$\hat{\mu}^{(2)}_{2\to 1}(x_1) = \sum_{x_4, x_6} f_2(x_1, x_4, x_6)\mu^{(2)}_{4\to 2}(x_4)\mu^{(2)}_{6\to 2}(x_6)$$
$$= \sum_{x_4, x_6} f_2(x_1, x_4, x_6) f_3(x_4)\sum_{x_5} f_4(x_4, x_5)$$
$$= f''_2(x_1)$$

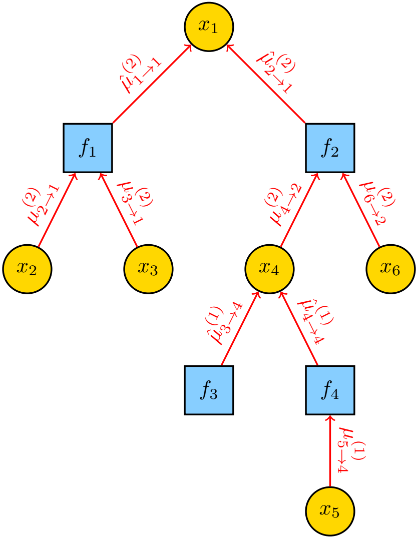


Marginalization via Belief Propagation: Example

iteration 2: variable marginal

$$\begin{aligned} \mu_1^{(3)}(x_1) &= \hat{\mu}_{1\to 1}^{(2)}(x_1)\hat{\mu}_{2\to 1}^{(2)}(x_1) \\ &= f_1''(x_1) f_2''(x_2) \end{aligned}$$

Same answer as peeling but from a distributed parallel algorithm



Outline

Sudoku: A Factor Graph for the Masses

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

rows are permutations of $\{1, 2, \ldots, 9\}$

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columns are permutations of $\{1, 2, \ldots, 9\}$

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subblocks are permutations of $\{1, 2, \ldots, 9\}$

Sudoku: A Factor Graph for the Masses

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	1		9		7		6	

rows are permutations of $\{1, 2, \ldots, 9\}$
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subblocks are permutations of $\{1, 2, \ldots, 9\}$

x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}
x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{39}
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}	x_{49}
x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}	x_{58}	x_{59}
x_{61}	x_{62}	x_{63}	x_{64}	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}
x_{71}	x_{72}	x_{73}	x_{74}	x_{75}	x_{76}	x_{77}	x_{78}	x_{79}
x_{81}	x_{82}	x_{83}	x_{84}	x_{85}	x_{86}	x_{87}	x_{88}	x_{89}
x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	x_{98}	x_{99}

implied factor graph has
81 variable and 27 factor nodes

$$f(\underline{x}) = \left(\prod_{i=1}^{9} f_\sigma(x_{i*})\right) \left(\prod_{j=1}^{9} f_\sigma(x_{*j})\right) \left(\prod_{k=1}^{9} f_\sigma(x_{B(k)})\right) \prod_{(i,j)\in O} \mathbb{I}(x_{ij} = y_{ij})$$

Solving Sudoku with a Factor Graph

- Consider any constraint satisfaction problem with observed entries
  - One can write $f(\underline{x})$ as the product of indicator functions
  - Some factors force $\underline{x}$ to be valid (i.e., satisfy constraints)
  - Other factors force $\underline{x}$ to be compatible with observed values
  - Summing over $\underline{x}$ counts the $\#$ of valid compatible sequences

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  - Summing over $\underline{x}$ counts the $\#$ of valid compatible sequences
- Low-complexity peeling solution
  - Set elements of $\underline{x}$ one at a time
  - Each step looks for $i \in [n]$ and $x' \in \mathcal{X}$ such that:
    - For currently set variables, $f(\underline{x}) = 0$ for all $x_i \in \mathcal{X} \setminus x'$
  - Sudoku's unique solution implies that $x_i = x'$ correct
  - Fix $x_i = x'$ and repeat until all values fixed

Boolean Satisfiability: K-SAT

- One instance of 3-SAT is given, for example, by

$$f(\underline{x}) = (\overline{x}_1 \vee \overline{x}_3 \vee x_7) \wedge (x_1 \vee \overline{x}_2 \vee x_5) \wedge (x_2 \vee \overline{x}_4 \vee x_6)\,.$$

  - In the FG, clause $a \in [m]$ is enforced by the function f_a

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- Marginalization allows uniform sampling from valid set
  - For $i = 1, 2, \ldots, n$, fix x_j for $j < i$ and compute marginal

$$g_i(x_i) = \frac{1}{Z_i} \sum_{x_{i+1},\ldots,x_n} f(\underline{x}) = \mathbb{P}\left(X_i = x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\right)$$

  - Then, sample $x_i \sim g_i(\cdot)$ and repeat

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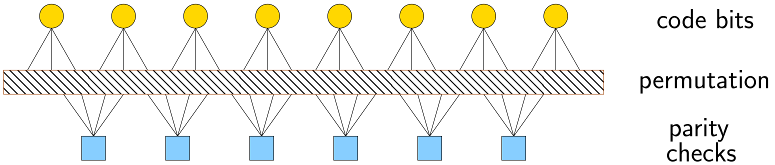
  - Then, sample $x_i \sim g_i(\cdot)$ and repeat

- This algorithm has low complexity if factor graph forms a tree
  - If not a tree, use approximate marginal from belief propagation
  - This is related to BP-guided decimation [MM09]

Low-Density Parity-Check (LDPC) Codes



- Linear codes defined by $\underline{x}H^T = \underline{0}$ for all c.w. $\underline{x} \in \mathcal{C} \subset \{0,1\}^n$
  - H is an $m \times n$ sparse parity-check matrix for the code
  - Code bits and parity checks associated with cols/rows of H

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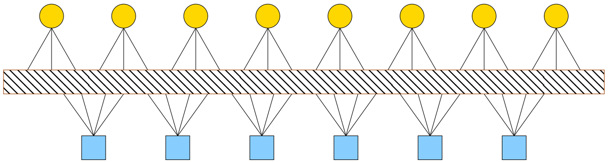

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  - H is an $m \times n$ sparse parity-check matrix for the code
  - Code bits and parity checks associated with cols/rows of H
- Factor graph: H is the biadjacency matrix for variable/factor nodes
  - Ensemble defined by configuration model for random graphs
  - Checks define factors: $f_{\text{even}}(x_1^d) = \mathbb{I}(x_1 \oplus \cdots \oplus x_d = 0)$
  - Let $\underline{x}_{\partial a}$ be the subvector of variables in the a-th check and

$$f(x_1, \ldots, x_n) = \left(\prod_{a=1}^{m} f_{\text{even}}(\underline{x}_{\partial a}) \right) \left(\prod_{i=1}^{n} P_{Y|X}(y_i|x_i) \right)$$

A Little History

Robert Gallager

introduced LDPC codes in 1962 paper

1962 *IRE TRANSACTIONS ON INFORMATION THEORY* 21

Low-Density Parity-Check Codes*

R. G. GALLAGER†

Summary—A low-density parity-check code is a code specified by a parity-check matrix with the following properties: each column contains a small fixed number $j \geq 3$ of 1's and each row contains a small fixed number $k > j$ of 1's. The typical minimum distance of these codes increases linearly with block length for a fixed rate and fixed j. When used with maximum likelihood decoding on a sufficiently quiet binary-input symmetric channel, the typical probability of decoding error decreases exponentially with block length for a fixed rate and fixed j.

A simple but nonoptimum decoding scheme operating directly from the channel a posteriori probabilities is described. Both the

equations. We call the set of digits contained in a parity-check equation a parity-check set. For example, the first parity-check set in Fig. 1 is the set of digits (1, 2, 3, 5).

The use of parity-check codes makes coding (as distinguished from decoding) relatively simple to implement. Also, as Elias [3] has shown, if a typical parity-check code of long block length is used on a binary symmetric channel, and if the code rate is between *critical rate* and channel capacity, then the probability of decoding error

Judea Pearl

defined general belief-propagation in 1986 paper

Fusion, Propagation, and Structuring in Belief Networks*

Judea Pearl
Cognitive Systems Laboratory, Computer Science Department, University of California, Los Angeles, CA 90024, U.S.A.

Recommended by Patrick Hayes

ABSTRACT

Belief networks are directed acyclic graphs in which the nodes represent propositions (or variables), the arcs signify direct dependencies between the linked propositions, and the strengths of these dependencies are quantified by conditional probabilities. A network of this sort can be used to represent the generic knowledge of a domain expert, and it turns into a computational architecture if the links are used not merely for storing factual knowledge but also for directing and activating the data flow in the computations which manipulate this knowledge.

Simple Message-Passing Decoding for the BEC

- Constraint nodes define the valid patterns
  - Circles represent a single value shared by factors
  - Squares assert attached variables sum to 0 mod 2
- Iterative decoding on the binary erasure channel (BEC)
  - Messages passed in phases: bit-to-check and check-to-bit
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  - Each message is either the correct value or an erasure

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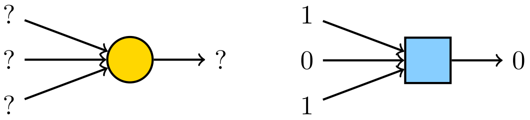
  - Checks pass the correct value only if all other inputs are correct



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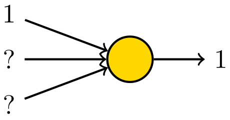
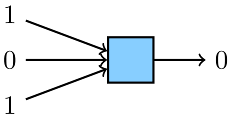
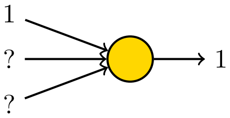
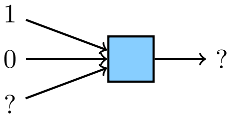




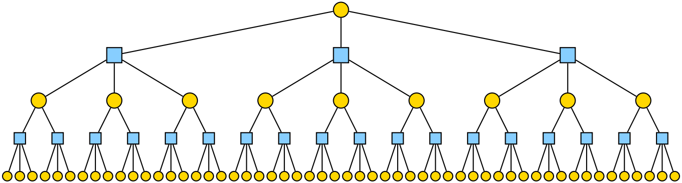
Simple Message-Passing Decoding for the BEC

- Constraint nodes define the valid patterns
  - Circles represent a single value shared by factors
  - Squares assert attached variables sum to 0 mod 2
- Iterative decoding on the binary erasure channel (BEC)
  - Messages passed in phases: bit-to-check and check-to-bit
  - Each output message depends on other input messages
  - Each message is either the correct value or an erasure
- Message passing rules for the BEC
  - Bits pass an erasure only if all other inputs are erased
  - Checks pass the correct value only if all other inputs are correct





Computation Graph and Density Evolution



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$$y_2 = 1-(1-x_2)^3$$

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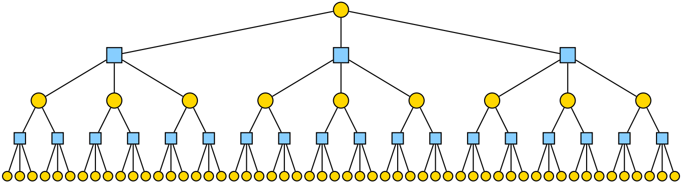

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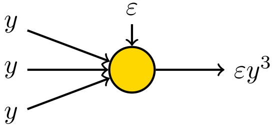

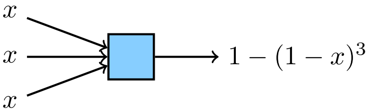

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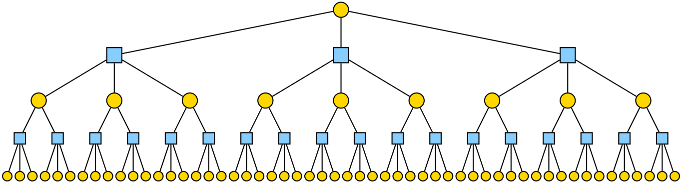

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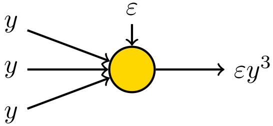

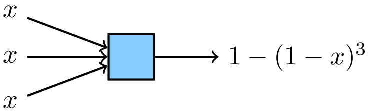

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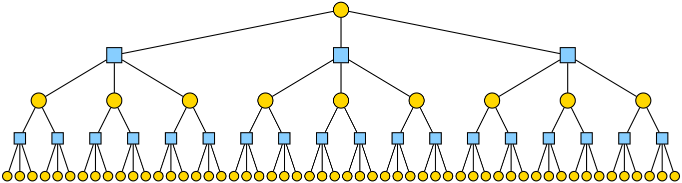

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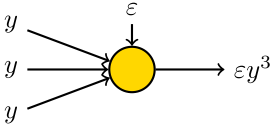

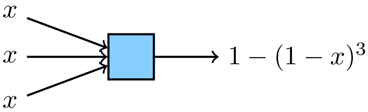

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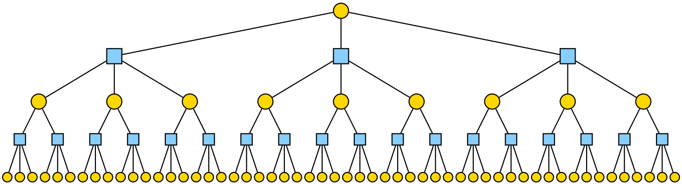

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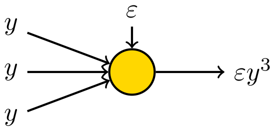

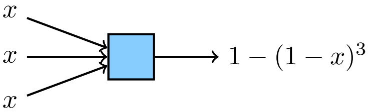

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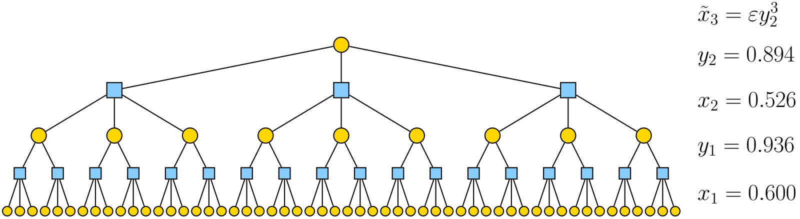

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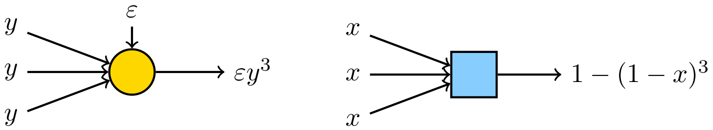

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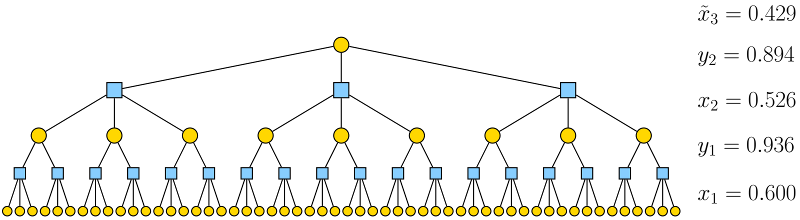

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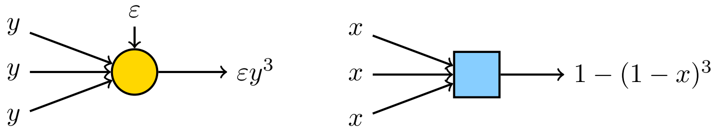
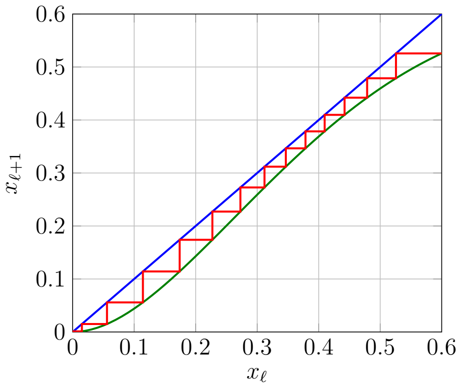


Density Evolution (DE) for LDPC Codes

(3,4) LDPC Code with $\varepsilon = 0.6$



Density evolution for a $(3,4)$-regular LDPC code:

$$x_{\ell+1} = \varepsilon \left(1 - (1 - x_\ell)^3\right)^2$$

Decoding Thresholds:

$$\varepsilon^{\mathrm{BP}} \approx 0.647$$
$$\varepsilon^{\mathrm{MAP}} \approx 0.746$$
$$\varepsilon^{\mathrm{Sh}} = 0.750$$

- Binary erasure channel (BEC) with erasure prob. ε
- DE tracks bit-to-check msg erasure rate x_ℓ after ℓ iterations
- Defines noise threshold $\varepsilon^{\mathrm{BP}}$ for the large system limit
  - Easily computed numerically for given code ensemble

EXtrinsic Information Transfer (EXIT) Curves

- Introduced by ten Brink in 1999 to understand iterative decoding
  - For the BEC, the MAP EXIT curve is

$$h^{\mathrm{MAP}}(\varepsilon) \triangleq \frac{1}{n}\sum_{i=1}^{n} H(X_i|\underline{Y}_{\sim i}(\varepsilon))$$

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$$h^{\mathrm{BP}}(\varepsilon) \triangleq \frac{1}{n}\sum_{i=1}^{n} H\big(X_i|\Phi_i^{\mathrm{BP}}(\underline{Y}_{\sim i}(\varepsilon))\big)$$

    - where $\Phi_i^{\mathrm{BP}}(Z)$ is the BP estimate of X_i given Z
    - Data processing inequality: $h^{\mathrm{BP}}(\varepsilon) \geq h^{\mathrm{MAP}}(\varepsilon)$

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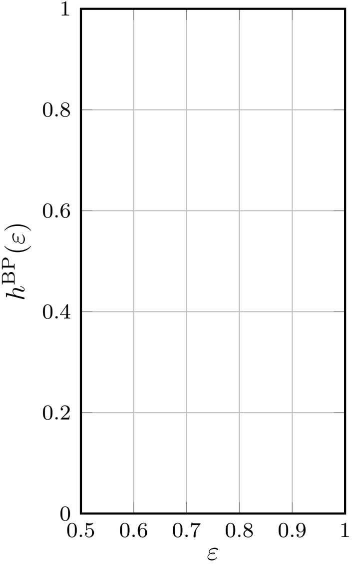

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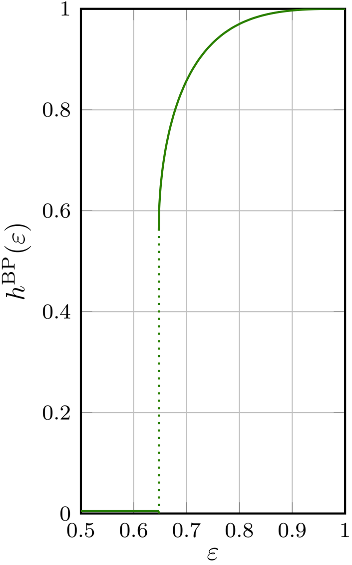

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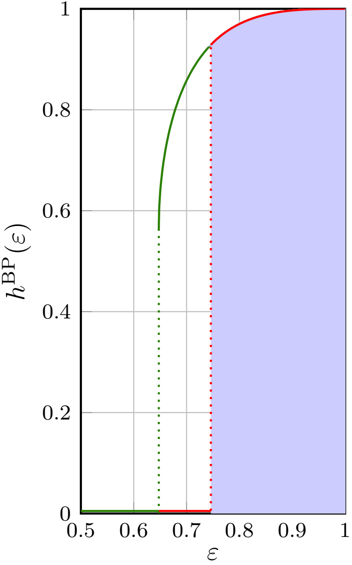

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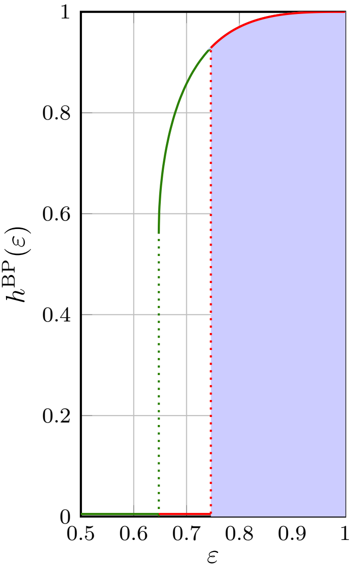

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  - ε s.t. area under BP EXIT is R

Outline

Introduction

Factor Graphs

Message Passing

Applications of Factor Graphs

Applications of EXIT Curves

Spatially-Coupled Factor Graphs

Universality for Multiuser Scenarios

Abstract Formulation of Threshold Saturation

Properties of the MAP EXIT Curve

- For linear codes, the recovery of X_i from $\underline{Y} = \underline{y}$
  - is independent of the transmitted codeword $\underline{X}$
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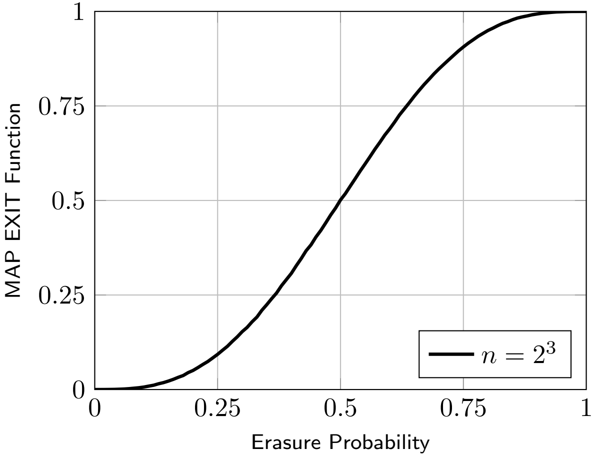
- A sequence of rate-R codes achieves capacity iff
  - $P_b(\varepsilon) \to 0$ for all $\varepsilon < 1 - R$
  - $h^{\mathrm{MAP}}(\varepsilon) \to 0$ for all $\varepsilon < 1 - R$
  - $h^{\mathrm{MAP}}(\varepsilon)$ transitions sharply from 0 to 1

The MAP EXIT Curve of a Capacity-Achieving Code



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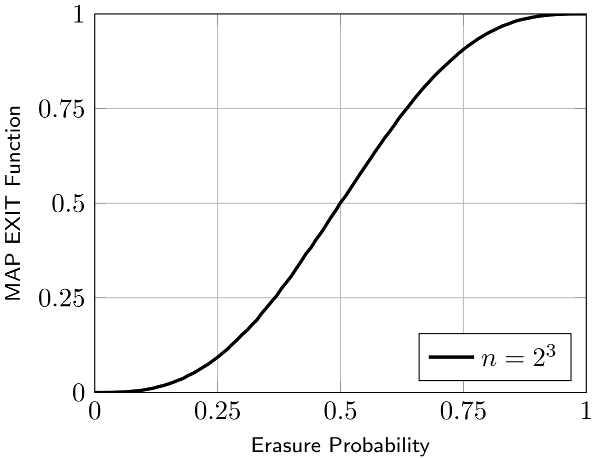

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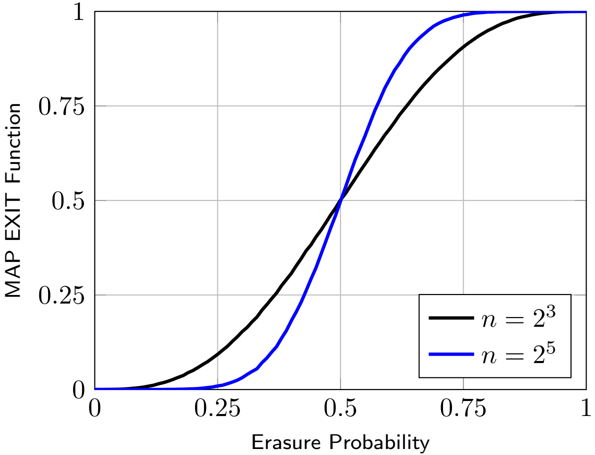

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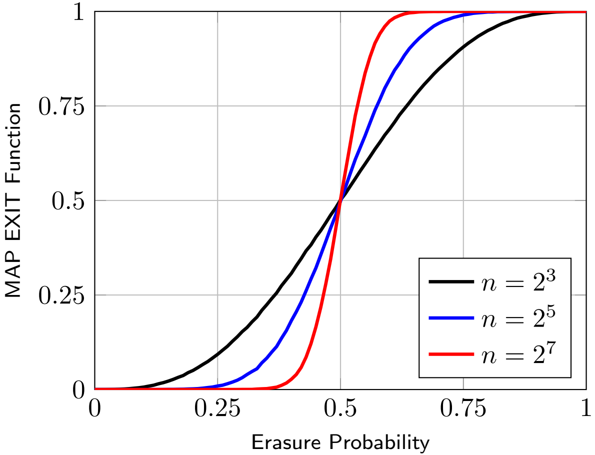

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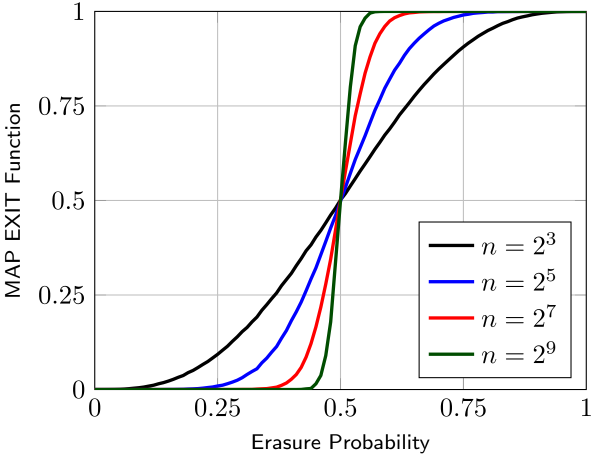

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- When do EXIT curves have a sharp transition? [KKMPSU15]
  - If the code's permutation group is doubly transitive!
  - For example, Reed-Muller and prim. narrow-sense BCH codes

---

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- Open problems
  - Generalize the Reed-Muller result to have weaker conditions and/or apply to more general channels/problems
  - Find a purely information-theoretic proof of the Reed-Muller result for the BEC

Outline

What is Spatial Coupling?

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

			6	5	4									
			7	3	9									
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8	7	6	1		5	9								
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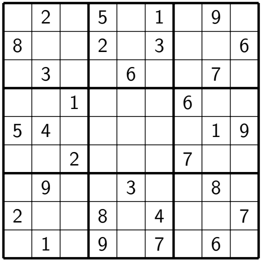
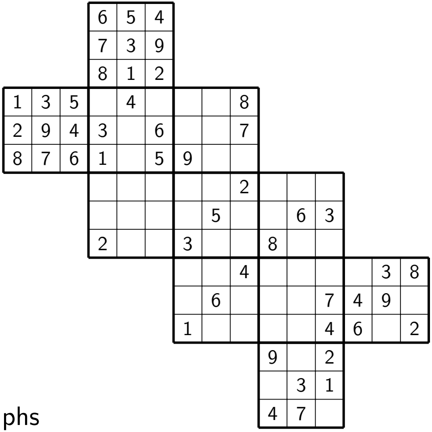
								4					3	8
							6				7	4	9	
						1					4	6		2
									9		2			
										3	1			
									4	7				

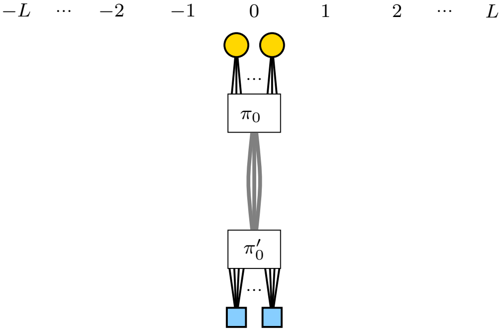
What is Spatial Coupling?





- Spatially-Coupled Factor Graphs
  - Variable nodes have a natural global orientation
  - Boundaries help variables to be recovered in an ordered fashion

Spatially-Coupled LDPC Codes: (l, r, L, w) Ensemble



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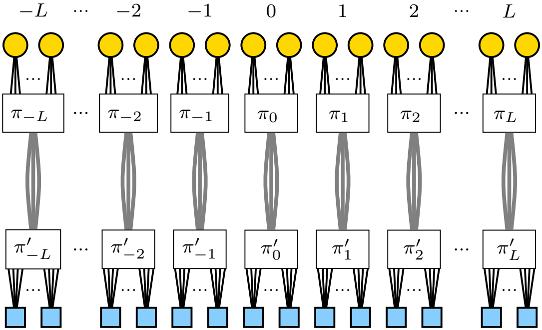

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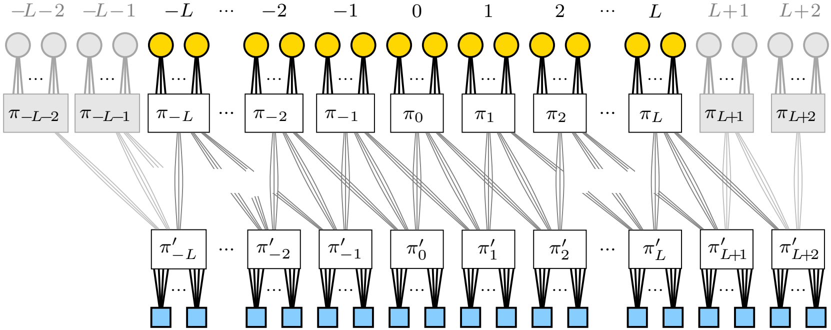

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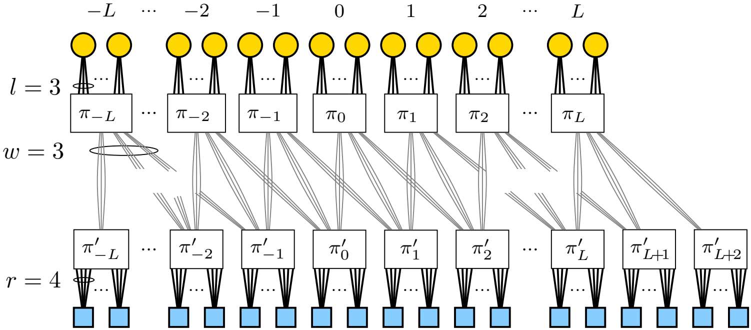


- Historical Notes
  - LDPC convolutional codes introduced by FZ in 1999
  - Shown to have near optimal noise thresholds by LSZC in 2005
  - (l, r, L, w) ensemble proven to achieve capacity by KRU in 2011

The LDPCC Gang

5274 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 56, NO. 10, OCTOBER 2010

Iterative Decoding Threshold Analysis for LDPC Convolutional Codes

Michael Lentmaier, *Member, IEEE*, Arvind Sridharan, *Member, IEEE*, Daniel J. Costello, Jr., *Life Fellow, IEEE*, and Kamil Sh. Zigangirov, *Fellow, IEEE*

The Spatial Coupling KRU

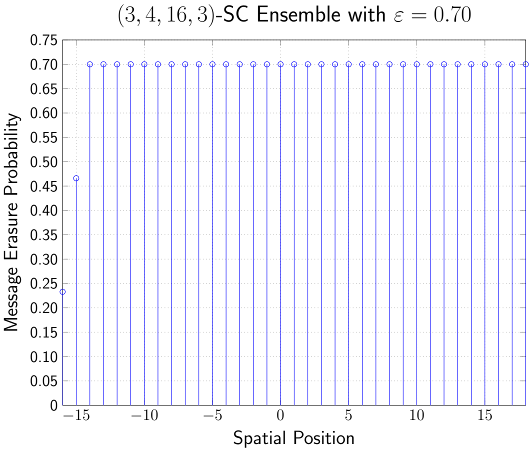
IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 57, NO. 2, FEBRUARY 2011 803

Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform So Well over the BEC

Shrinivas Kudekar, *Member, IEEE*, Thomas J. Richardson, *Fellow, IEEE*, and Rüdiger L. Urbanke

Density Evolution for the (l, r, L, w)-SC LDPC Ensemble



$$x_i^{(\ell+1)} = \frac{1}{w} \sum_{k=0}^{w-1} \varepsilon \left(\frac{1}{w} \sum_{j=0}^{w-1} \left(1 - (1 - x_{i+j-k}^{(\ell)})^{r-1}\right) \right)^{l-1} \mathbf{1}_{[-L, L+w-1]}(i-k)$$

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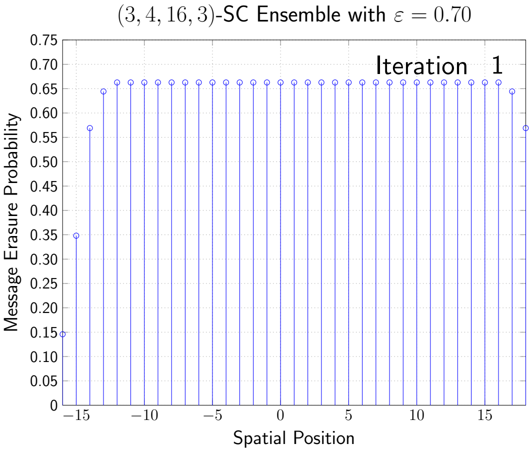

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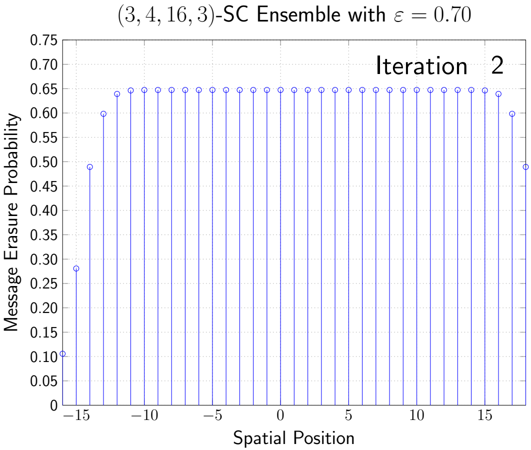

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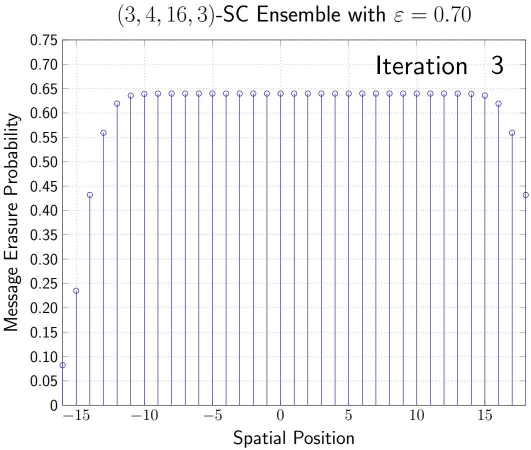

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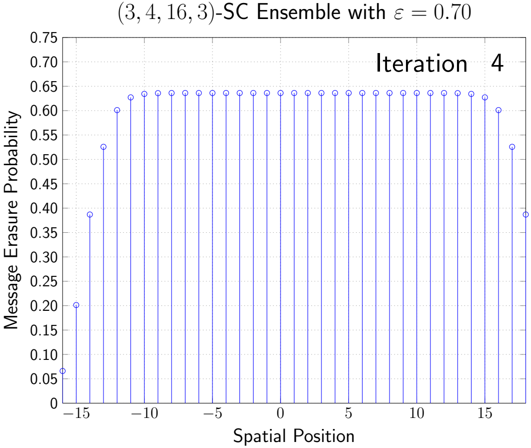

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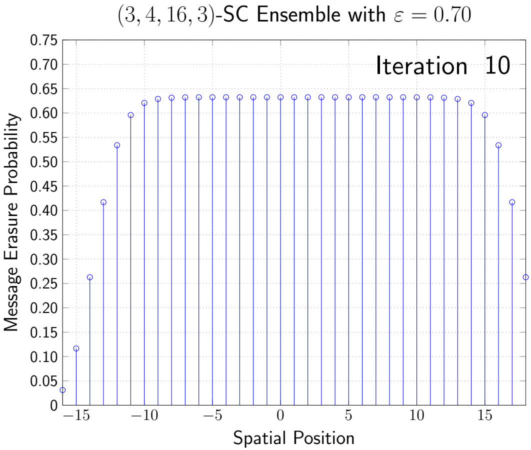

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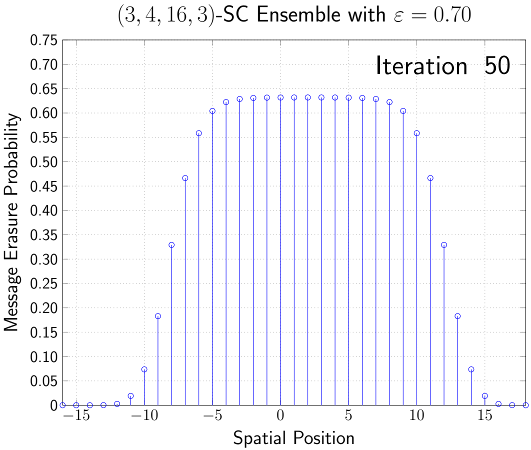

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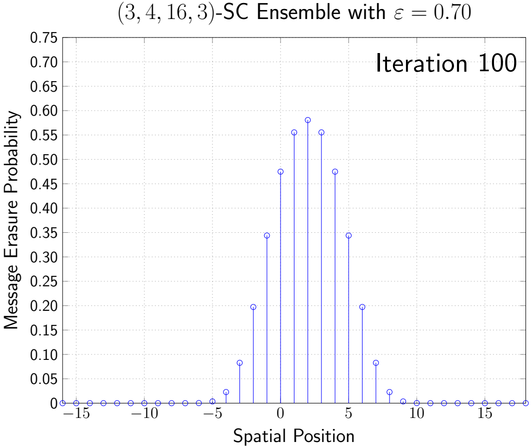

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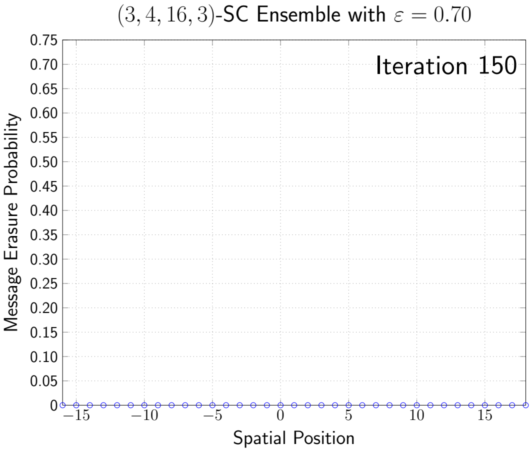

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Properties of Threshold Saturation

l	r	$\varepsilon^{\mathrm{BP}}$	$\varepsilon^{\mathrm{MAP}}$
3	6	0.4294	0.4882
4	8	0.3834	0.4977
5	10	0.3416	0.4995
6	12	0.3075	0.4999
7	14	0.2798	0.5000

- Spatial coupling achieves the MAP threshold as $w \to \infty$
  - BP threshold typically decreases after $l = 3$
  - MAP threshold is increasing in l, r for fixed rate
- Benefits and Drawbacks
  - For fixed L, minimum distance grows linearly with block length
  - Rate loss of $O(w/L)$ is a big obstacle in practice

Threshold Saturation via Spatial Coupling

- **General Phenomenon** (observed by Kudekar, Richardson, Urbanke)
  - BP threshold of the spatially-coupled system converges to the MAP threshold of the uncoupled system
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- Connection to statistical physics
  - Factor graph defines system of coupled particles
  - Valid sequences are ordered crystalline structures
- Between BP and MAP threshold, system acts as supercooled liquid
  - Correct answer (crystalline state) has minimum energy
  - Crystallization (i.e., decoding) does not occur without a seed
  - Ex.: ice melts at $0\,^{\circ}$C but freezing w/o a seed requires $-48.3\,^{\circ}$C

http://www.youtube.com/watch?v=Xe8vJrIvDQM

Why is Spatial Coupling Interesting?

- Breakthroughs: first practical constructions of
  - universal codes for binary-input memoryless channels [KRU12]
  - information-theoretically optimal compressive sensing [DJM11]
  - universal codes for Slepian-Wolf and MAC problems [YJNP11]
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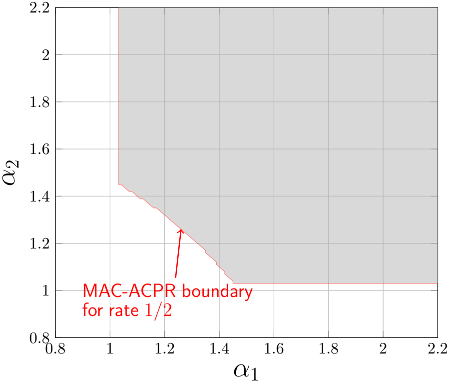
  - Our proof for increasing scalar/vector recursions [YJNP12/13]
- Spatial coupling as a proof technique [GMU13]
  - For a large random factor graph, construct a coupled version
  - Use DE to analyze BP decoding of coupled system
  - Compare uncoupled MAP with coupled BP via interpolation

Outline

Universality over Unknown Parameters

- The Achievable Channel Parameter Region (ACPR)
  - For a sequence of coding schemes involving one or more parameters, the parameter region where decoding succeeds in the limit
  - In contrast, a capacity region is a rate region for fixed channels



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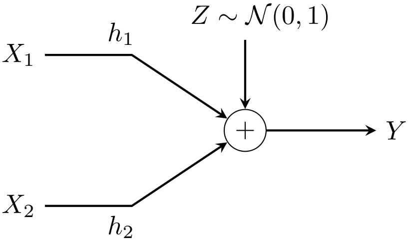
- Universality
  - A sequence of encoding/decoding schemes is called universal if:
    its ACPR equals the optimal ACPR
  - Channel parameters are assumed unknown at the transmitter
  - At the receiver, the channel parameters are easily estimated

2-User Binary-Input Gaussian Multiple Access Channel



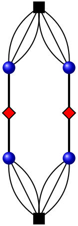
- Fixed noise variance
- Real channel gains h_1 and h_2 not known at transmitter
- Each code has rate R

- MAC-ACPR denotes the information-theoretic optimal region

A Little History: SC for Multiple-Access (MAC) Channels

- KK consider a binary-adder erasure channel (ISIT 2011)
  - SC exhibits threshold saturation for the joint decoder
- YNPN consider the Gaussian MAC (ISIT/Allerton 2011)
  - SC exhibits threshold saturation for the joint decoder
  - For channel gains h_1, h_2 unknown at transmitter, SC provides universality
- Others consider CDMA systems without coding
  - TTK show SC improves BP demod of standard CDMA
  - ST prove saturation for a SC protograph-style CDMA

Spatially-Coupled Factor Graph for Joint Decoder



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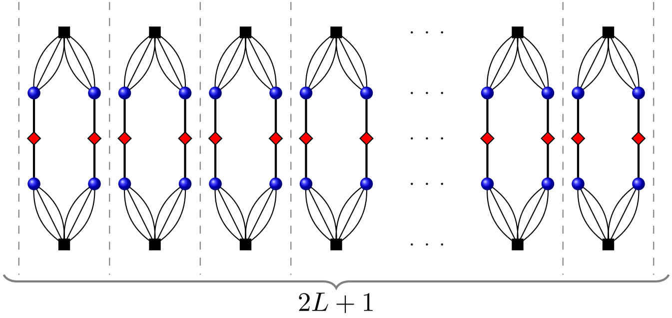

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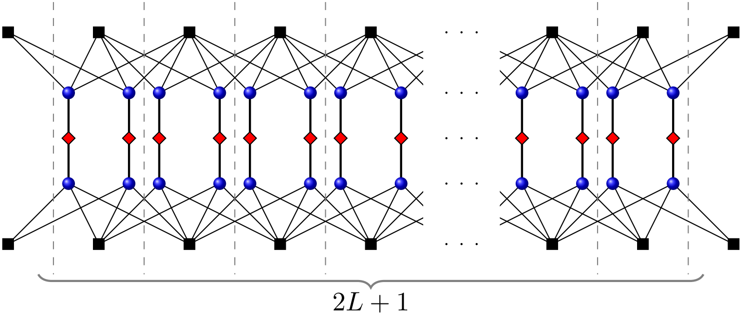
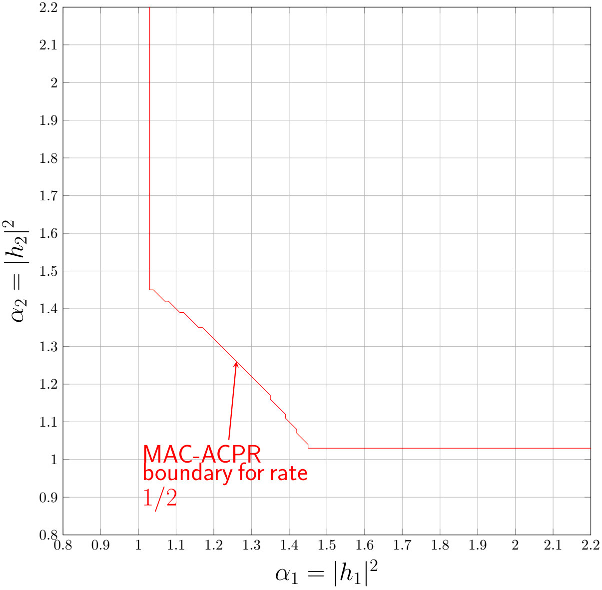


DE Performance of the Joint Decoder



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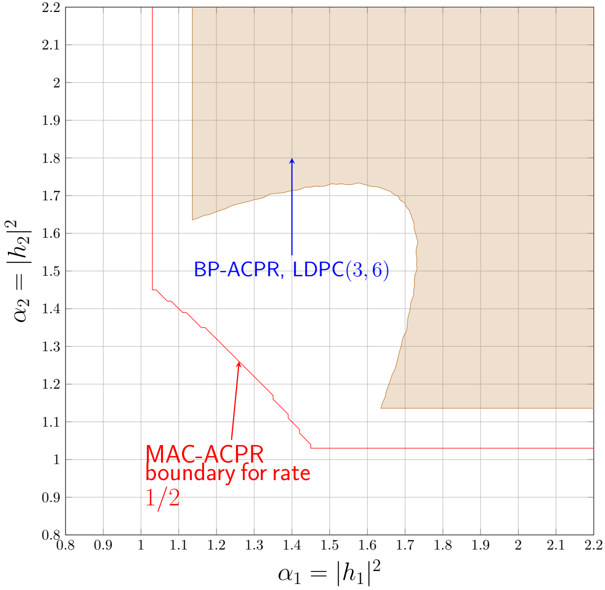

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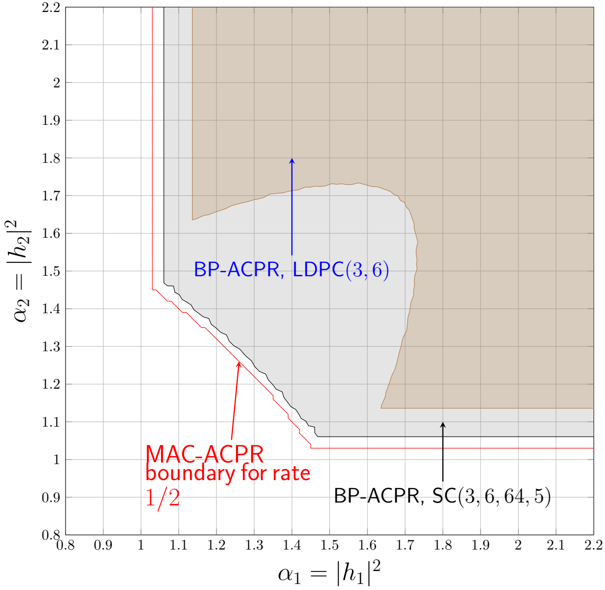

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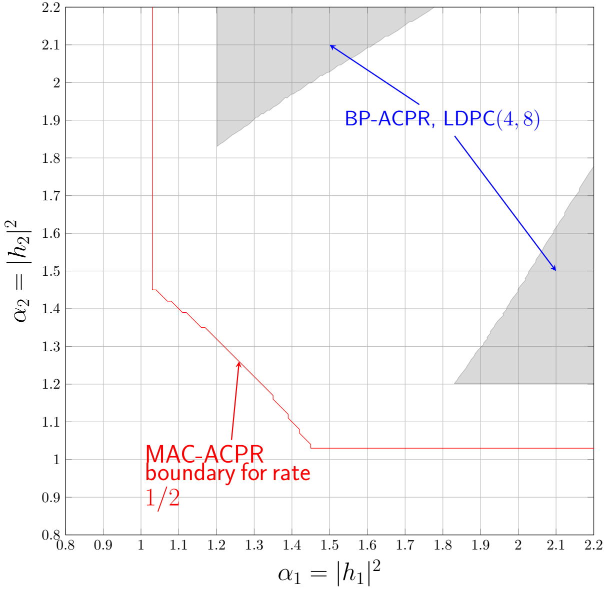

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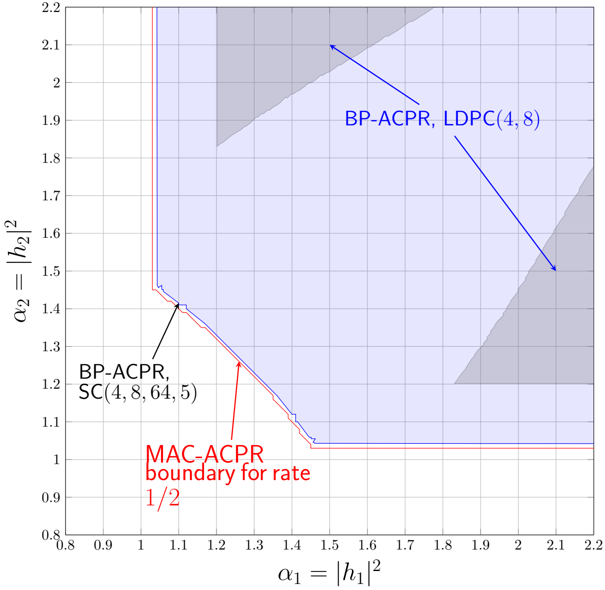
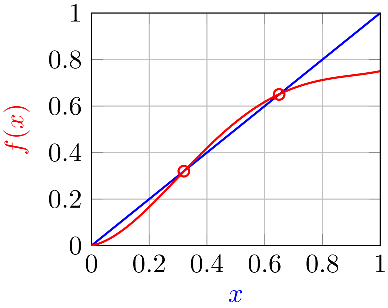


Outline

Single Monotone Recursion

- Smooth increasing $f\colon [0,1] \to [0,1]$



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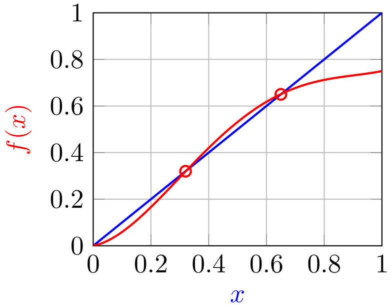

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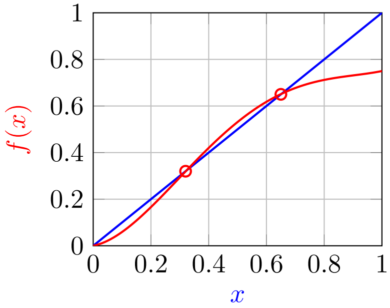

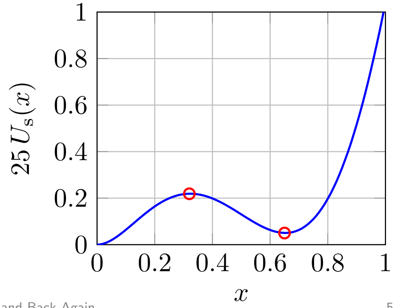

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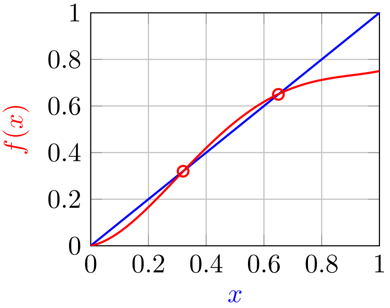

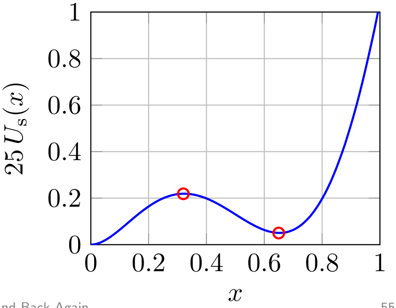

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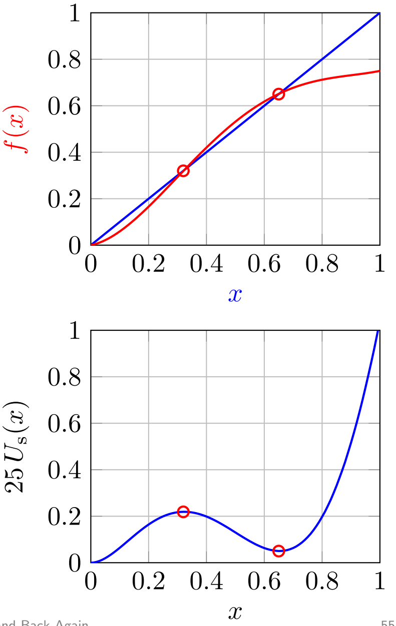

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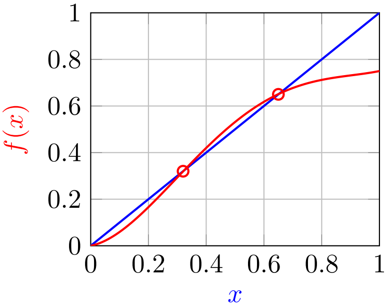
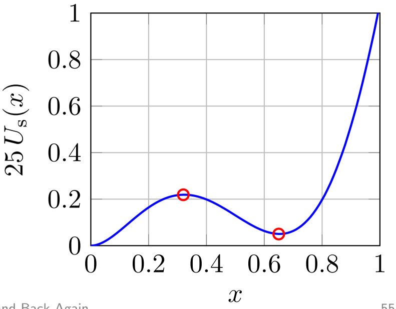
Both $\downarrow 0$ iff no fixed points in $(0,1]$





Coupled Monotone Recursion (1)

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$$T\underline{x} \triangleq A^\top \underline{f}\left(A\underline{x}\right),$$

where $[\underline{f}(\underline{x})]_i = f(x_i)$ and A averages w adjacent values

$$A = \frac{1}{w}\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots \\ 0 & 1 & 1 & \ddots & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

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- Coupled potential: $U_{\mathrm{c}}(\underline{x}) = \frac{1}{2}\sum_{i=0}^{\infty} x_i^2 - \sum_{i=0}^{\infty} F\left(\frac{1}{w}\sum_{j=0}^{w-1} x_{i+j}\right)$
  - Satisfies $\nabla U_{\mathrm{c}}(\underline{x}) = \underline{x} - A^\top \underline{f}(A\underline{x})$
  - Danger: there be ~~dragons~~ infinities

Coupled Monotone Recursion (2)

- Properties of T (note: $\underline{x} \preceq \underline{y} \Leftrightarrow x_i \leq y_i$ for all i)
  - T is monotone: $\underline{x} \preceq \underline{y}$ implies $T\underline{x} \preceq T\underline{y}$
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- For $\underline{x}^{(0)} = \underline{1}$, iterates $x_i^{(\ell)}$ are decreasing in ℓ and increasing in i
  - Spatial limit exists: $x_\infty^{(\ell)} = \lim_{i\to\infty} x_i^{(\ell)}$
  - Iteration limit exists: $x_i^{(\infty)} = \lim_{\ell\to\infty} x_i^{(\ell)}$
  - Iteration limit satisfies fixed point: $\underline{x}^{(\infty)} = T\underline{x}^{(\infty)}$
  - Double limit satisfies fixed point: $x_\infty^{(\infty)} = f\big(x_\infty^{(\infty)}\big)$

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- Between the BP and MAP threshold
  - decoding trajectory looks like a right-moving wave

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  - If $x_{i+1} \geq x_i$ for all i, then $V_{\underline{x}}(t)$ well-defined for $t \in [0,1]$
  - For $t = 1$, one gets a telescoping sum that shows

$$V_{\underline{x}}(1) \leq -U_{\mathrm{s}}(x_\infty)$$

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Theorem

If $f(0)=0$ and $f'(0)<1$ (0 is stable f.p.) with $U_{\mathrm{s}}(x)>0$ for $x\in(0,1]$, then $\exists w_0 < \infty$ such that $x_{\infty}^{(\infty)} = 0$ for all $w > w_0$.

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  - Telescoping sum for V shows $V_{\underline{z}}(1) \leq -U(z_\infty) < 0$
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  - Telescoping sum for V shows $V_{\underline{z}}(1)\leq -U(z_\infty)<0$
  - Taylor series for V shows $|V_{\underline{z}}(1)|\leq K\frac{1}{w}\left(1+\sup_{x\in[0,1]}|f'(x)|\right)$
  - Thus, we get a contradiction for sufficiently large w

History of Threshold Saturation Proofs

- the BEC in 2010 [KRU11]
  - Established many properties and tools used by later approaches
- the Curie-Weiss model of physics in 2010 [HMU12]
- CDMA using a GA in 2011 [TTK12]
- CDMA with outer code via GA in 2011 [Tru12]
- compressive sensing using a GA in 2011 [DJM13]
- regular codes on BMS channels in 2012 [KRU13]
- increasing scalar and vector recursions in 2012 [YJNP14]
- irregular LDPC codes on BMS channels in 2012 [KYMP14]
- non-decreasing scalar recursions in 2012 [KRU15]
- non-binary LDPC codes on the BEC in 2014 [AG16]
- and more since 2014...

Summary and Open Problems

- Factor Graphs
  - Useful tool for modeling dependent random variables
  - Low-complexity algorithms for approximate inference
  - Density evolution can be used to analyze performance
- Spatial Coupling
  - Powerful technique for designing and understanding FGs.
  - Related to the statistical physics of supercooled liquids
  - Simple proof of threshold saturation for scalar recursions
- Interesting Open Problems
  - Code constructions that reduce the rate-loss due to termination
  - Compute the scaling exponent for SC codes
  - Finding new problems where SC provides benefits

Thanks for your attention

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