Spatially Coupled LDPC Codes: From Theory to Practice

Daniel J. Costello, Jr.

Dept. of Electrical Engineering,
University of Notre Dame

European School of Information Theory
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Outline

- **Introduction: From Shannon to Modern Coding Theory**
  - ➡ Channel capacity, structured codes, random codes, LDPC codes
- **LDPC Block Codes**
  - ➡ Parity-check matrix and Tanner graph representations, minimum distance bounds, iterative decoding thresholds, protograph-based constructions
- **Spatially Coupled LDPC Codes**
  - ➡ Protograph representation, edge-spreading construction, termination
  - ➡ Iterative decoding thresholds, threshold saturation, minimum distance
- **Practical Considerations**
  - ➡ Window decoding, performance, latency, and complexity comparisons to LDPC block codes, rate-compatibility, implementation aspects

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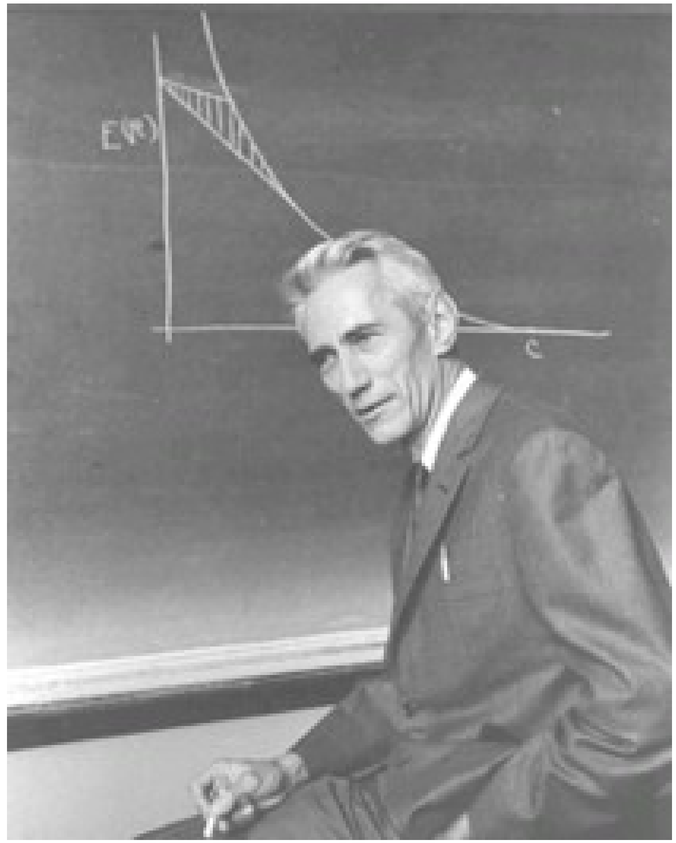
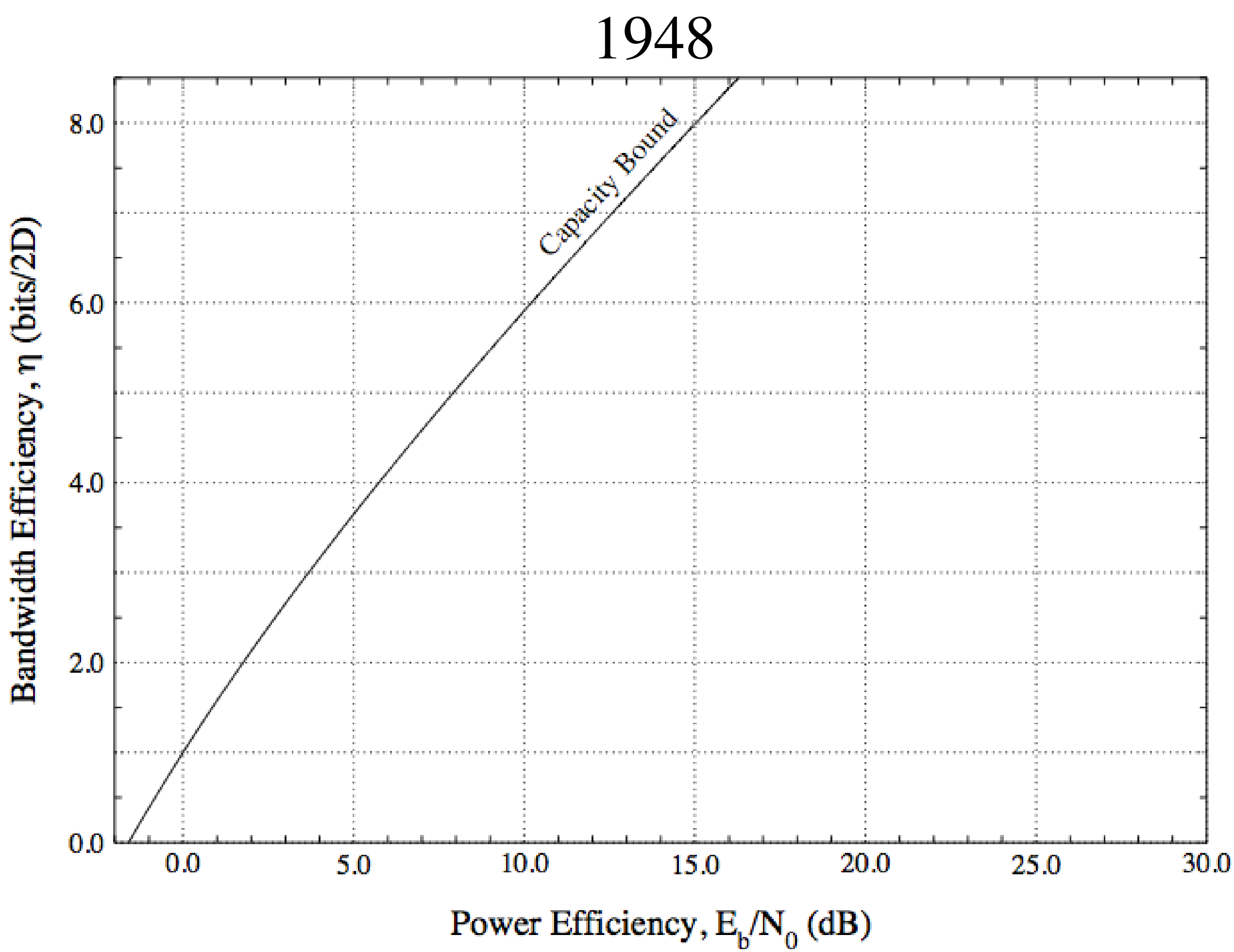
Shannon's Legacy



Claude Elwood Shannon
Apr. 30, 1916 – Feb. 24, 2001
Father of **Information Theory**

Channel Capacity (AWGNC)



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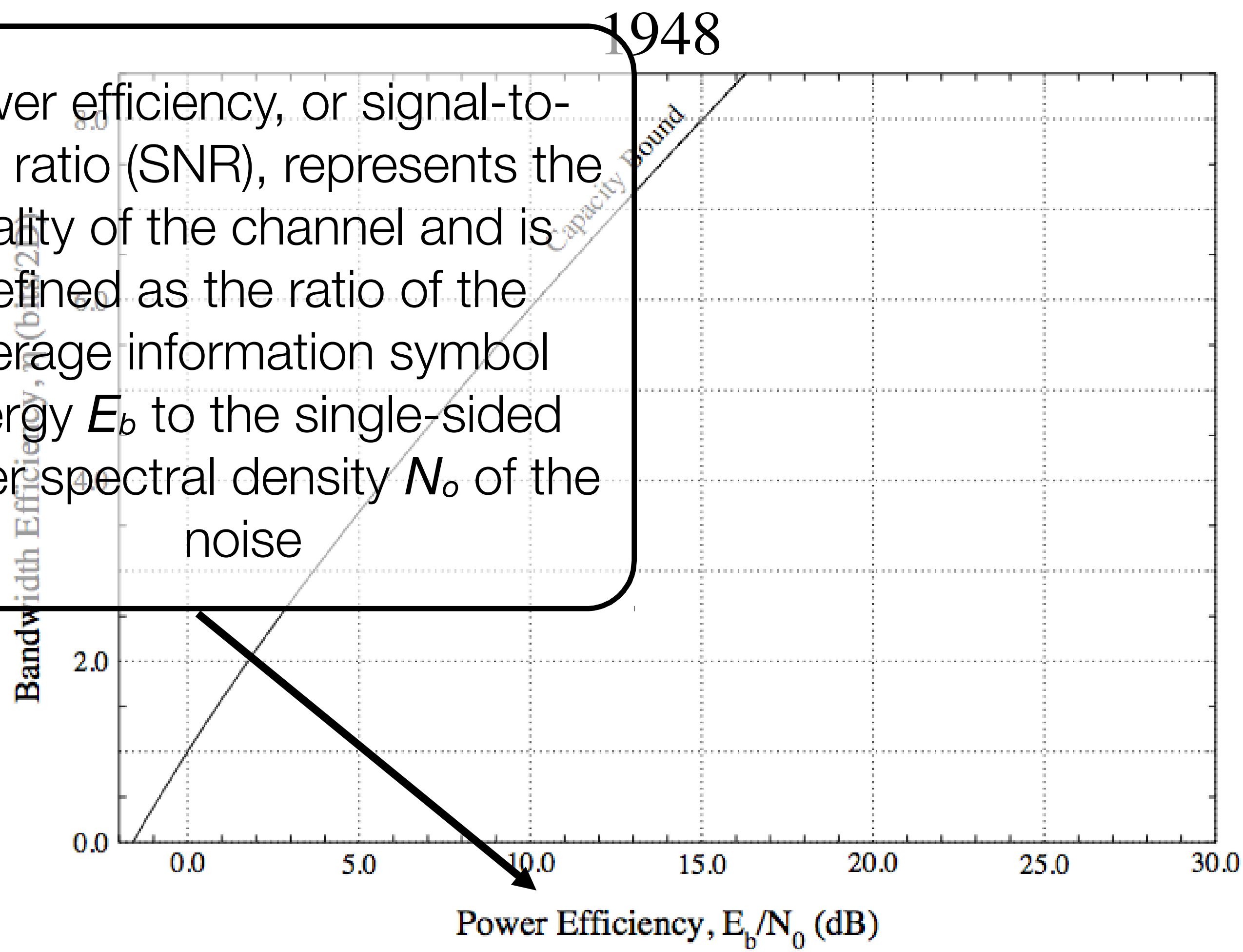

Power efficiency, or signal-to-noise ratio (SNR), represents the quality of the channel and is defined as the ratio of the average information symbol energy E_b to the single-sided power spectral density N_o of the noise

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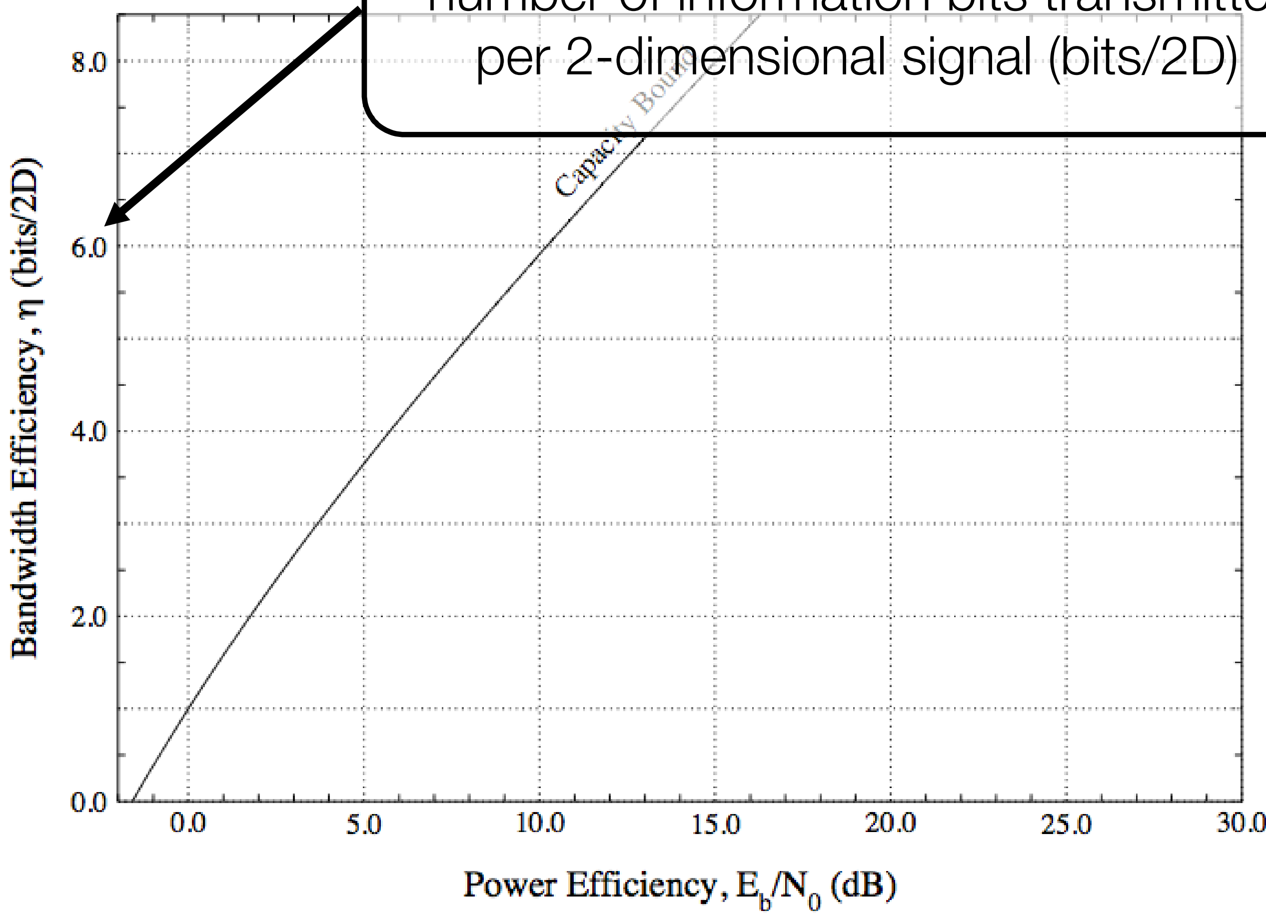

Bandwidth efficiency is normalized to the number of information bits transmitted per 2-dimensional signal (bits/2D)

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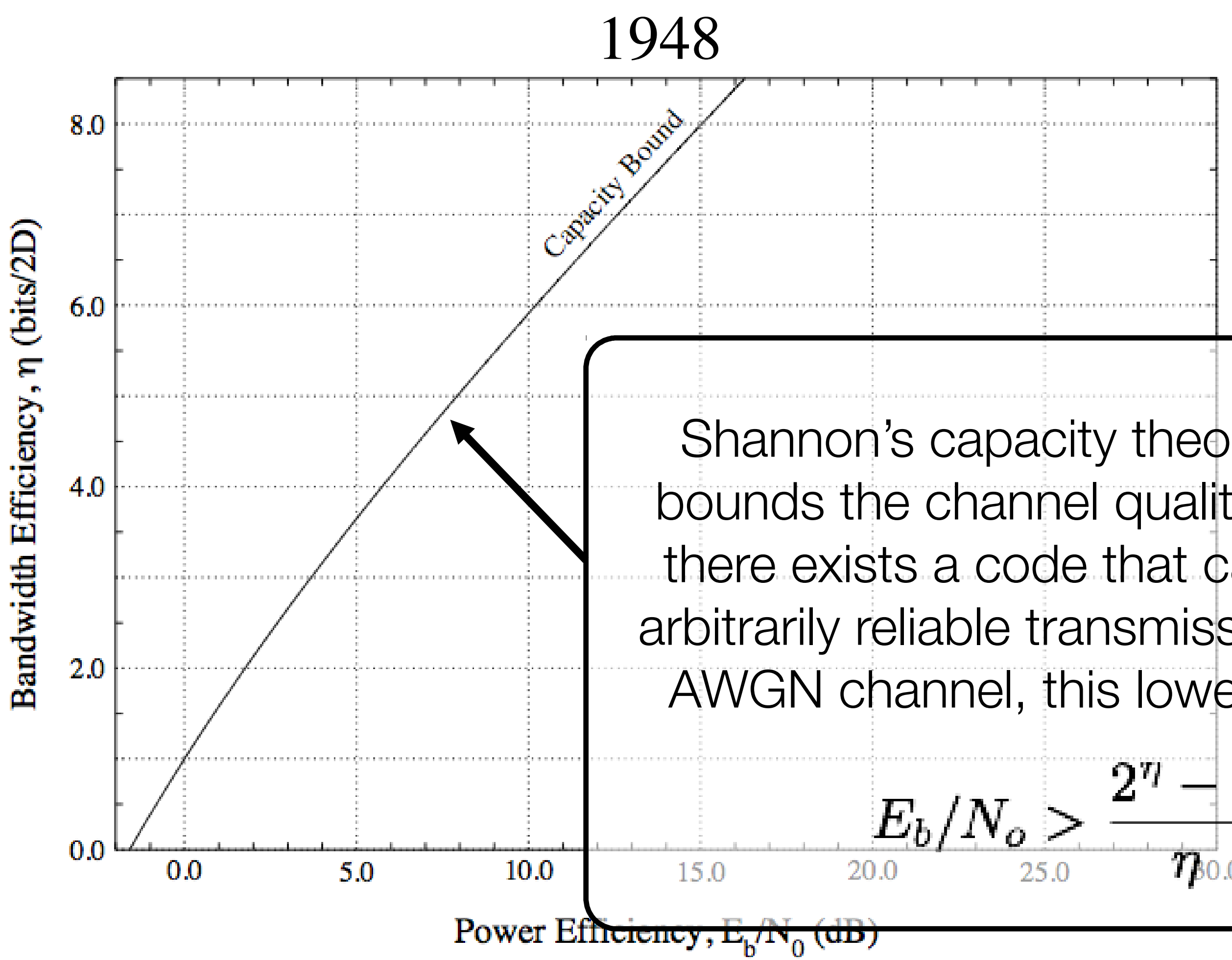

Shannon's capacity theorem lower bounds the channel quality for which there exists a code that can achieve arbitrarily reliable transmission. For the AWGN channel, this lower bound is

$$E_b/N_o > \frac{2^\eta - 1}{\eta}$$

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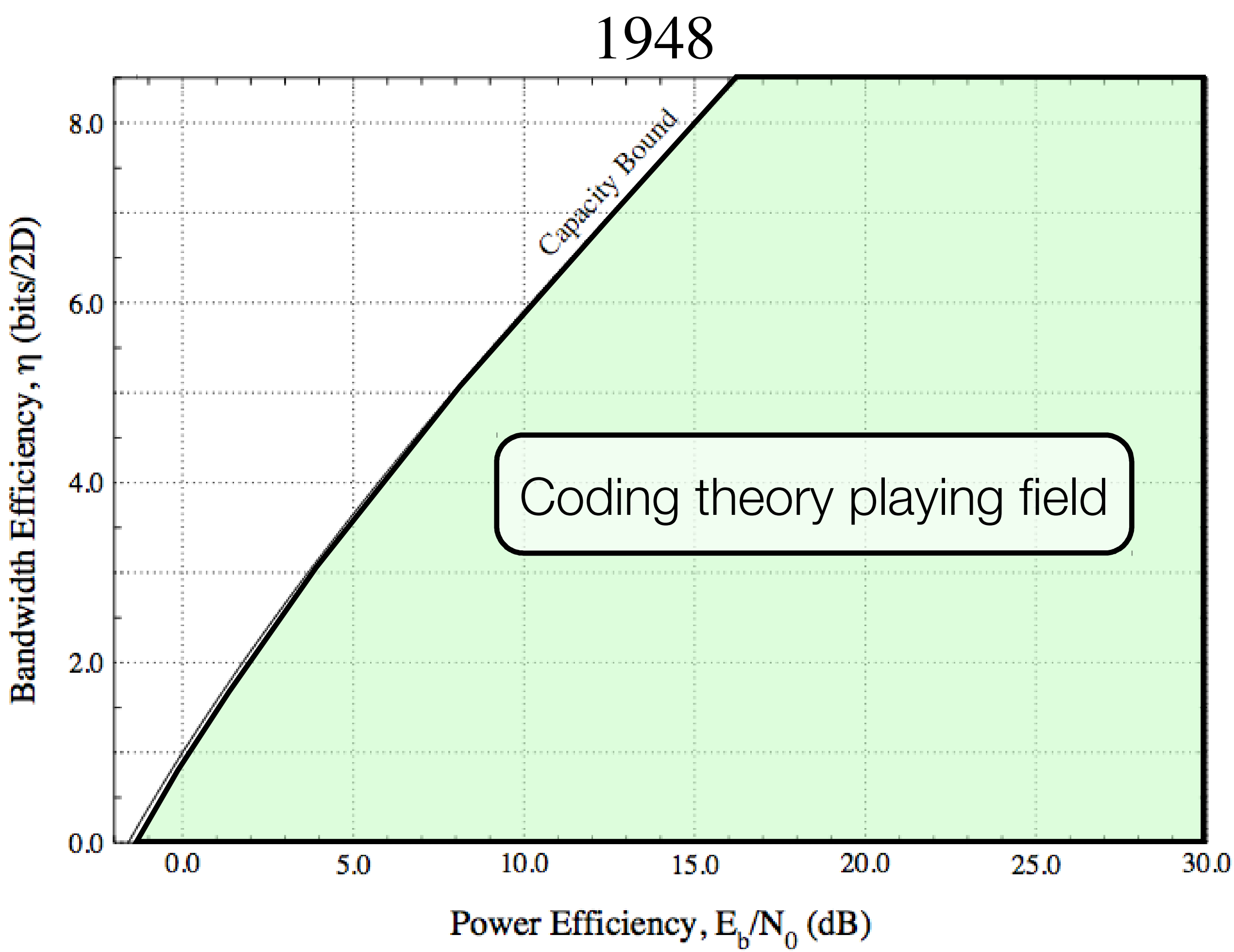
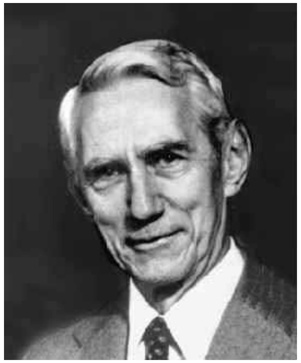


The coding dilemma

- Shannon showed that **random codes** with large block length can **achieve capacity**, but...



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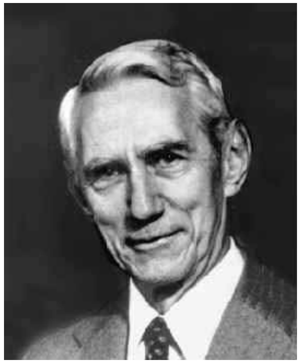

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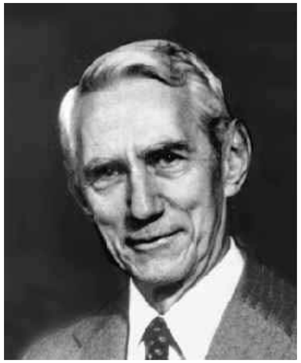

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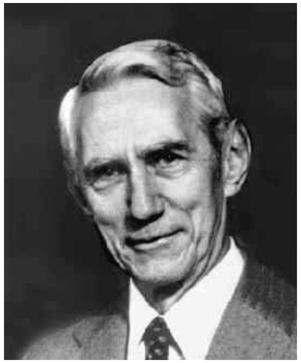

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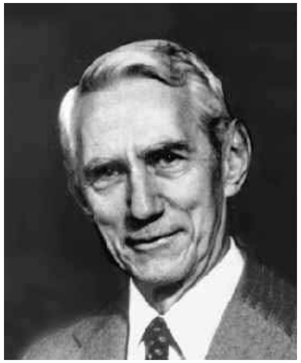

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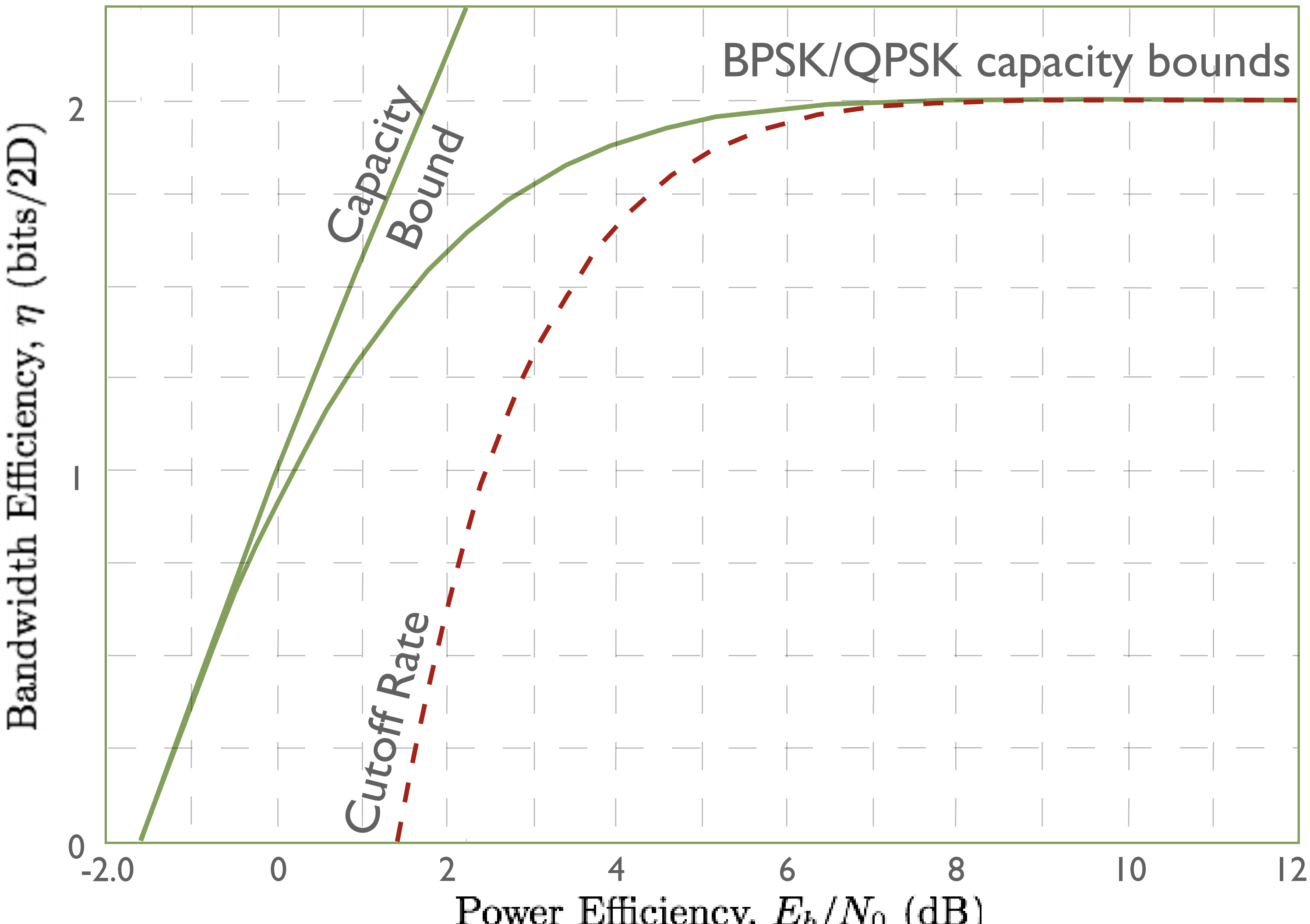
Solution: Construct random-like codes with just enough structure to allow efficient decoding

➡ Modern Coding Theory

LDPC Codes: motivation (for a target BER 10^{-5})



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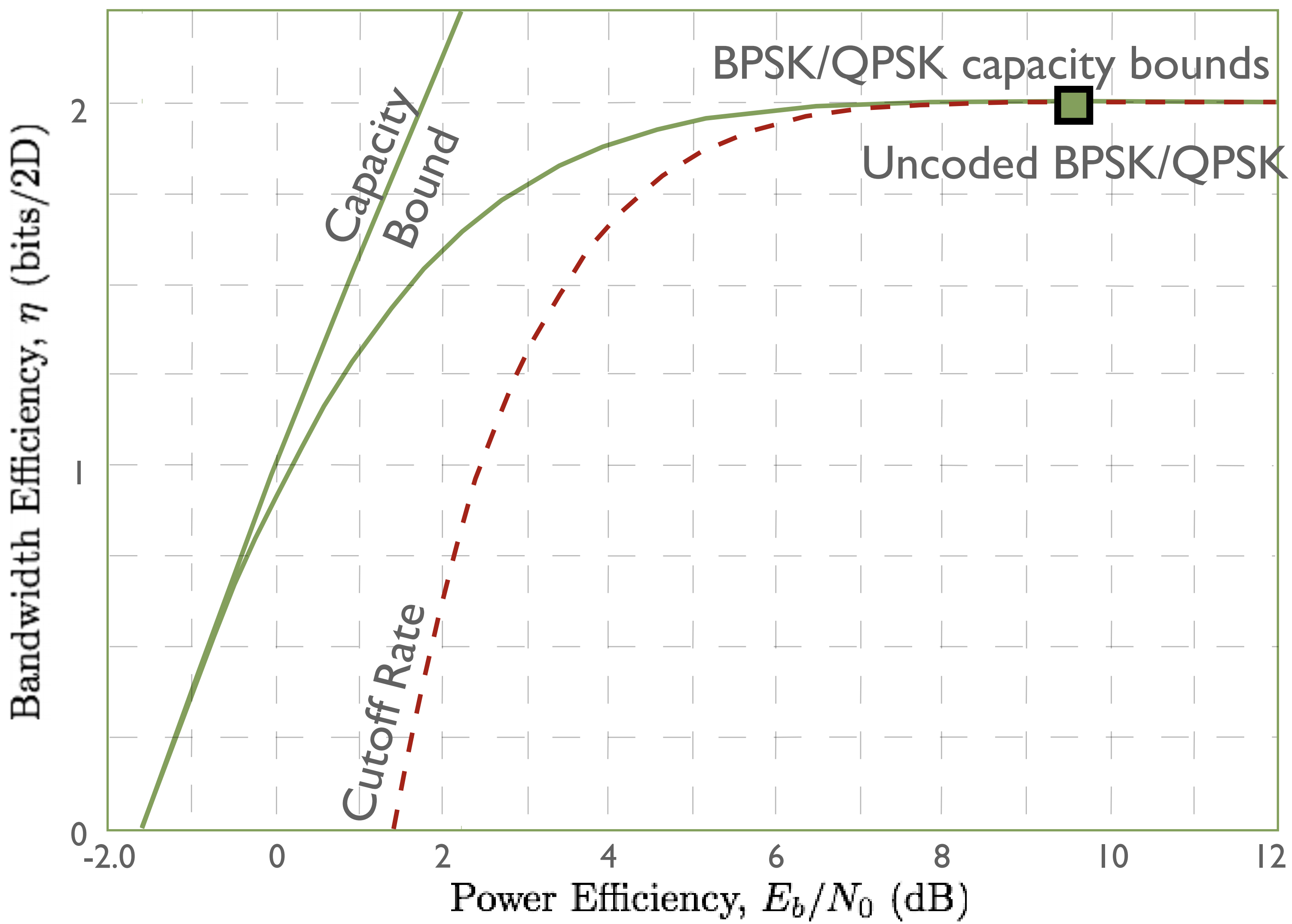

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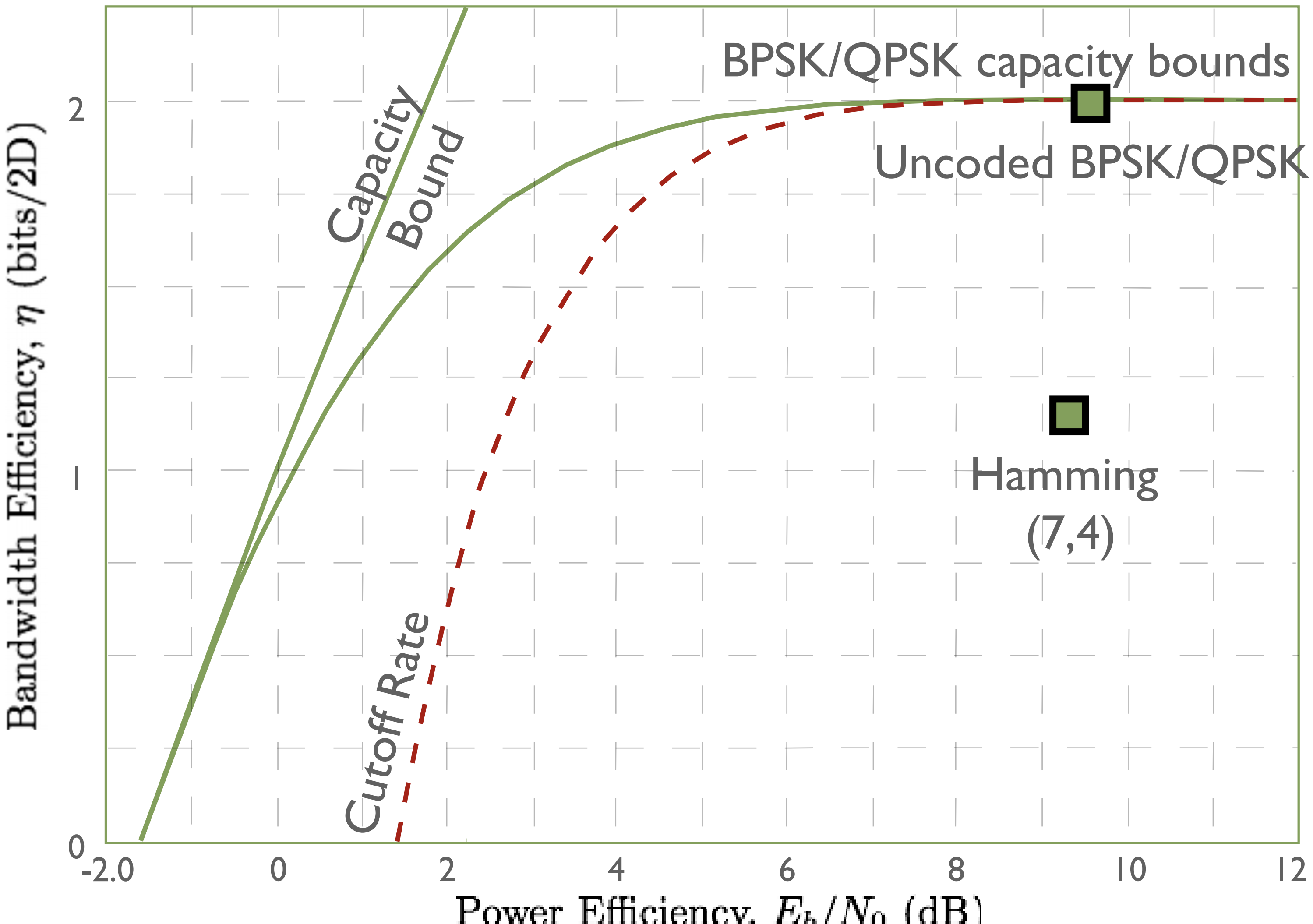

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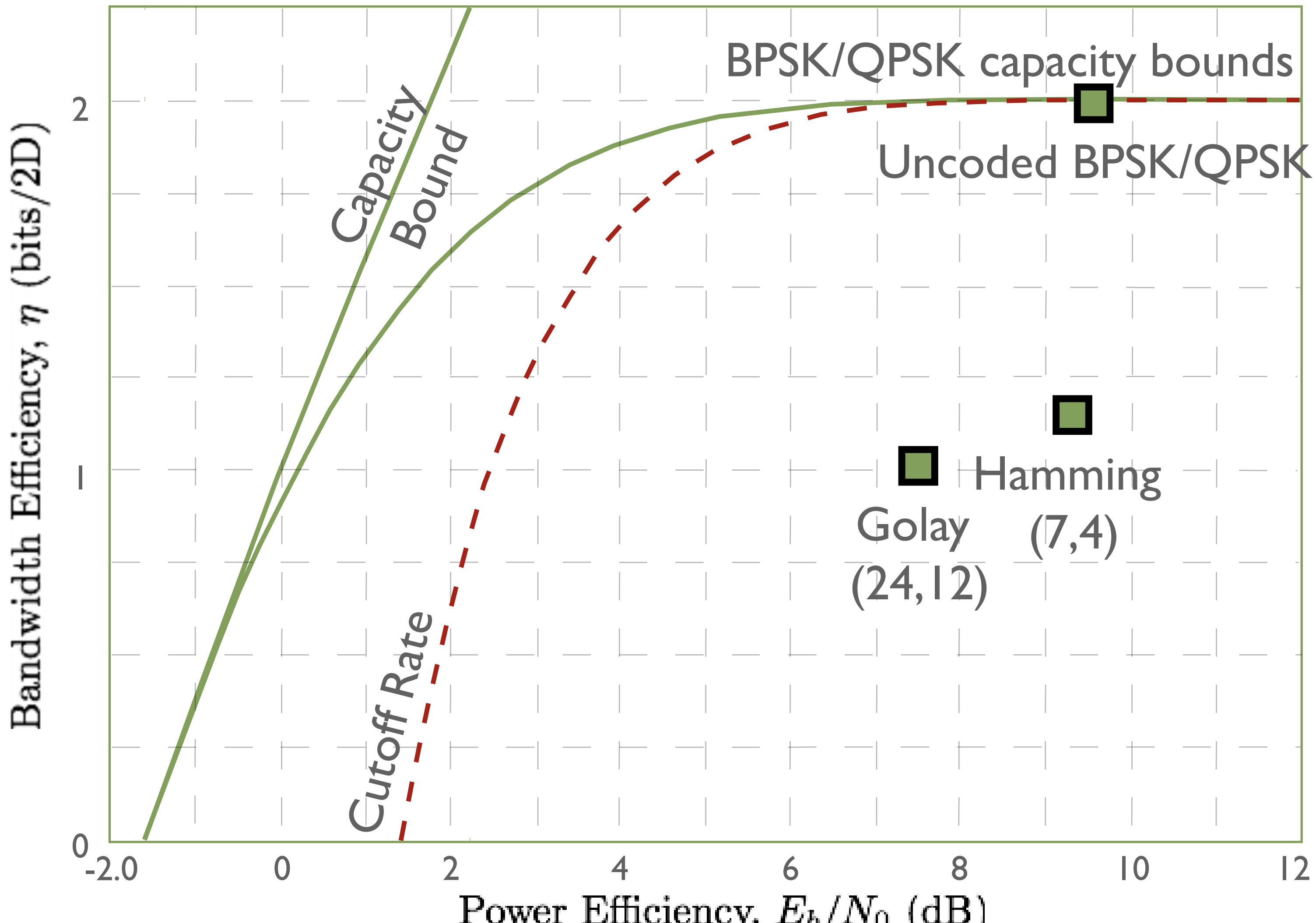

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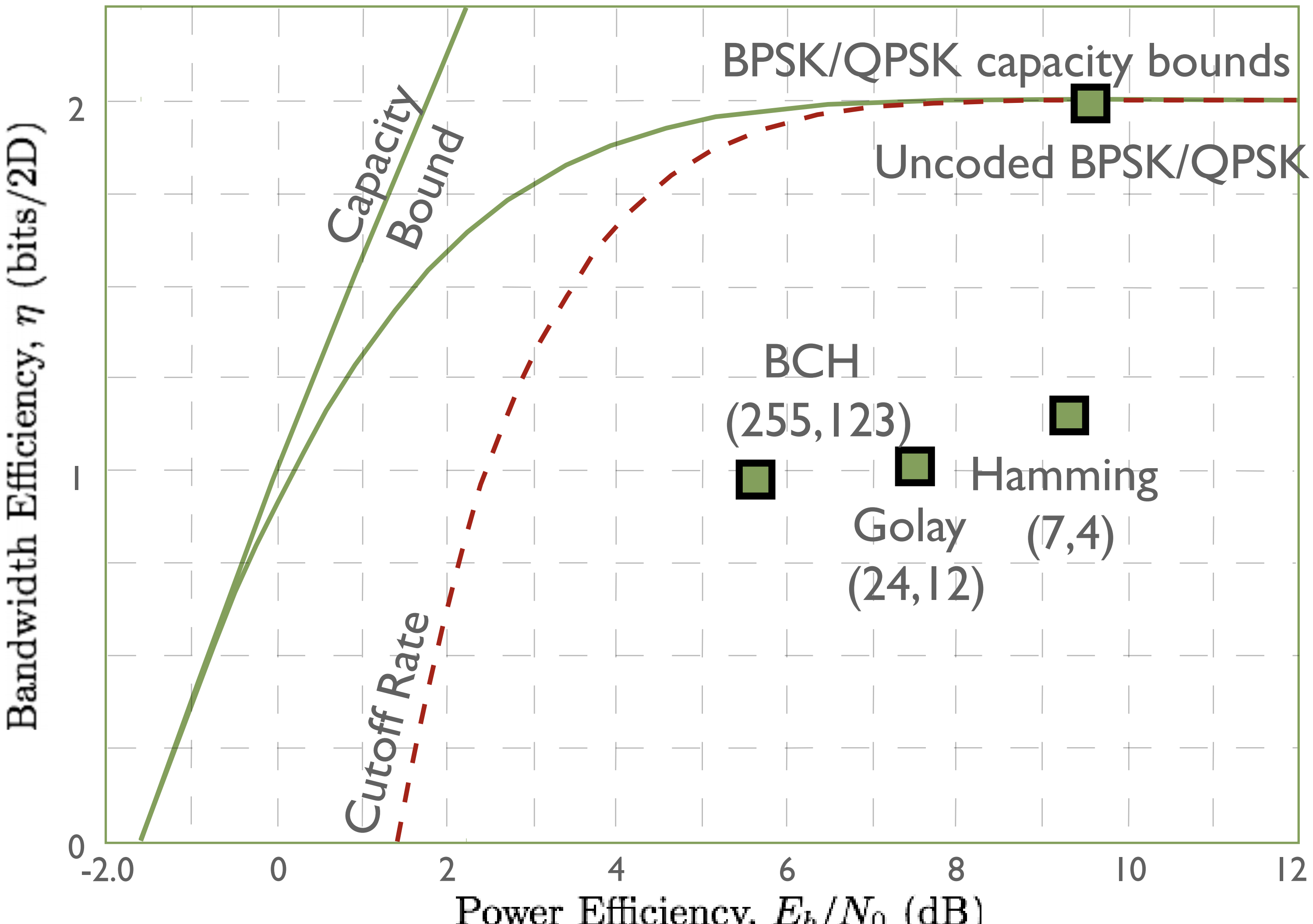

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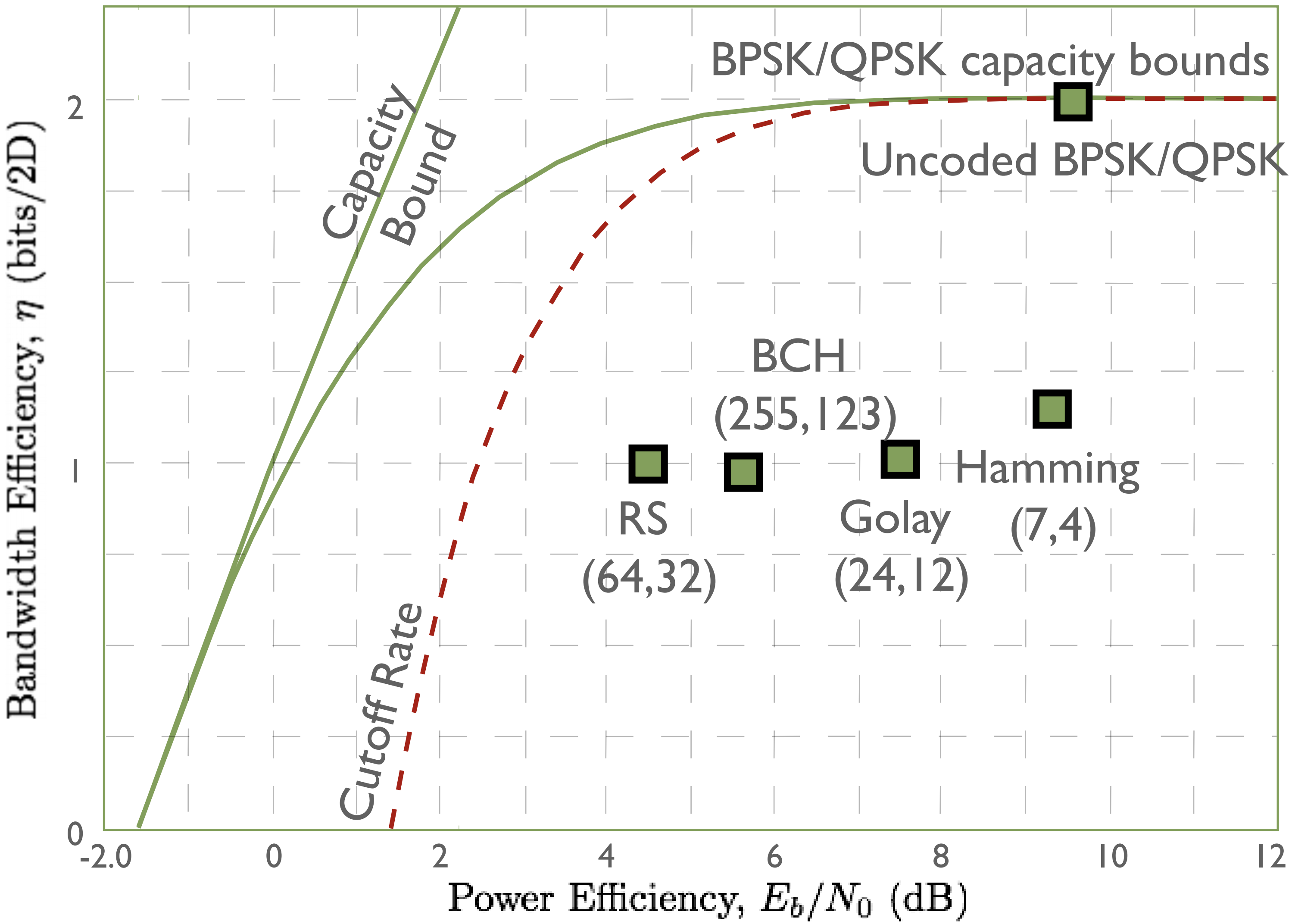

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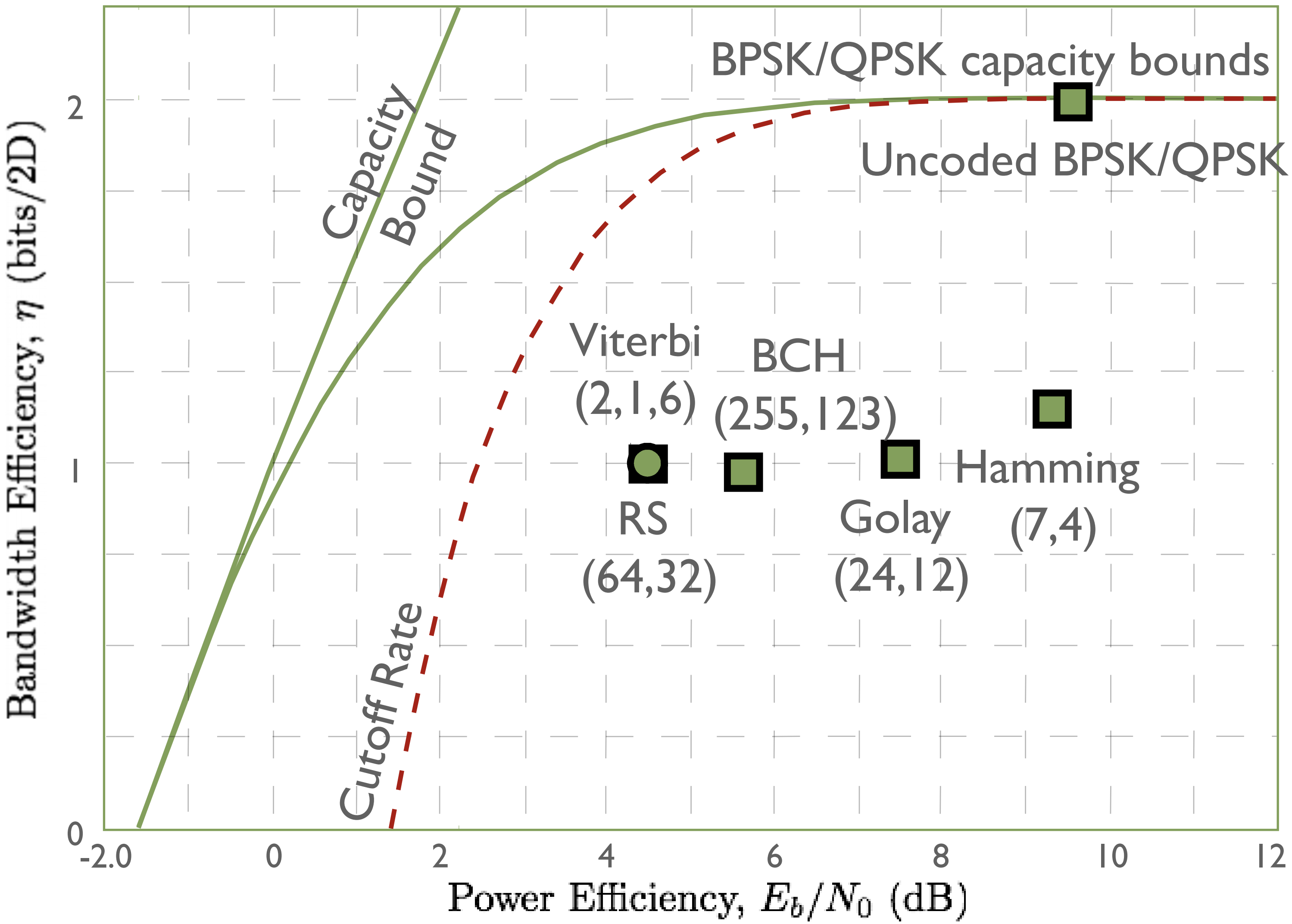

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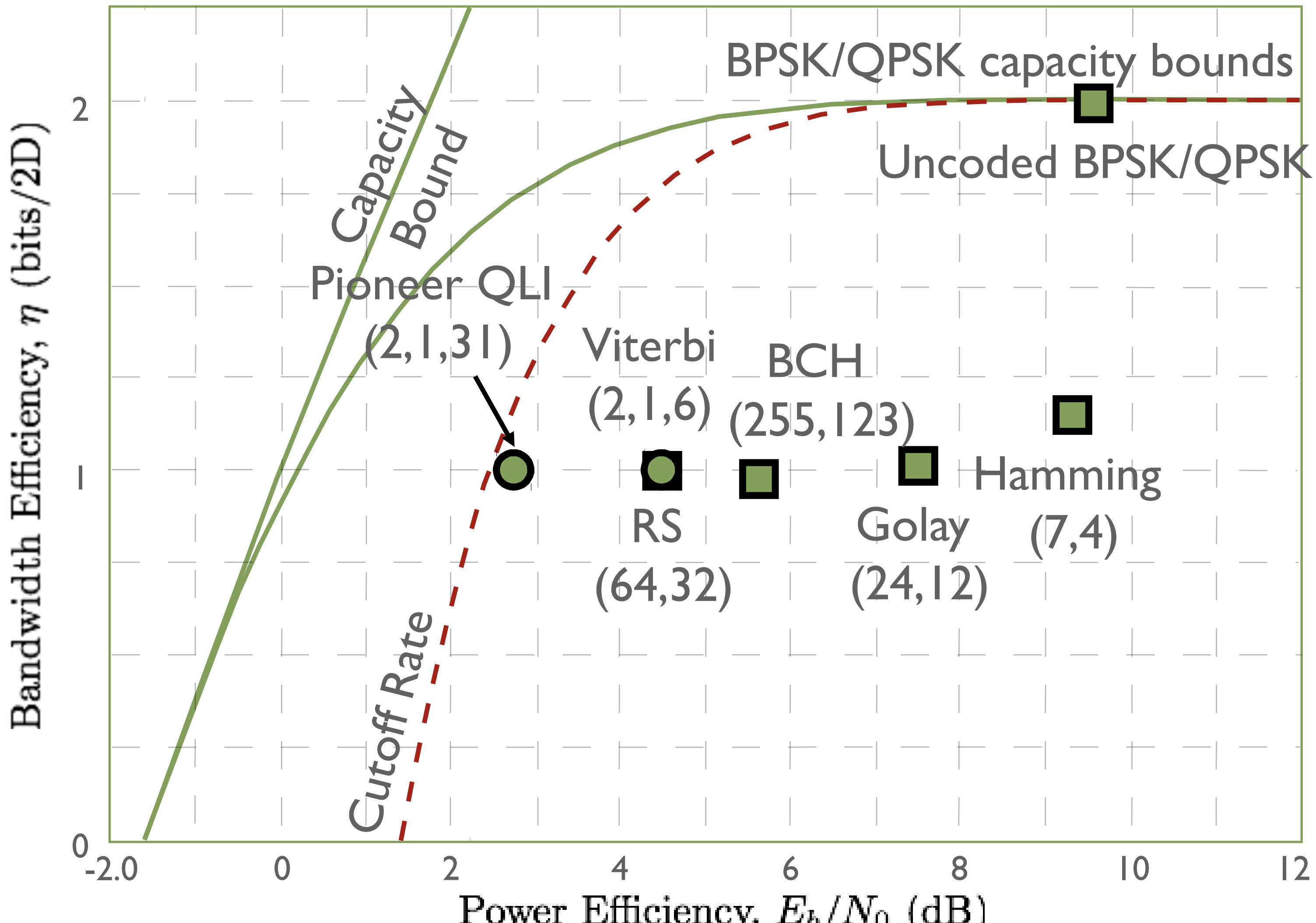

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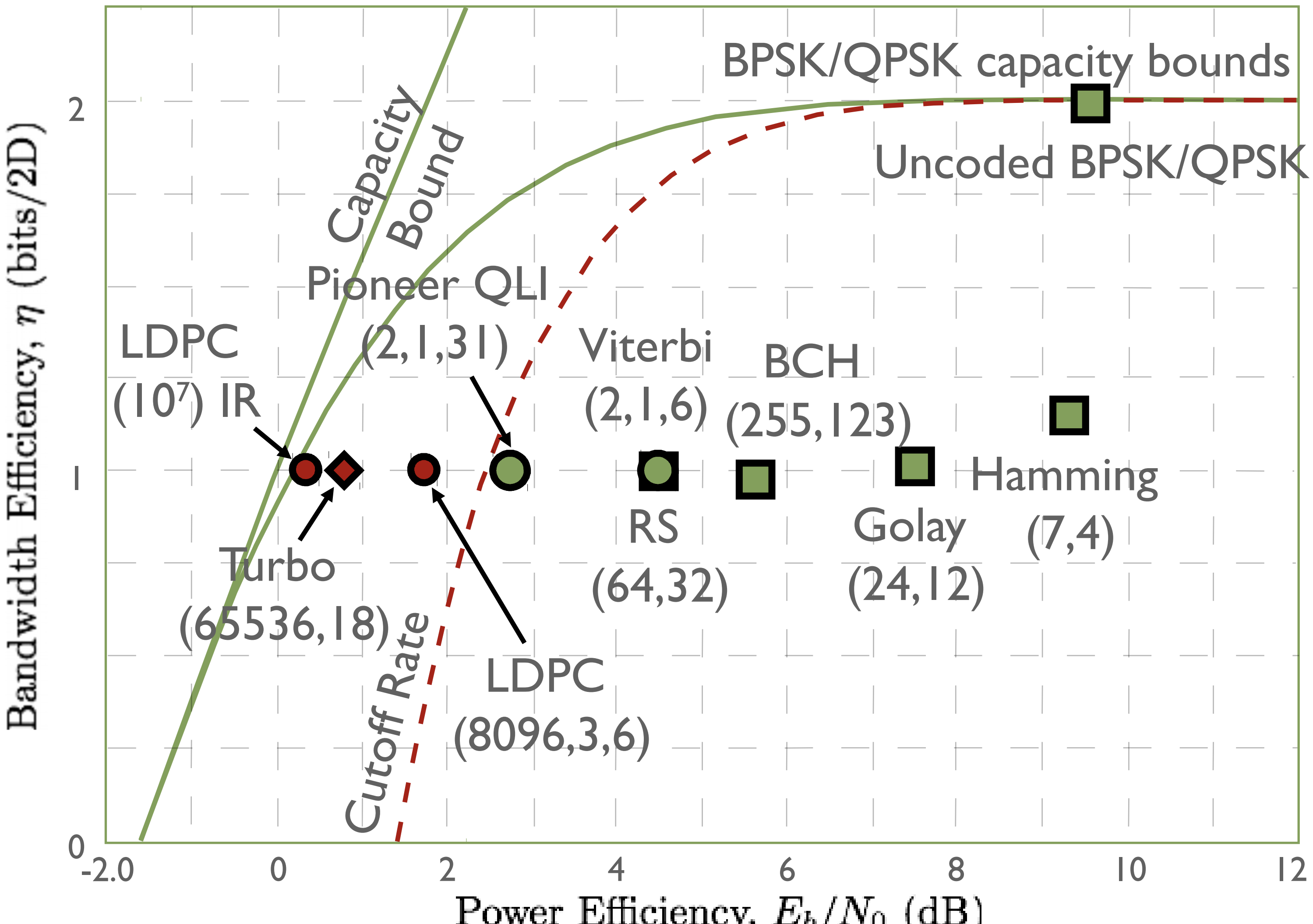
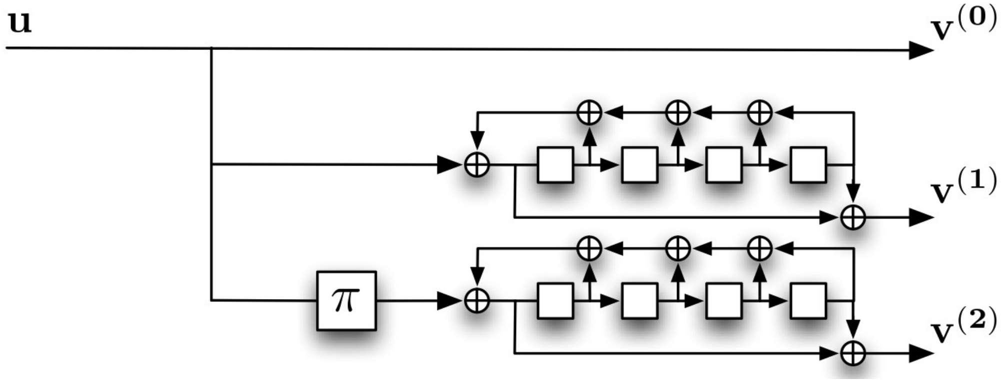


Random-like codes (2000s - today)

- Turbo codes use a long pseudorandom interleaver

➡ 3G and 4G telephony standards HSPA, EV-DO, LTE, satellite DVB-RCS, Mars Reconnaissance Rover, WiMAX, and so on.



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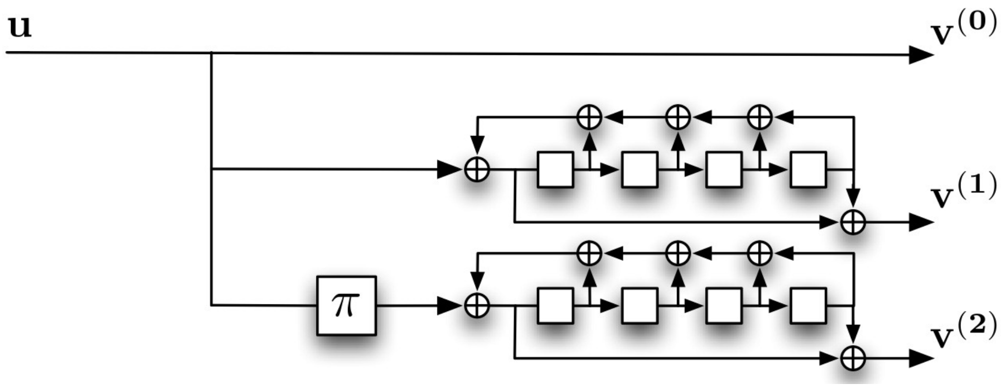

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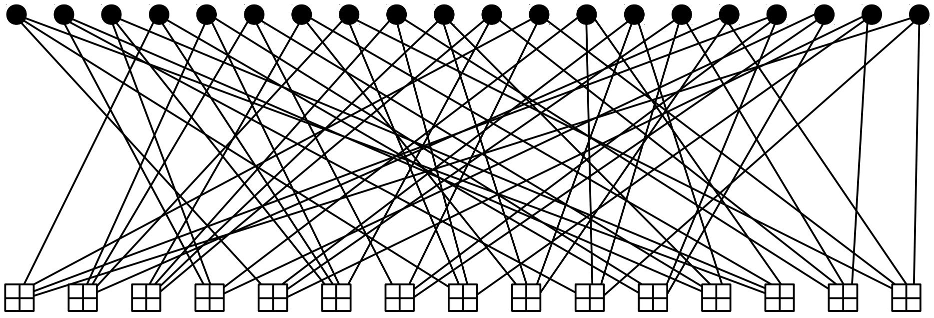
- Low-density parity-check (LDPC) codes are defined on a large sparse graph



➡ DVB-S2, ITU-T G.hn standard (data networking over power lines, phone lines, and coaxial cables), 10GBase-T Ethernet, Wi-Fi standards 802.11, and so on.

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Definitions

- Recall, an (n,k) binary linear code C is a k-dimensional subspace of $\{0,1\}^n$
  - G is a generator matrix for C if its rows span C. (G is an $l \times n$ matrix where $l \geq k$.)
  - H is a parity-check matrix for C if its rows span $C^{\perp}$, i.e., $v \in C$ iff $vH^T = 0$. (H is an $m \times n$ matrix where $m \geq n-k$.)

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Definition: A **low-density parity-check** (LDPC) code is a code for which the parity-check matrix of interest has a low density of ones.

- ➡ LDPC refers to the **representation** of a code, rather than the code itself.
- ➡ "low" is a vague term. (Complexity of decoding increases with the density of ones.)

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(2) Definition: An **irregular** (n,k) LDPC code is one in which the row and/or column weights of the parity-check matrix are not constant.

LDPC Block Codes

Definition by parity-check matrix: [Gallager, '62]

$$
H = \left[\begin{array}{cccccccccccccccccccc}
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0\\
\hline
0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0\\
1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0\\
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\hline
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0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1
\end{array}\right]_{15 \times 20}
$$

Code: $\{\mathbf{v} \mid \mathbf{v}\mathbf{H}^{\mathrm{T}} = \mathbf{0}\}$

(null space of a **sparse** parity-check matrix $\mathbf{H}$)

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Regular LDPC code:

Column weight: $J = 3$

Row weight: $K = 4$

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- If the row and column weights J and K are not constant, then the LDPC code is **irregular** (more later)

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- In the second class of nodes, there is one node for each of the m parity-checks (these are called the “check nodes”, “constraint nodes”, or “function nodes”).
- An edge connects a variable node to a check node if and only if that bit is included in that parity-check, i.e., iff H has a one in the associated position.

LDPC Block Codes

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Representation by bipartite graph: [Tanner, '81]

$$H = \left[\begin{array}{cccccccccccccccccccc}
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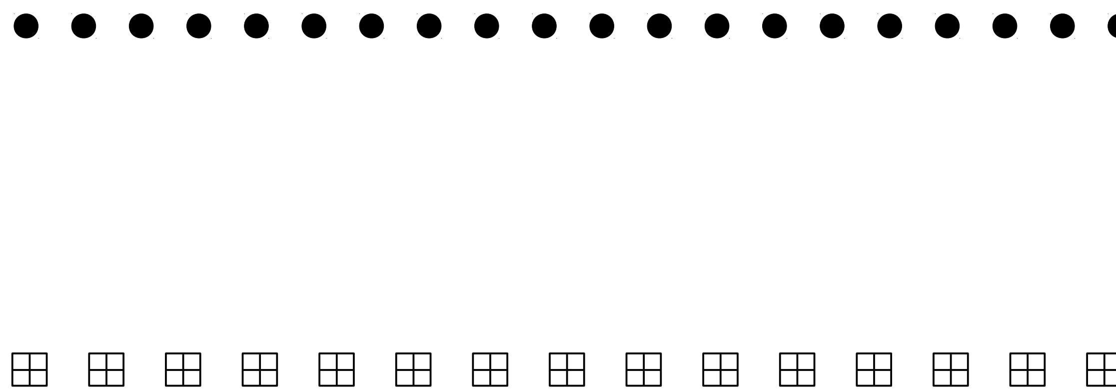
0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0\\
\hline
0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0\\
1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1\\
\hline
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1
\end{array}\right]_{15 \times 20}$$

$n = 20$ **variable nodes** of degree $J = 3$



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LDPC Block Codes

Representation by bipartite graph: [Tanner, '81]

$$H = \left[\begin{array}{cccccccccccccccccccc}
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0\\
\hline
0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0\\
1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1\\
\hline
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0\\
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15×20

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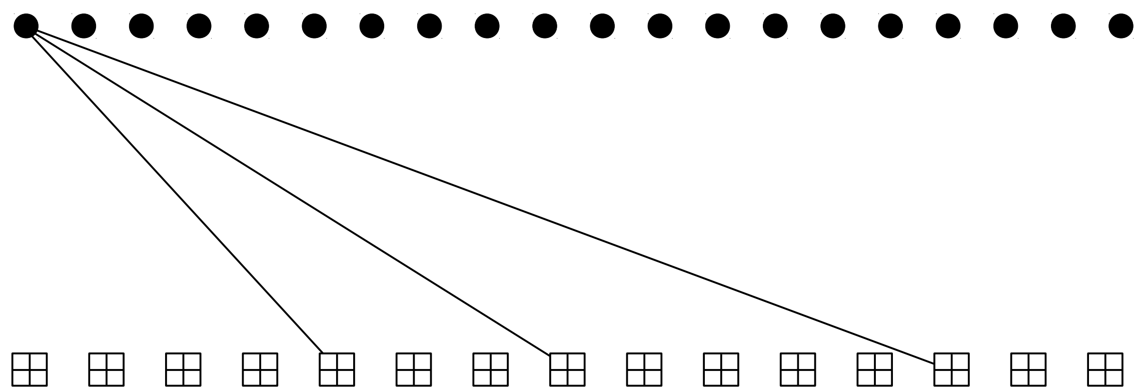

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\hline
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1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
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\hline
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
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1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1
\end{array}\right]_{15 \times 20}
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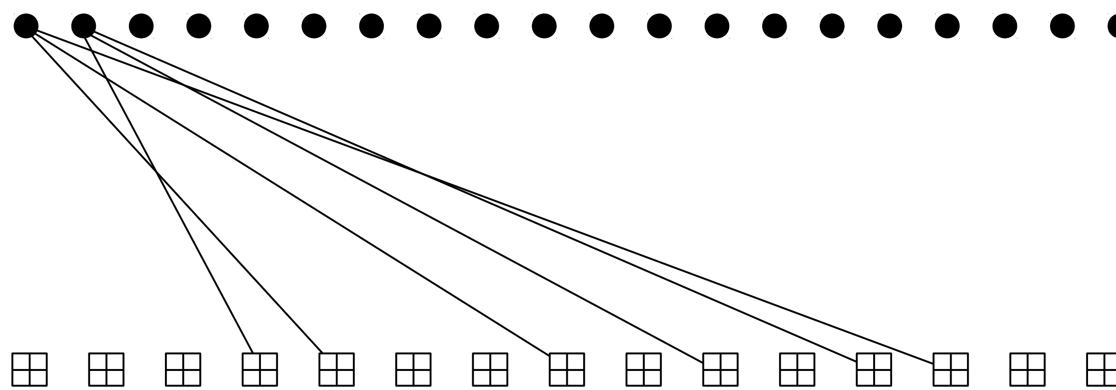

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0&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
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\hline
0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0\\
1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1\\
\hline
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0\\
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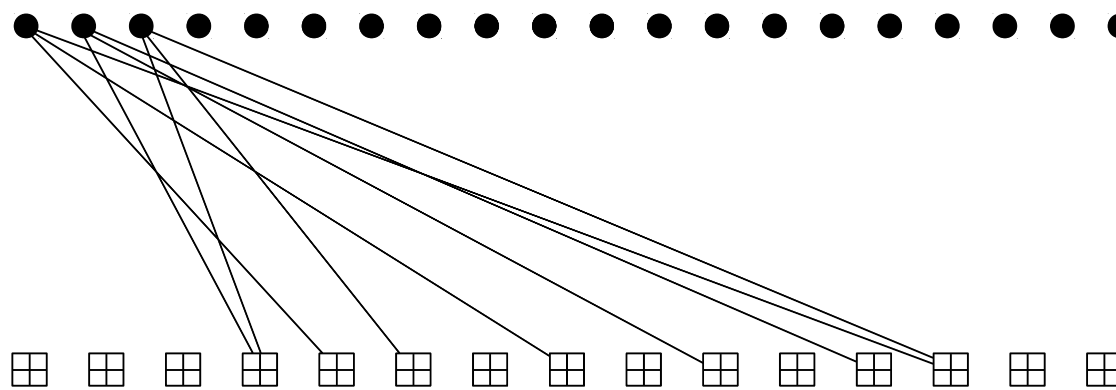

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0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0\\
\hline
0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0\\
1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1\\
\hline
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0\\
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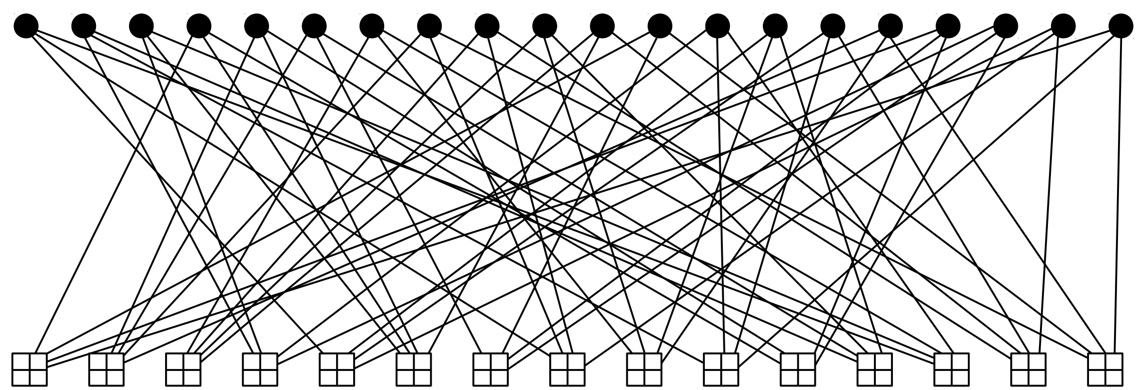

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Cycles in Tanner graphs

Definition: A **cycle** of length l in a Tanner graph is a path comprised of l edges from a node back to the same node.

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Cycles in Tanner graphs

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Definition: A **cycle** of length l in a Tanner graph is a path comprised of l edges from a node back to the same node.

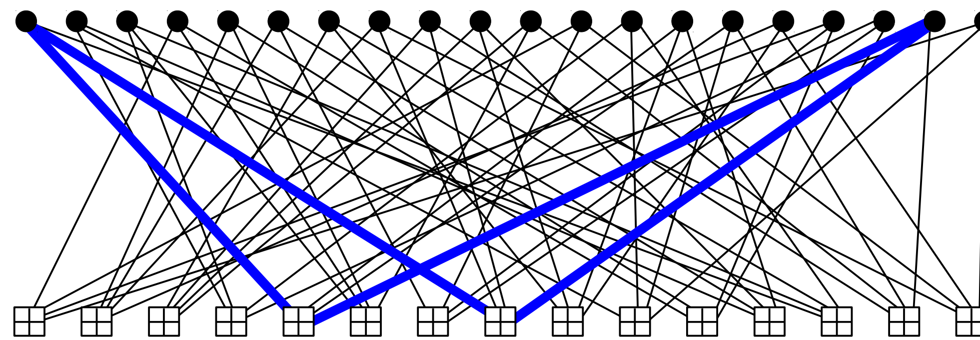
Definition: The **girth** of a Tanner graph is the minimum length of any cycle in the graph.

- The shortest possible cycle in any graph is 4 (indicated by a "rectangle" of four ones in $\boldsymbol{H}$).
- Short cycles are usually considered to be bad for message passing decoding (more later).

Cycles in Tanner graphs

- Tanner graphs typically contain **cycles**. The shortest cycle is called the **girth**.

$$H = \begin{bmatrix}
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0\\
\hline
0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0\\
1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1\\
\hline
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1
\end{bmatrix}_{15 \times 20}$$



cycle of length 4 (implies girth 4)

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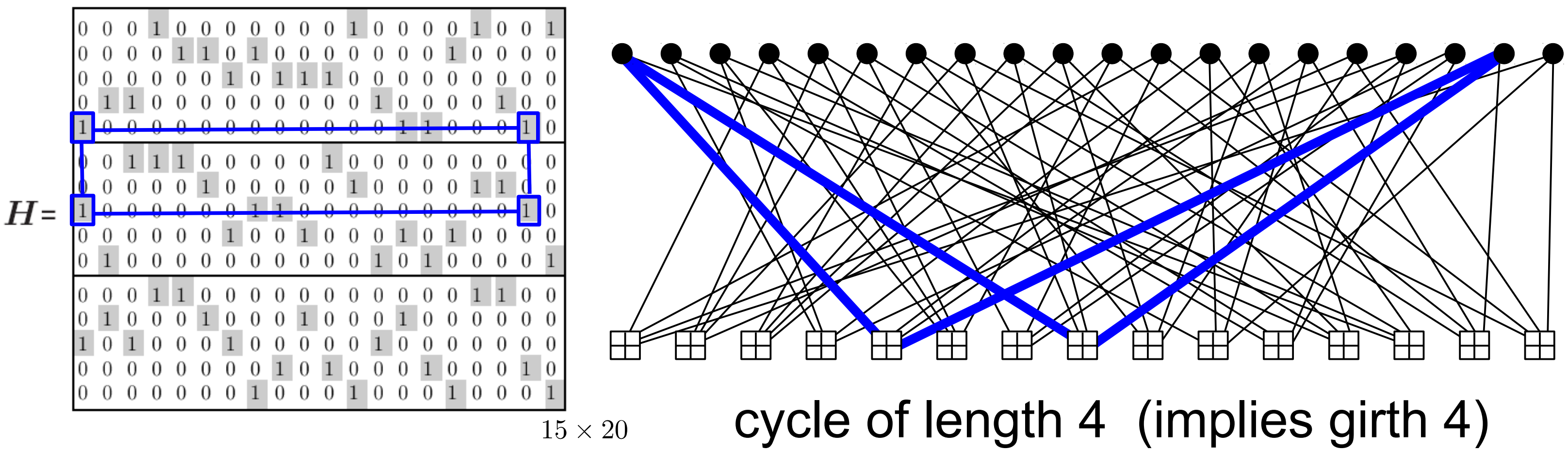

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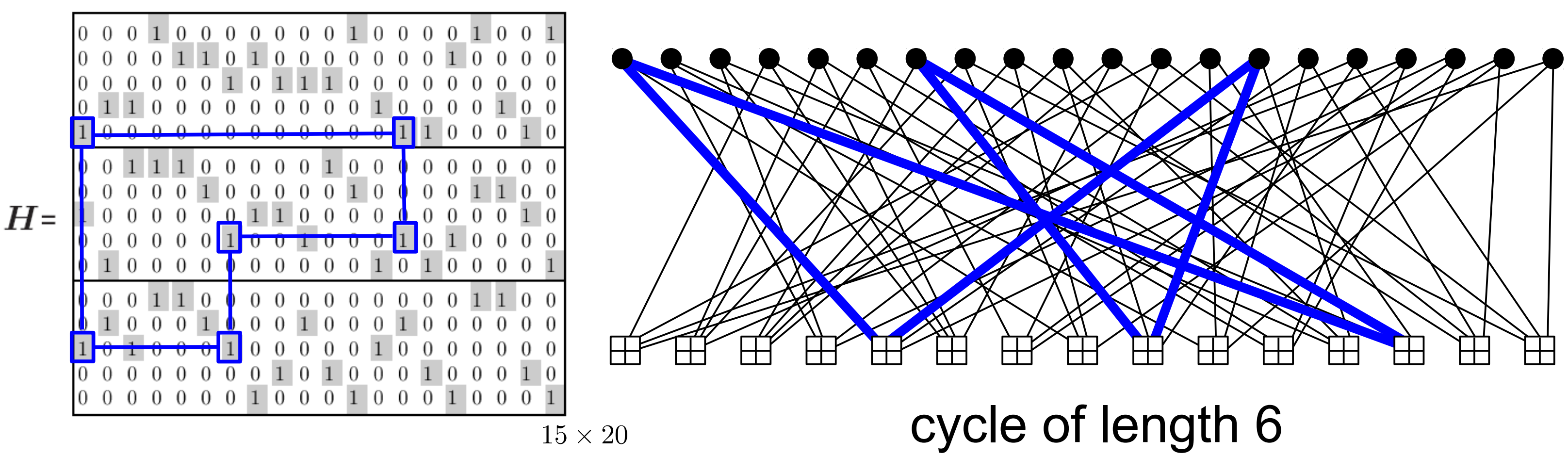
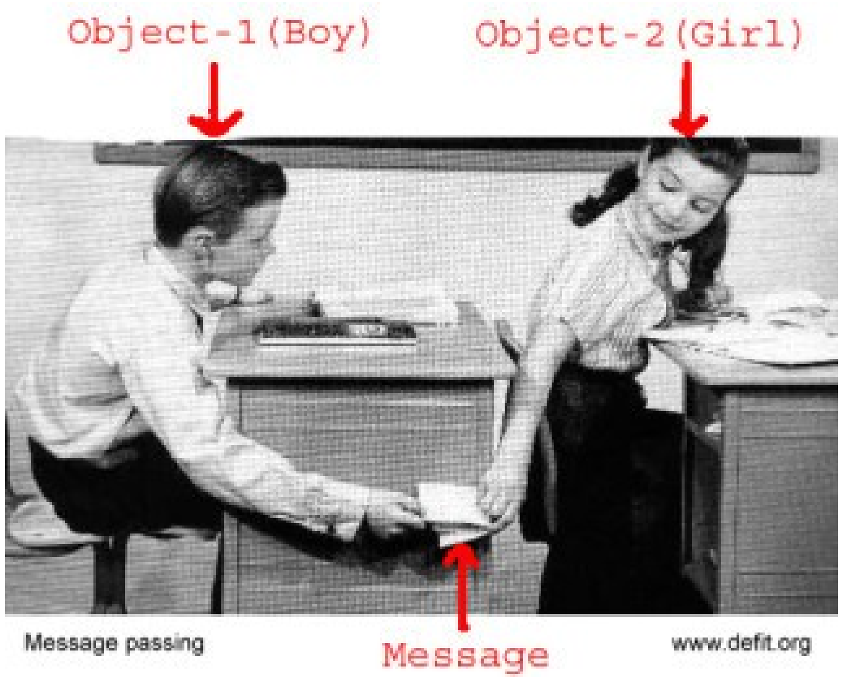


cycle of length 6

Message passing decoding

- The class of decoding algorithms used to decode LDPC codes are collectively called **message passing (MP)** algorithms since their operation can be explained by the iterative passing of messages between nodes in the Tanner graph.

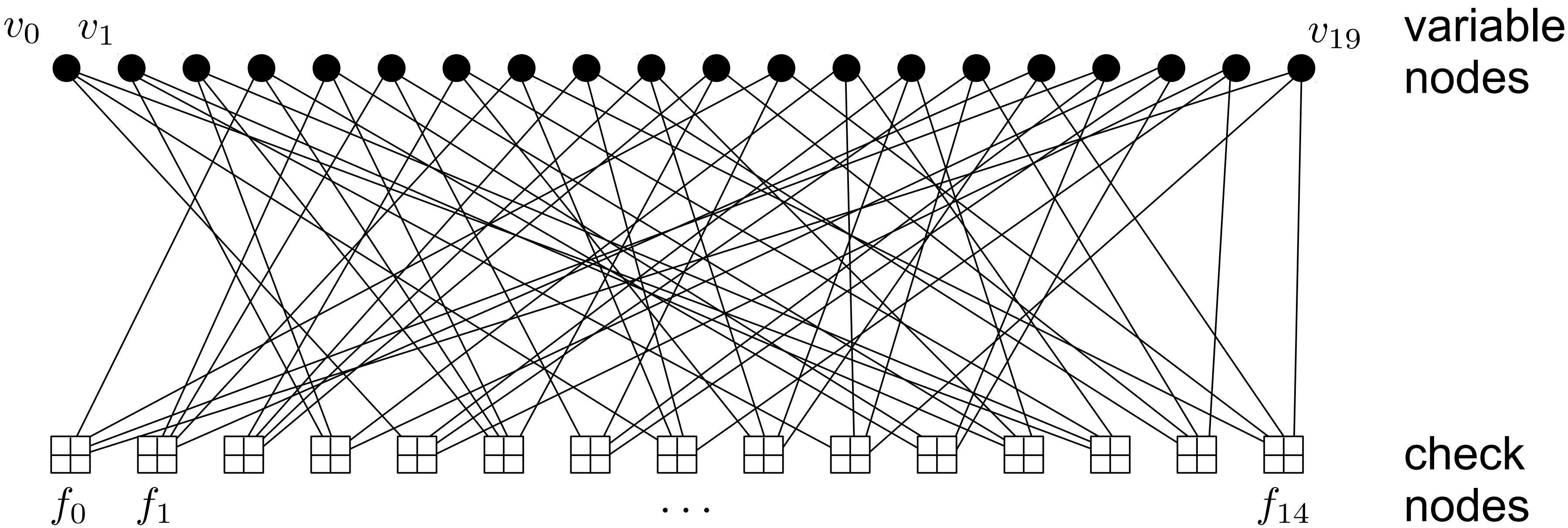


Message Passing Decoding

- Graph-based codes can be decoded **iteratively** with **low-complexity**

→ **Iterative decoding:** exchange of messages in Tanner graph

- We modify the graph as follows:



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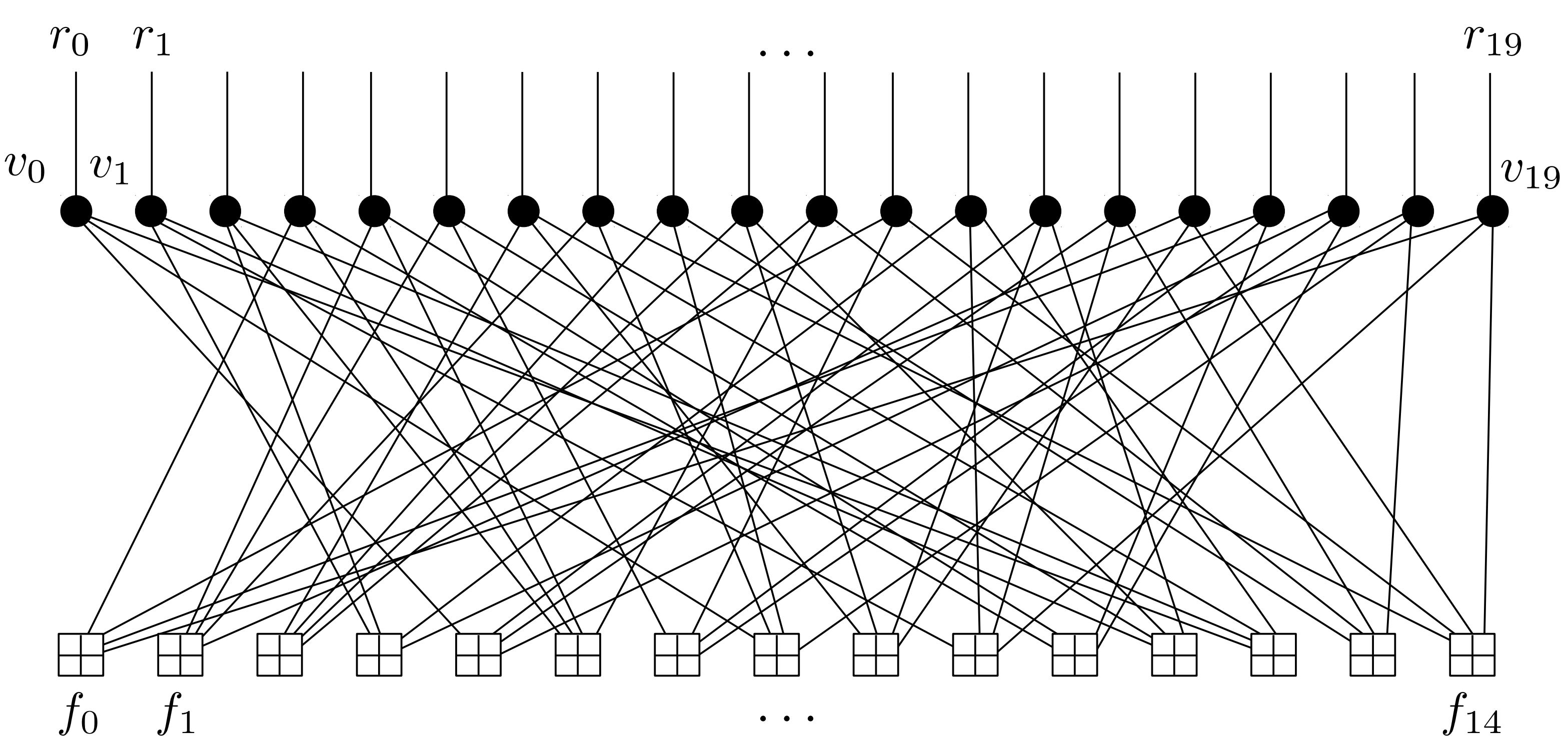

Belief Propagation (BP) decoding

- BP decoding is a **soft decision** algorithm, where the messages are **probabilities** which represent a level of belief about the value of the codeword bits

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- In the log domain, the algorithm is also referred to as the **sum product algorithm (SPA).**

Belief Propagation (BP) decoding

- The **log-likelihood ratio (LLR)** of a binary variable is

$$L(x) = \log\left(\frac{\mathbb{P}(x=0)}{\mathbb{P}(x=1)}\right) = \log(\mathbb{P}(x=0)) - \log(\mathbb{P}(x=1))$$

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➡ $\mathrm{sgn}(L(x))$ = hard decision

$|L(x)|$ determines the reliability

Belief Propagation (BP) decoding

Input: A priori probabilities on code bits, $\boldsymbol{H}$, I_{max}

Output: A posteriori probabilities (APPs) on code bits

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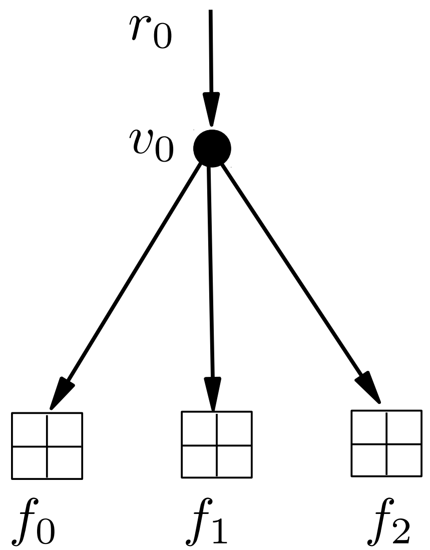
(4) We make hard decisions and terminate if $vH^T = 0$ or if I_{max}, the maximum number of iterations, is reached. Otherwise go to (1).

Belief Propagation (BP) decoding



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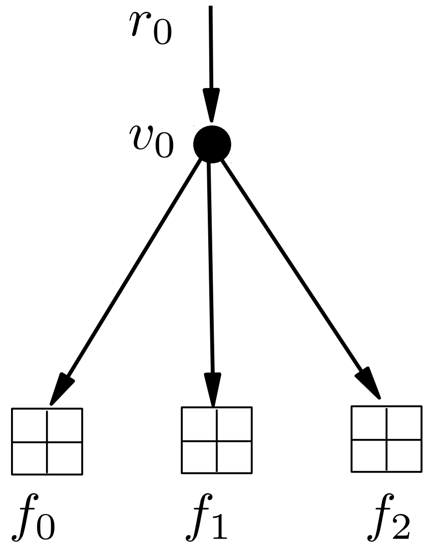

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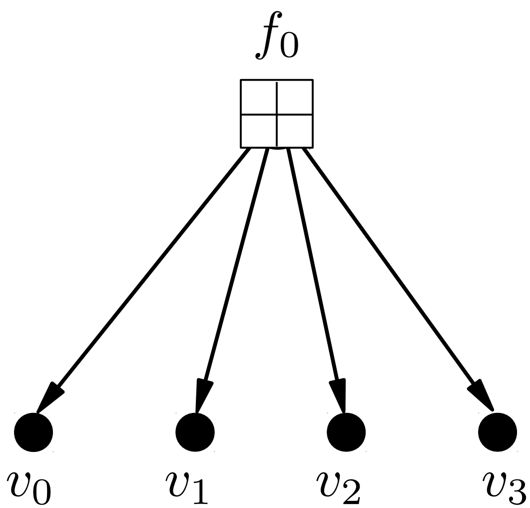

$q_{ji} =$ message passed from check node f_j to variable node v_i

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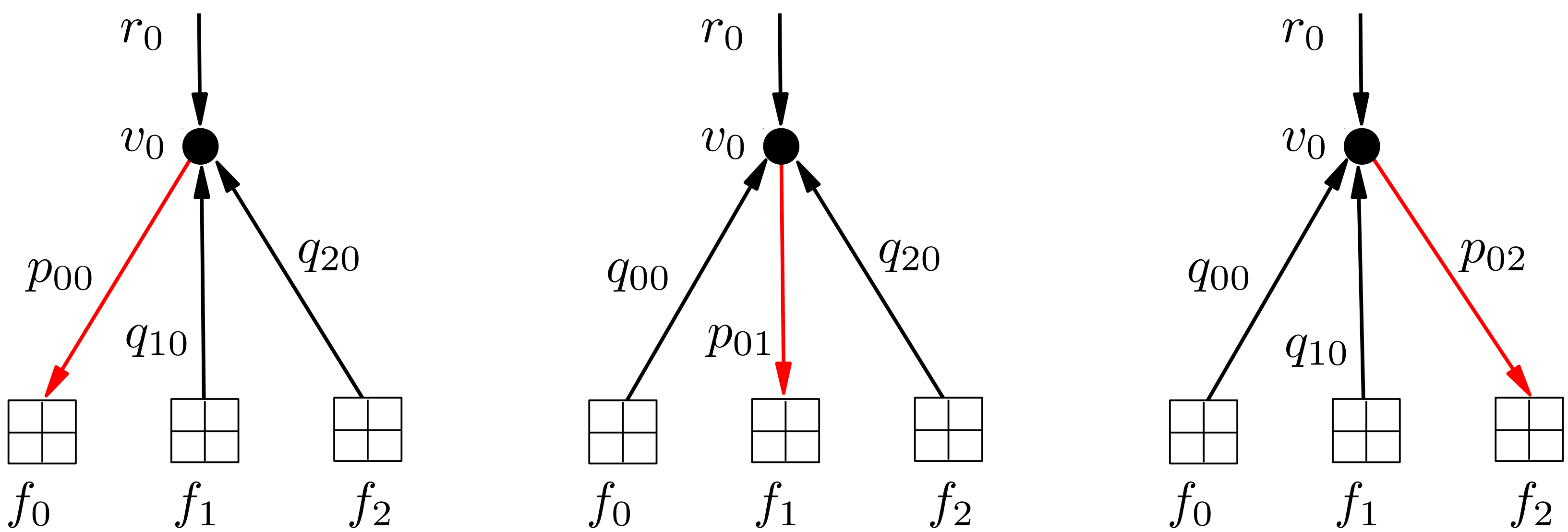
Message dependencies

- The message dependencies of the information flow into and out of a variable node look like this:



Message dependencies

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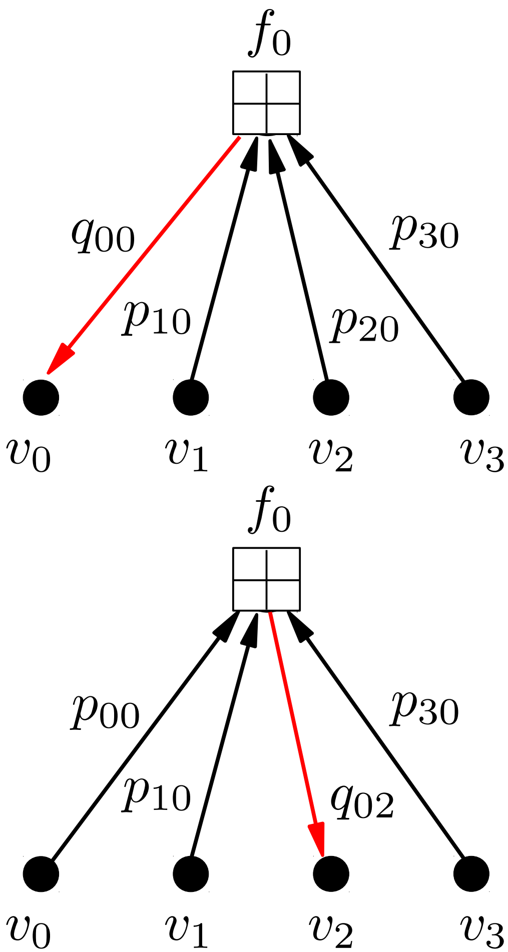

Is this globally optimal?

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Problem: The extrinsic information from a parity-check is **independent of the a priori probability** in the first iteration only **until it is returned back via a cycle** in the graph.

➡ Cycles cause a correlation in extrinsic information.

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➡ Cycles cause a correlation in extrinsic information.

BP decoding is a MAP algorithm if no cycles exist in the graph; otherwise it is sub-optimal!

Message passing on the BEC

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Recall: On the binary erasure channel (BEC), a bit is either received correctly or erased with probability ε

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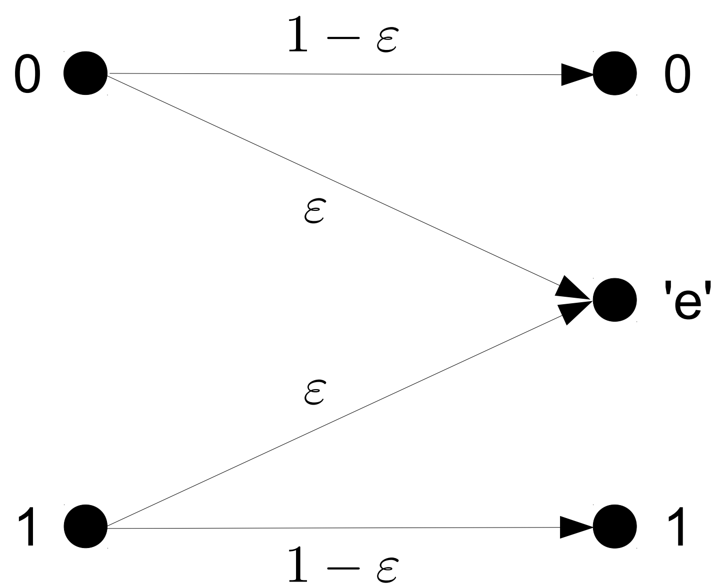
➡ The BEC is **much simpler** to analyze than the BSC or AWGN channels.



Belief Propagation - BEC

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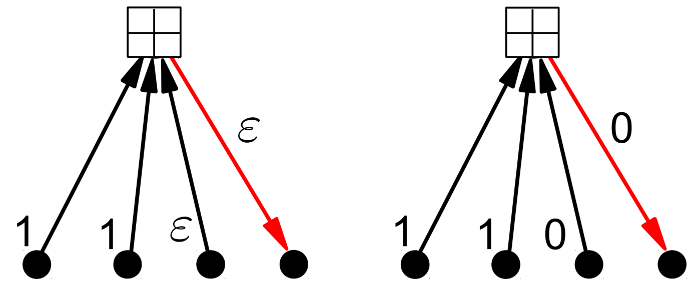

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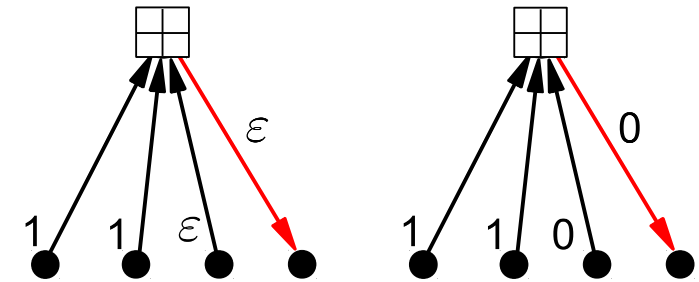
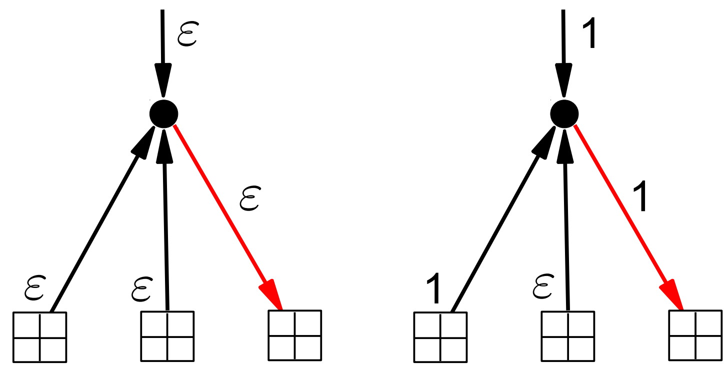


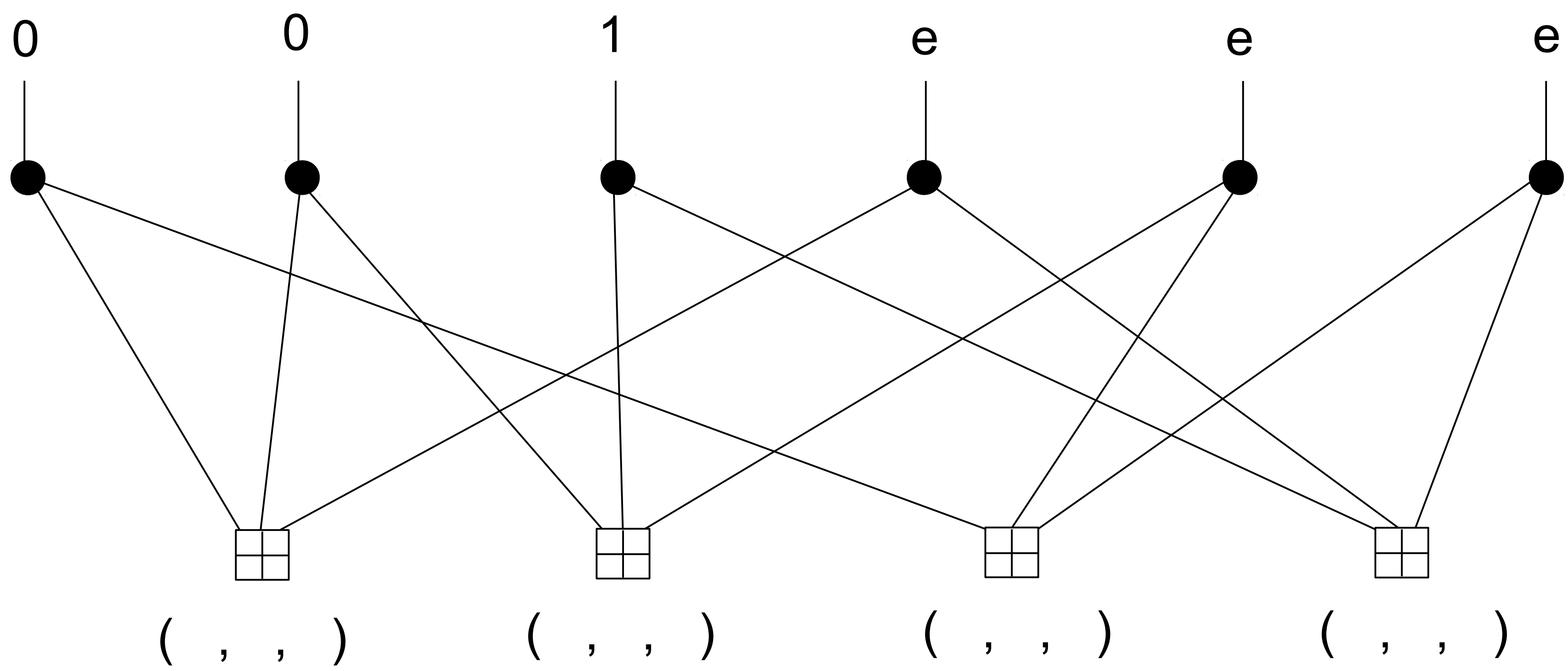
➡ At a **variable node**, the outgoing message is an erasure if **all** of the incoming messages are erasures; otherwise the message is the value of all non-erasures, which must agree.



Iterative Decoding Example (BEC)

Ex: $\mathbf{v} = [\,0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\,] \rightarrow \mathbf{r} = [\,0\ \ 0\ \ 1\ \ e\ \ e\ \ e\,]$

received (corrupted) codeword



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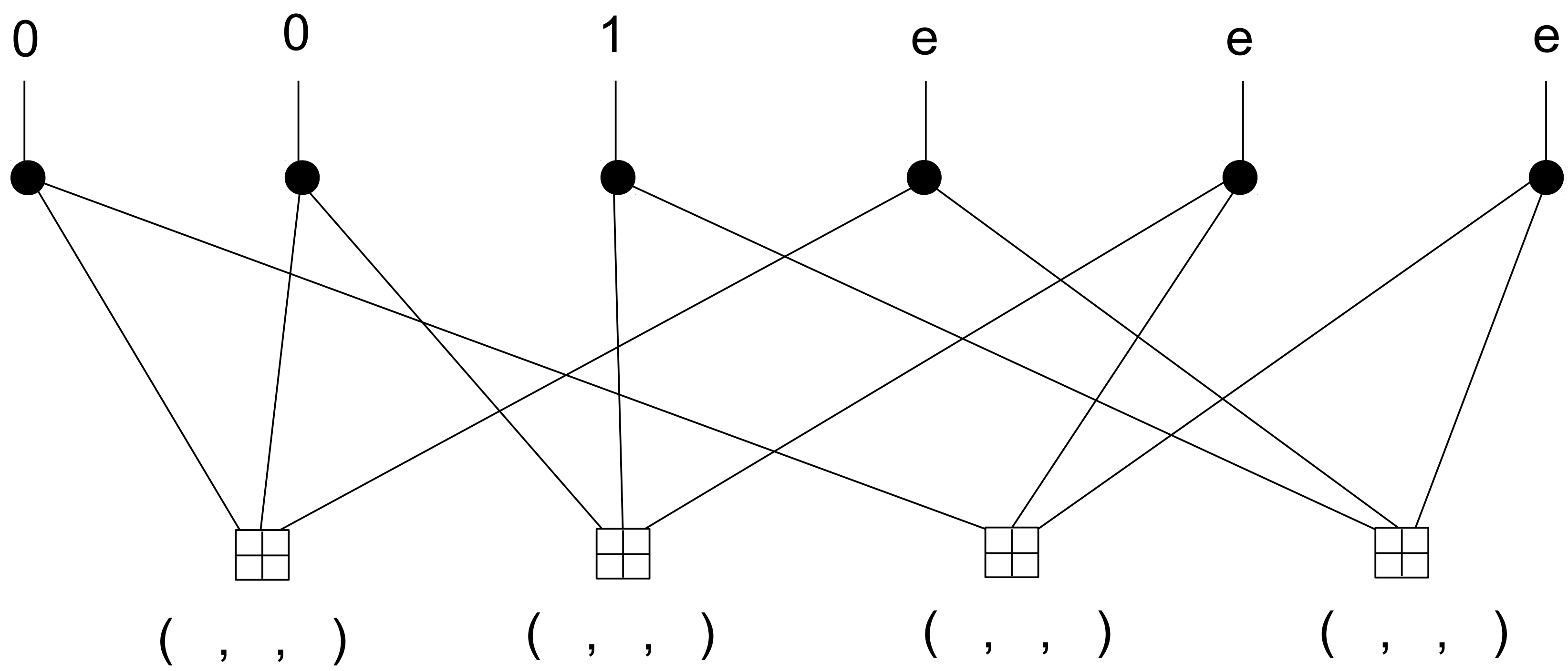

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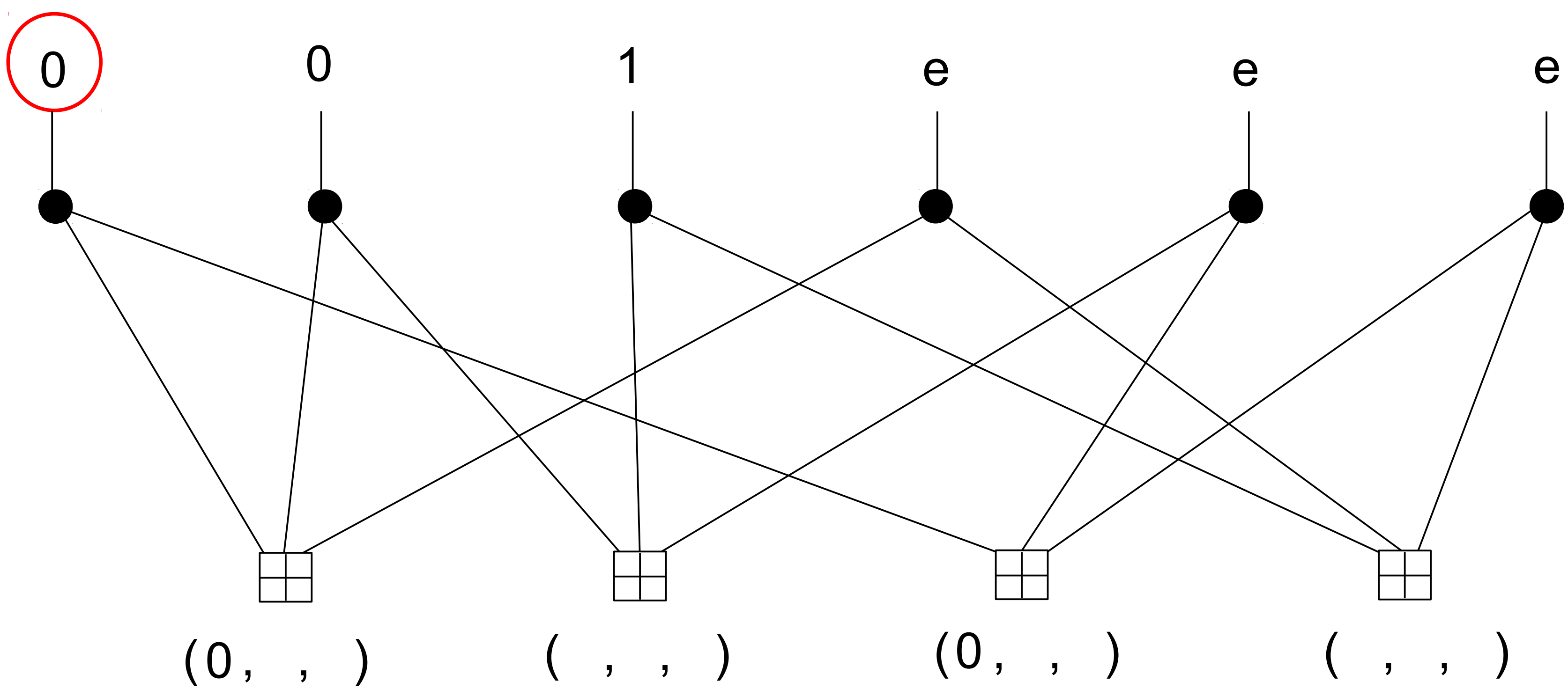

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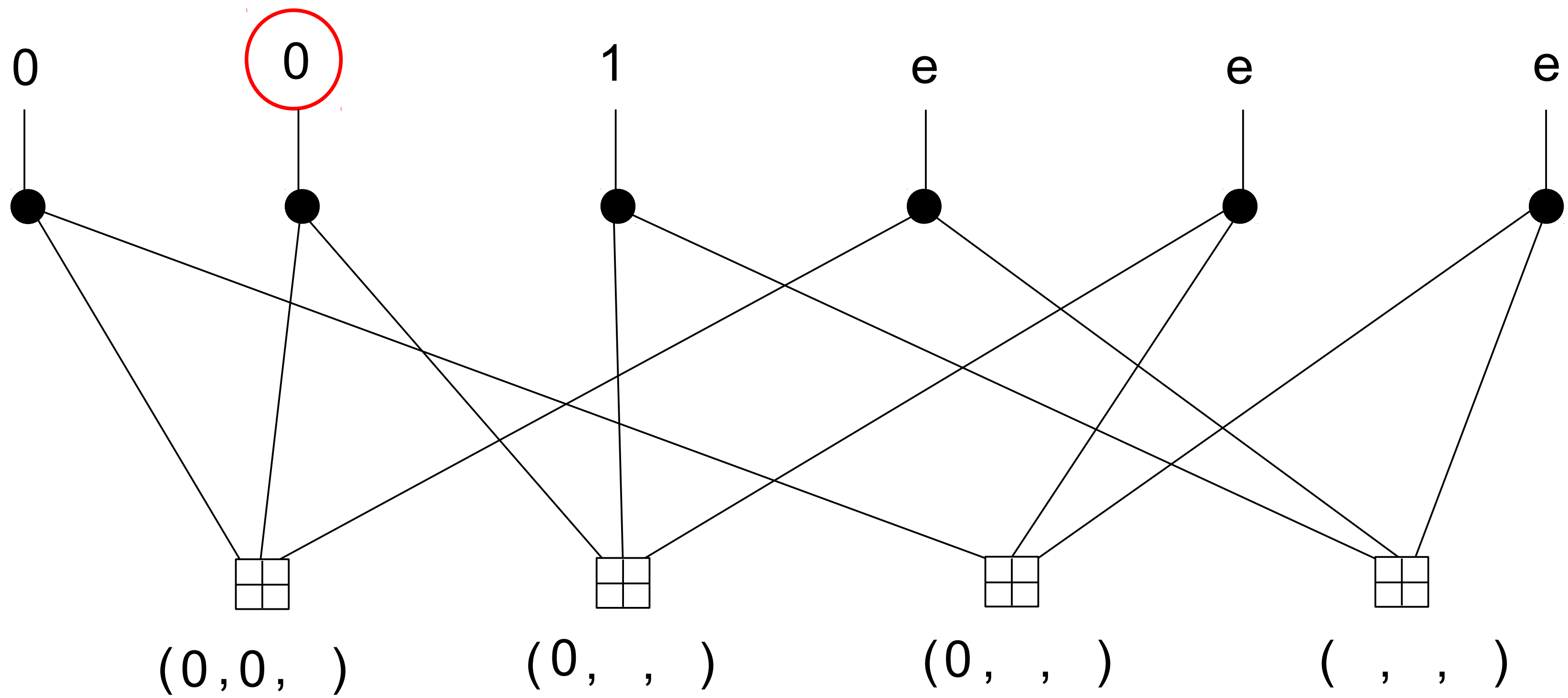

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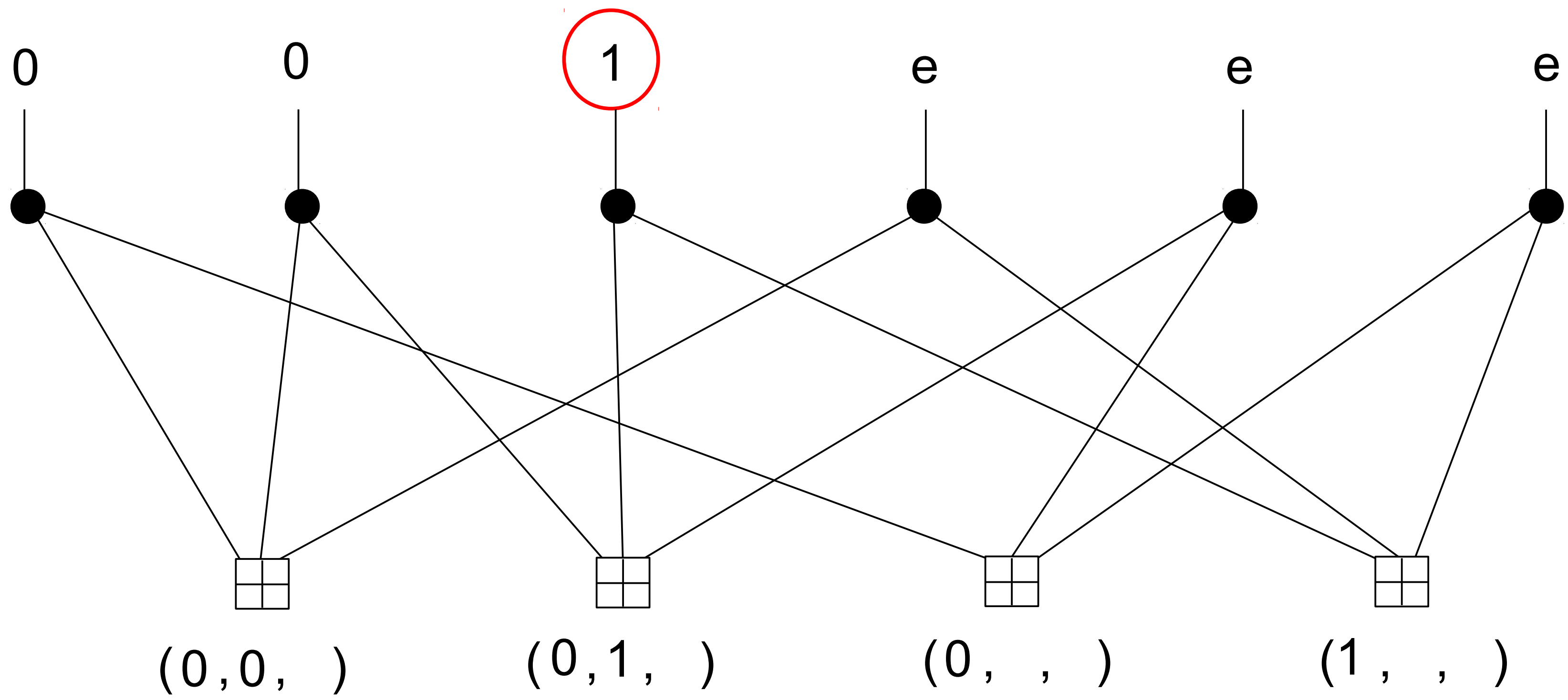

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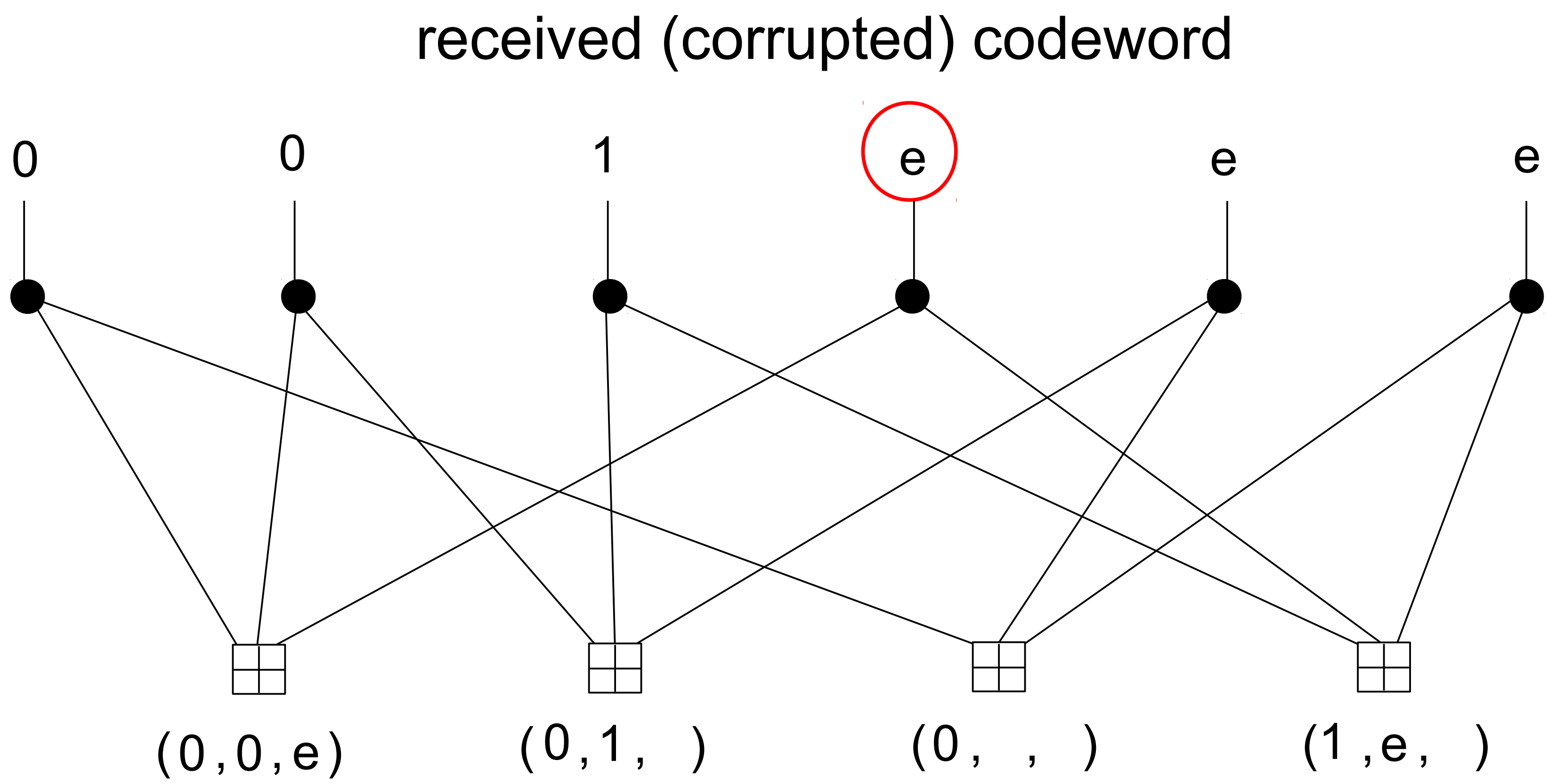

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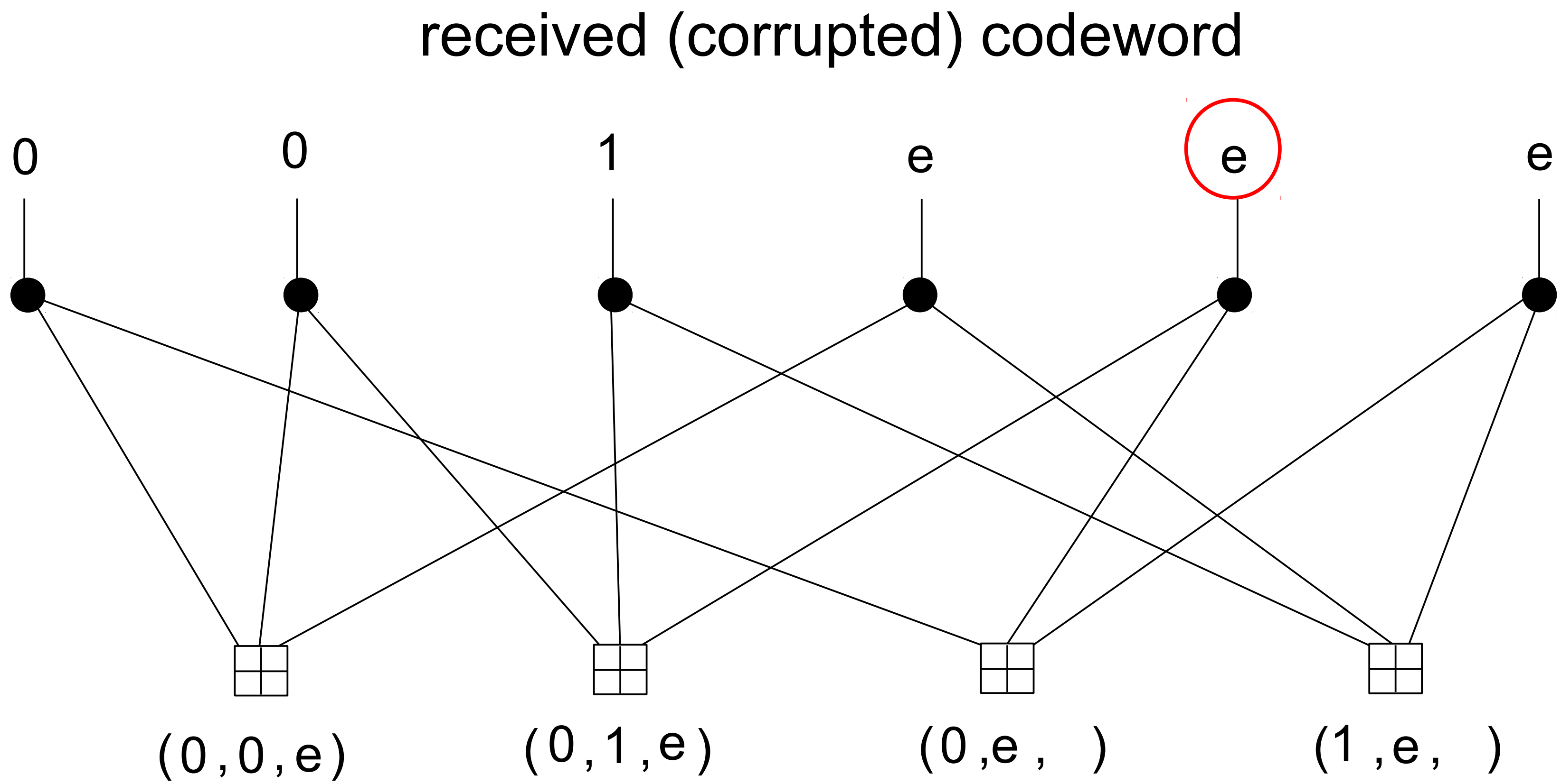

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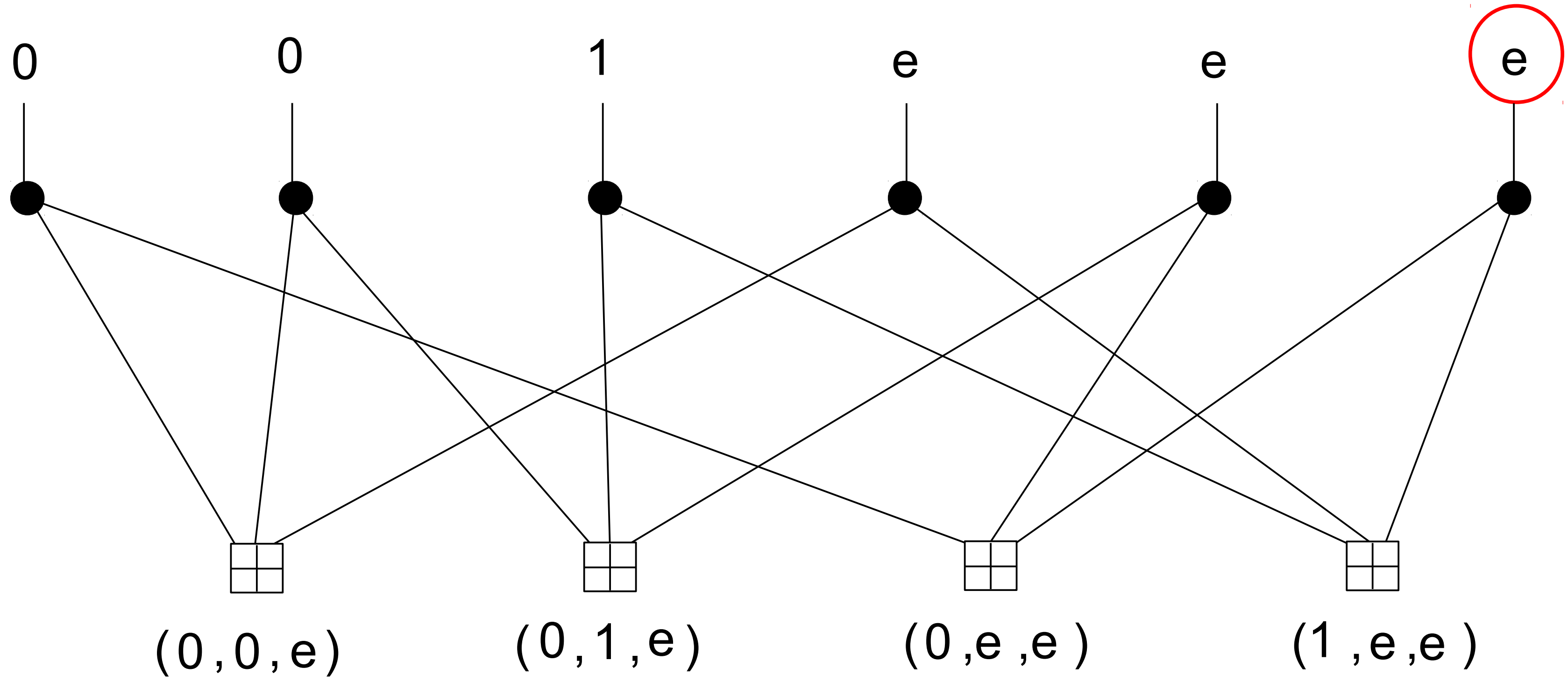

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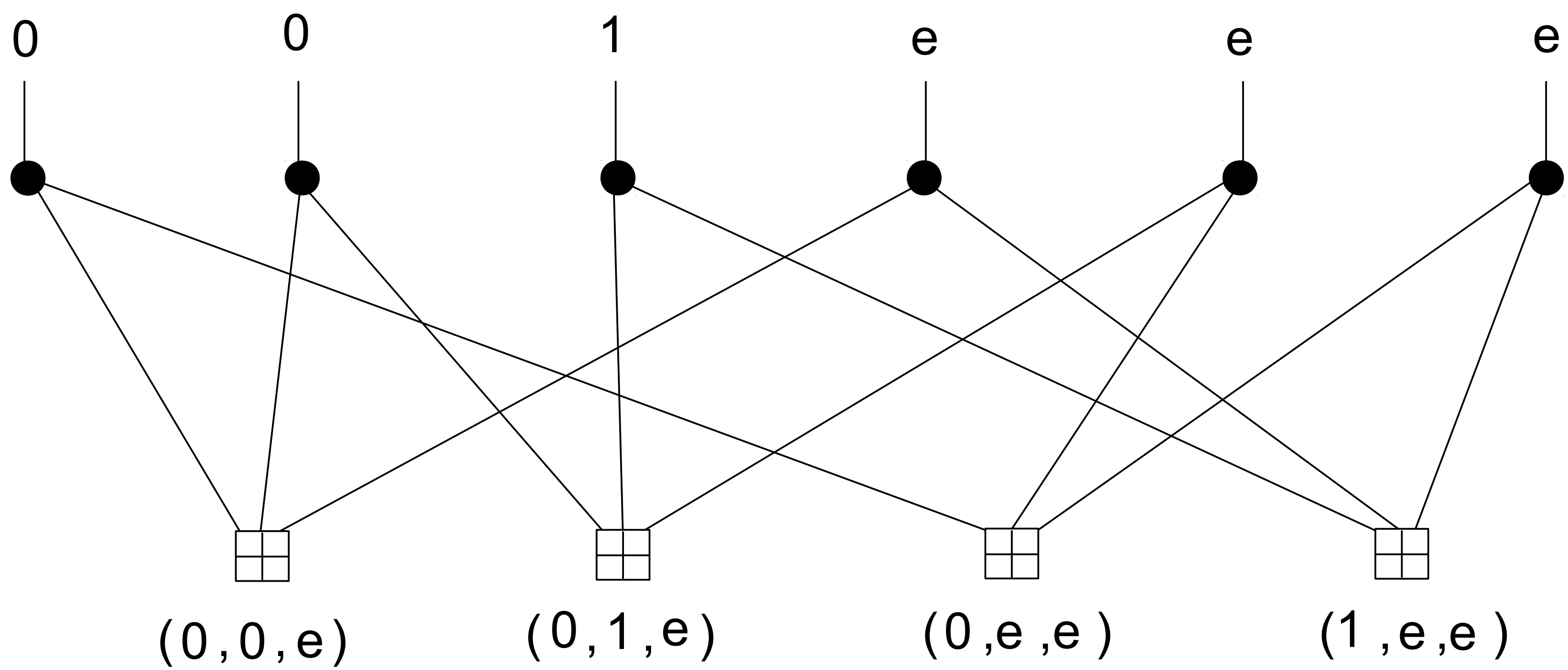

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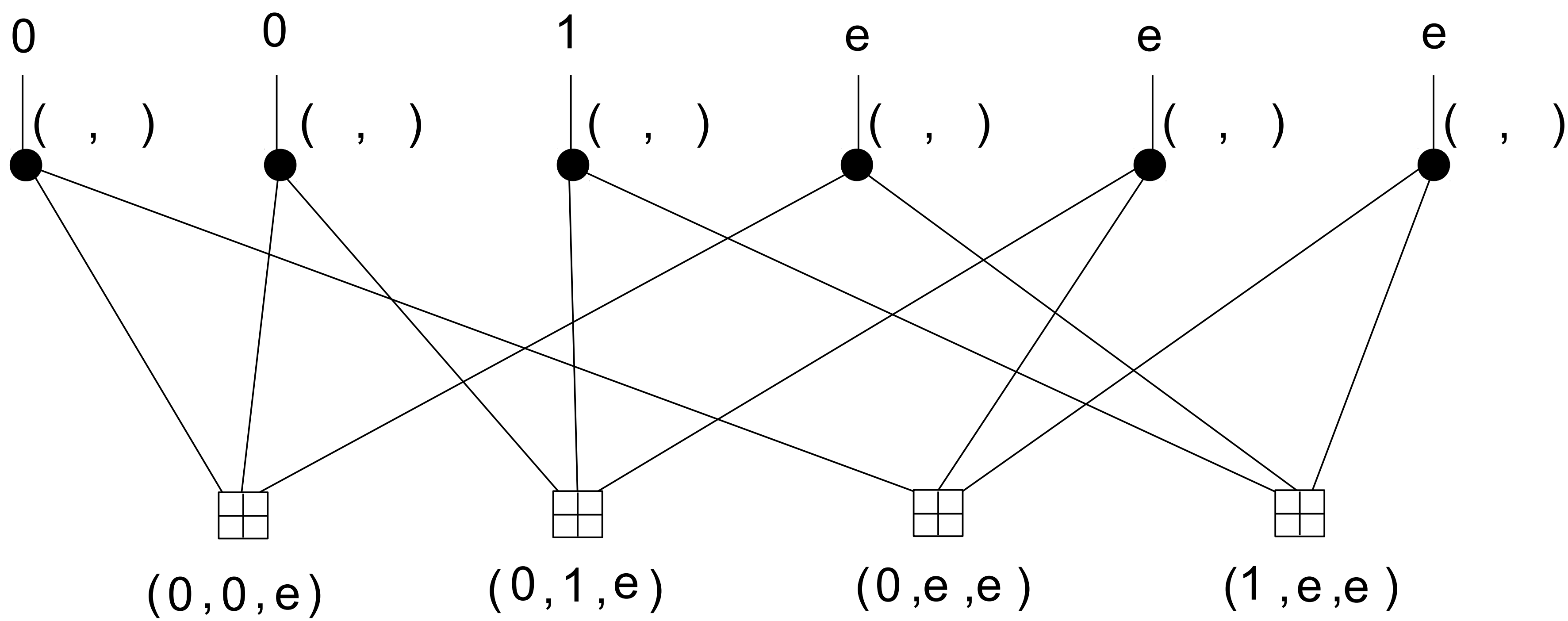

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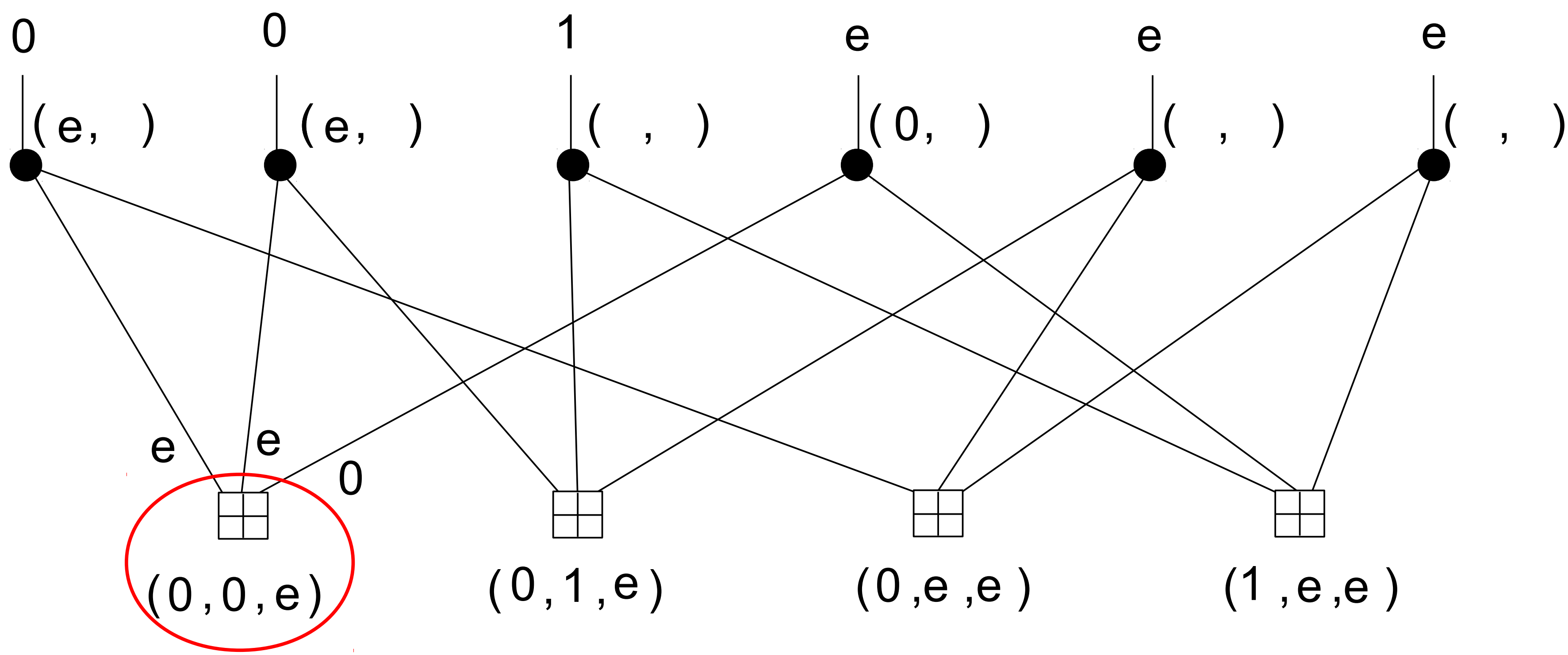

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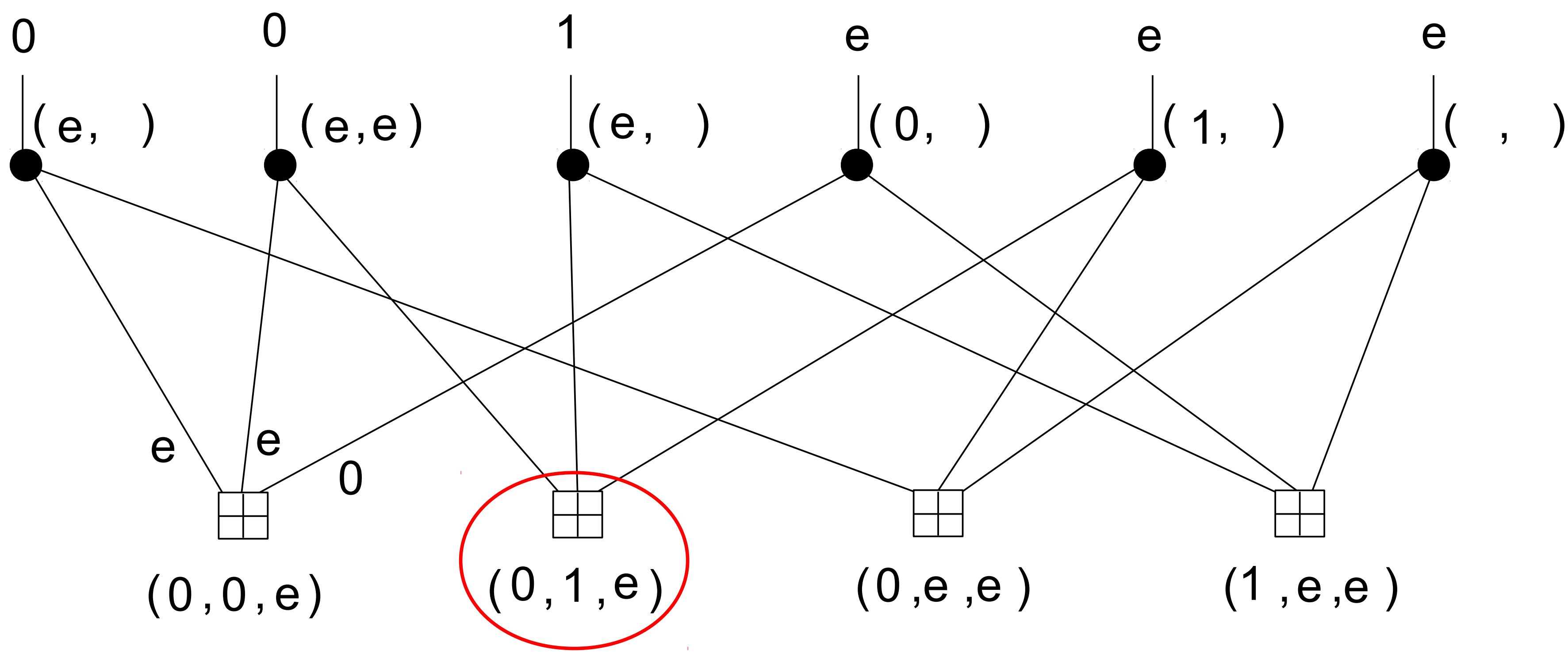

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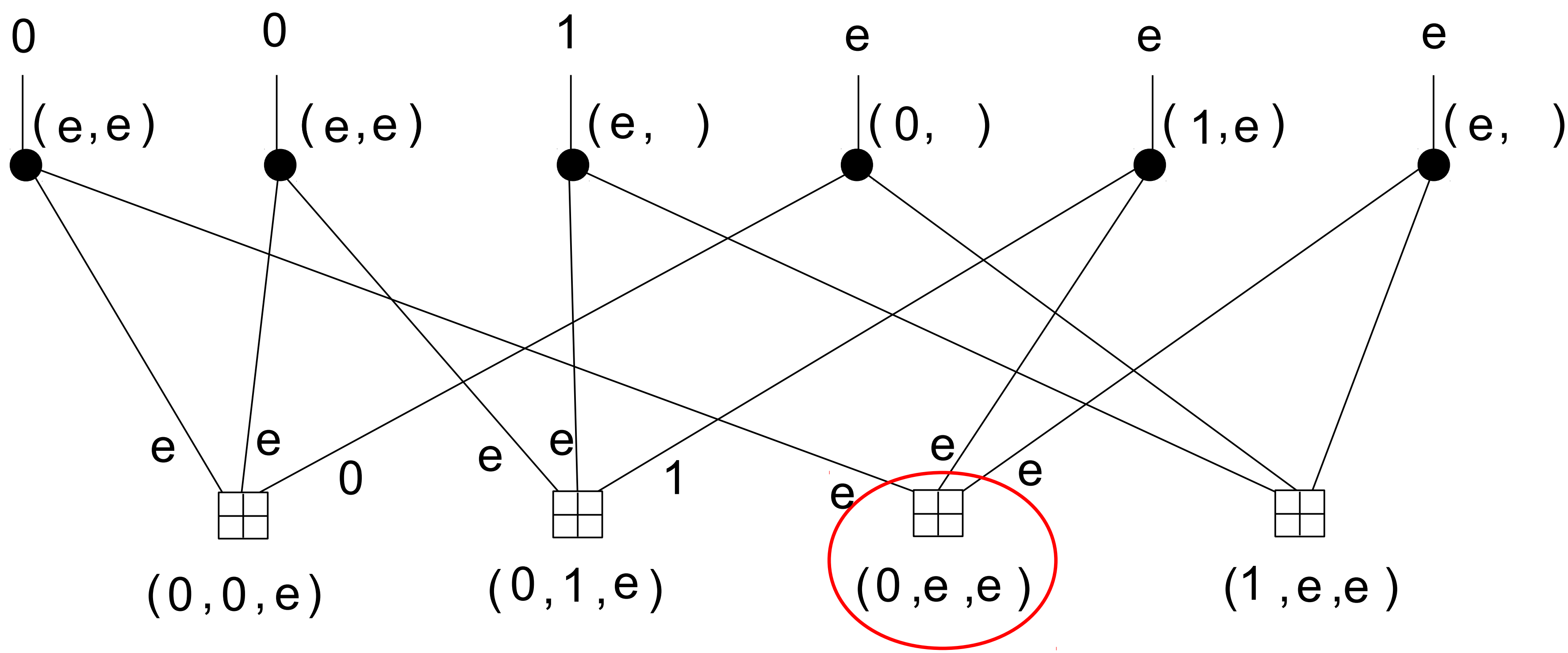

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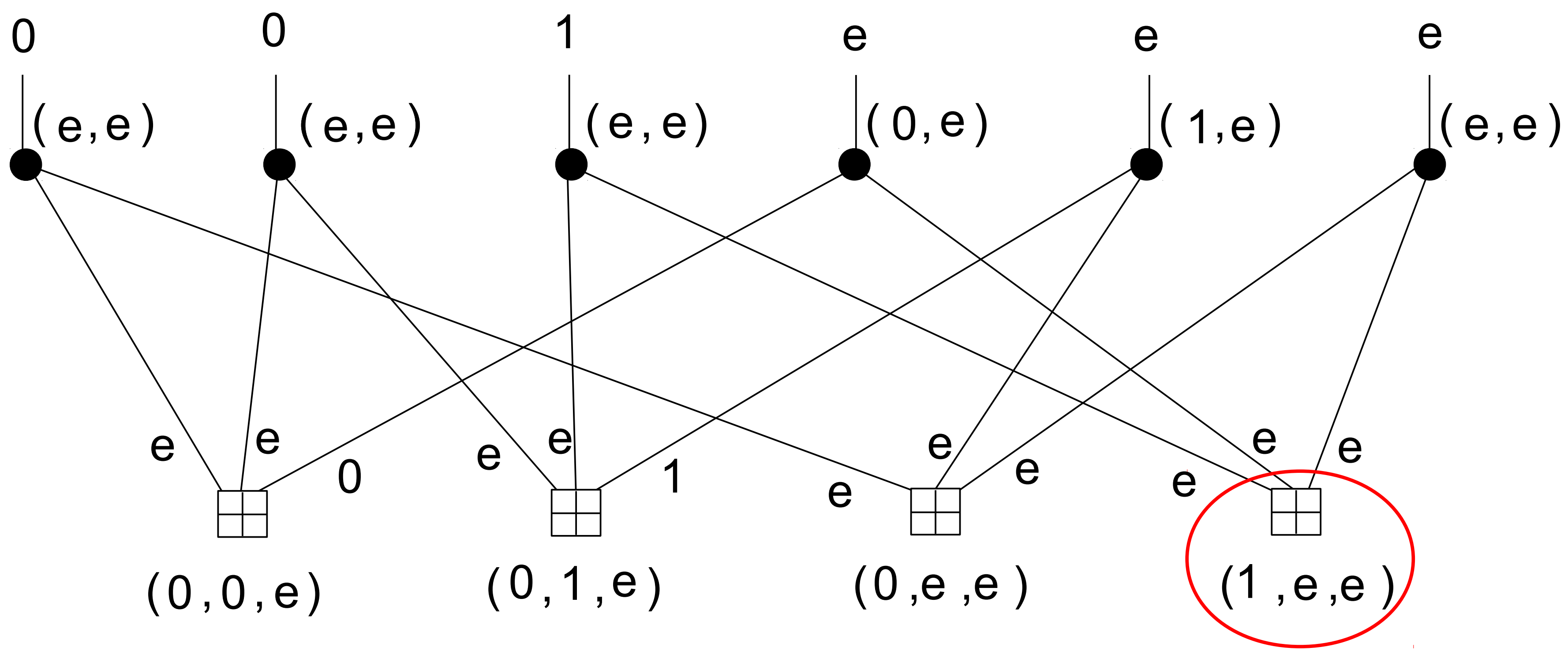

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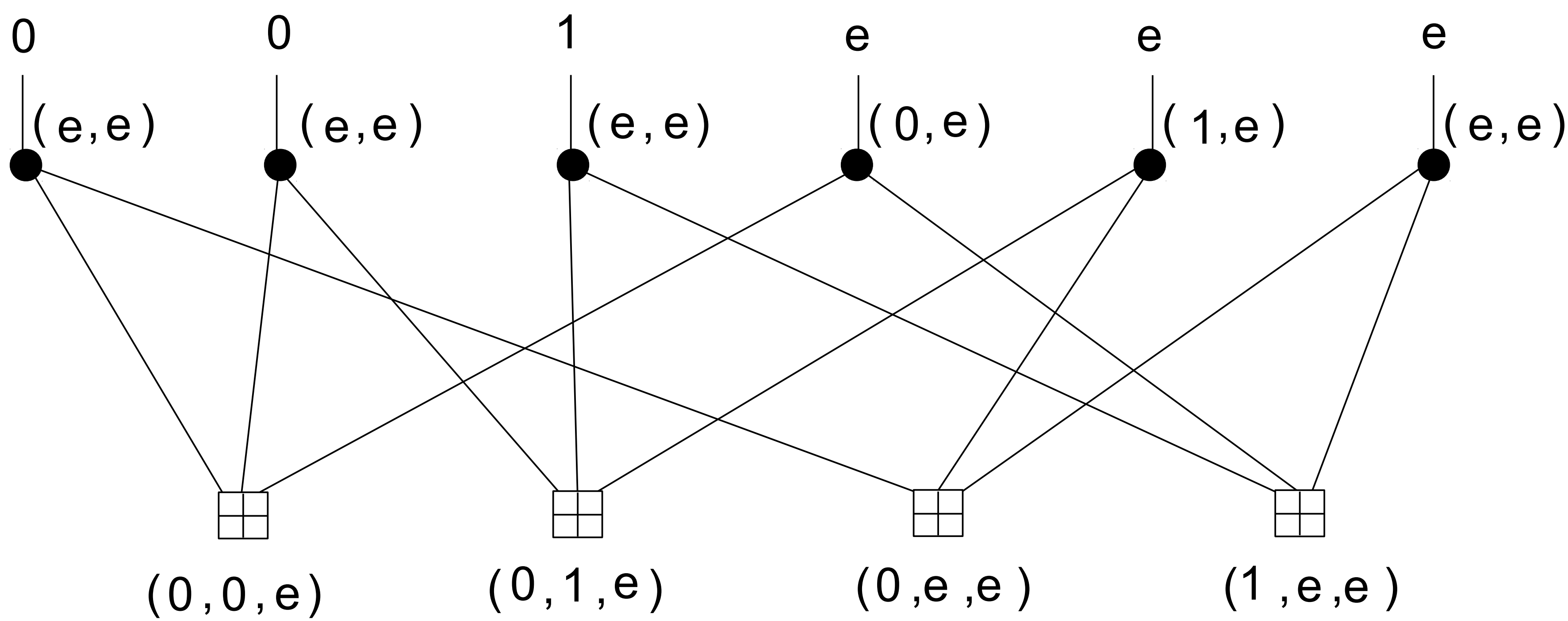

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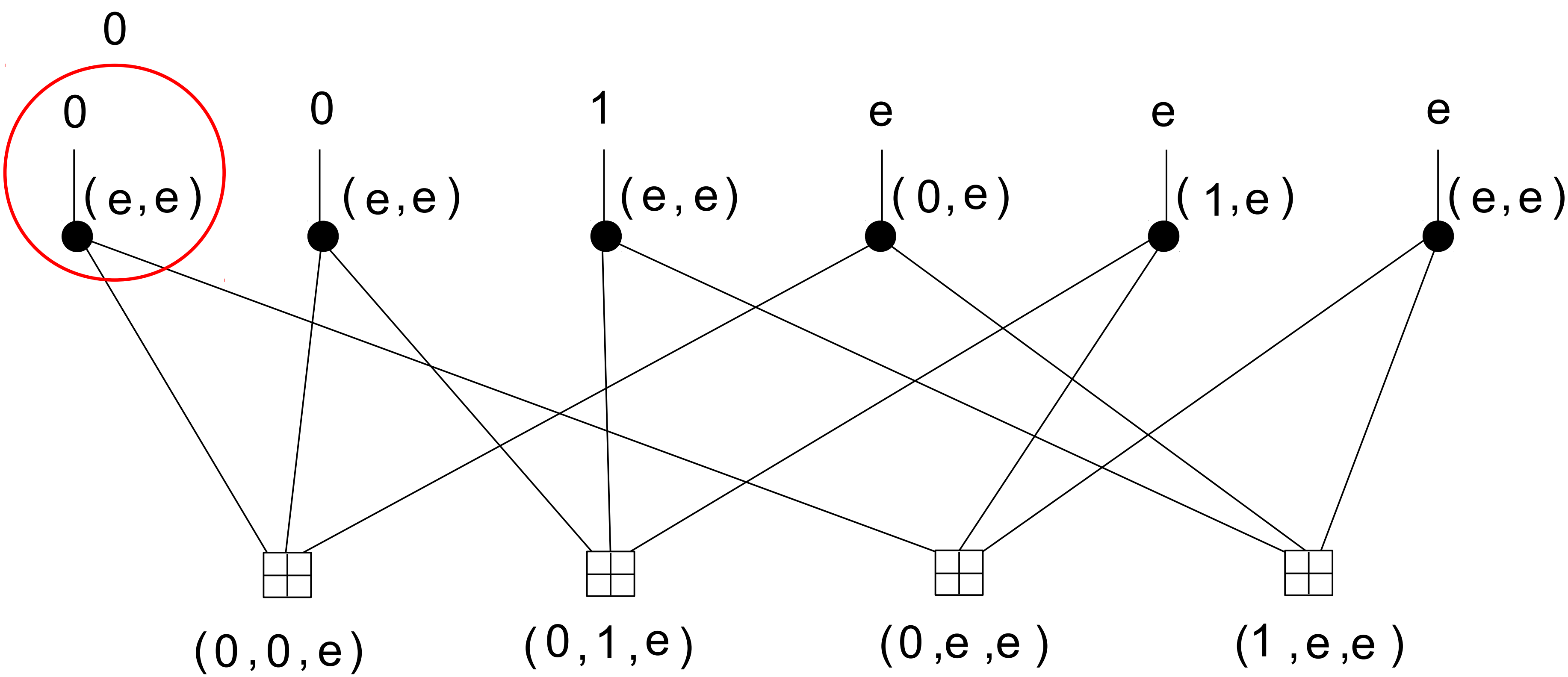

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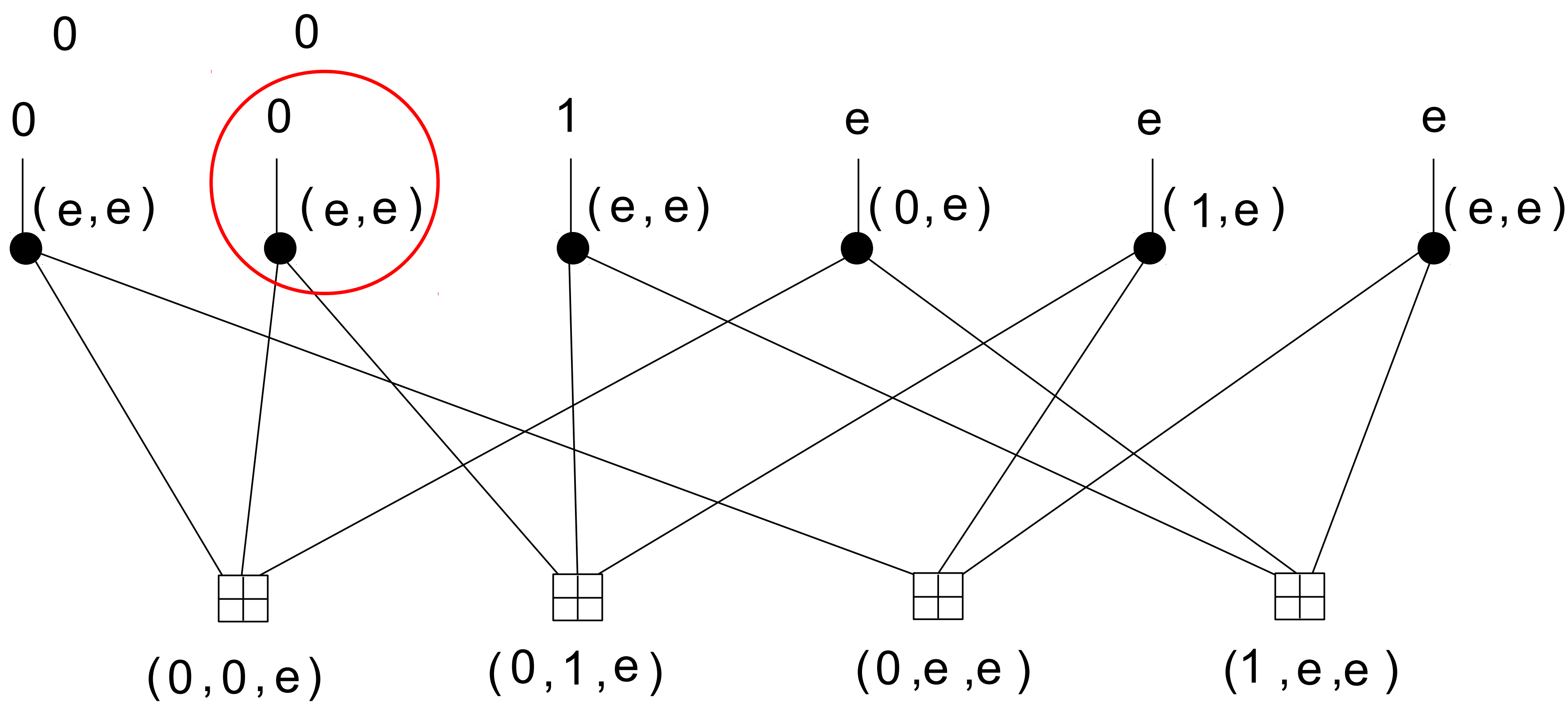

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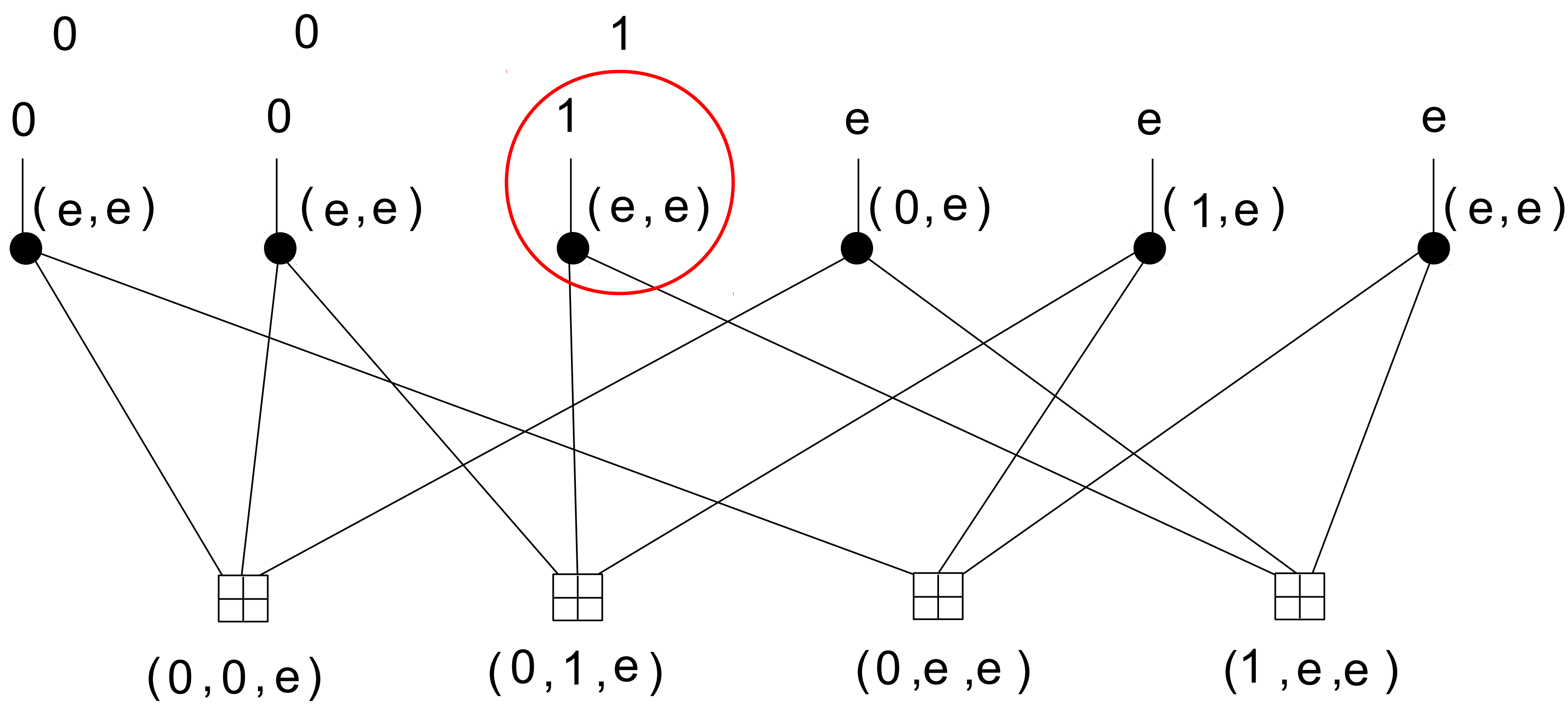

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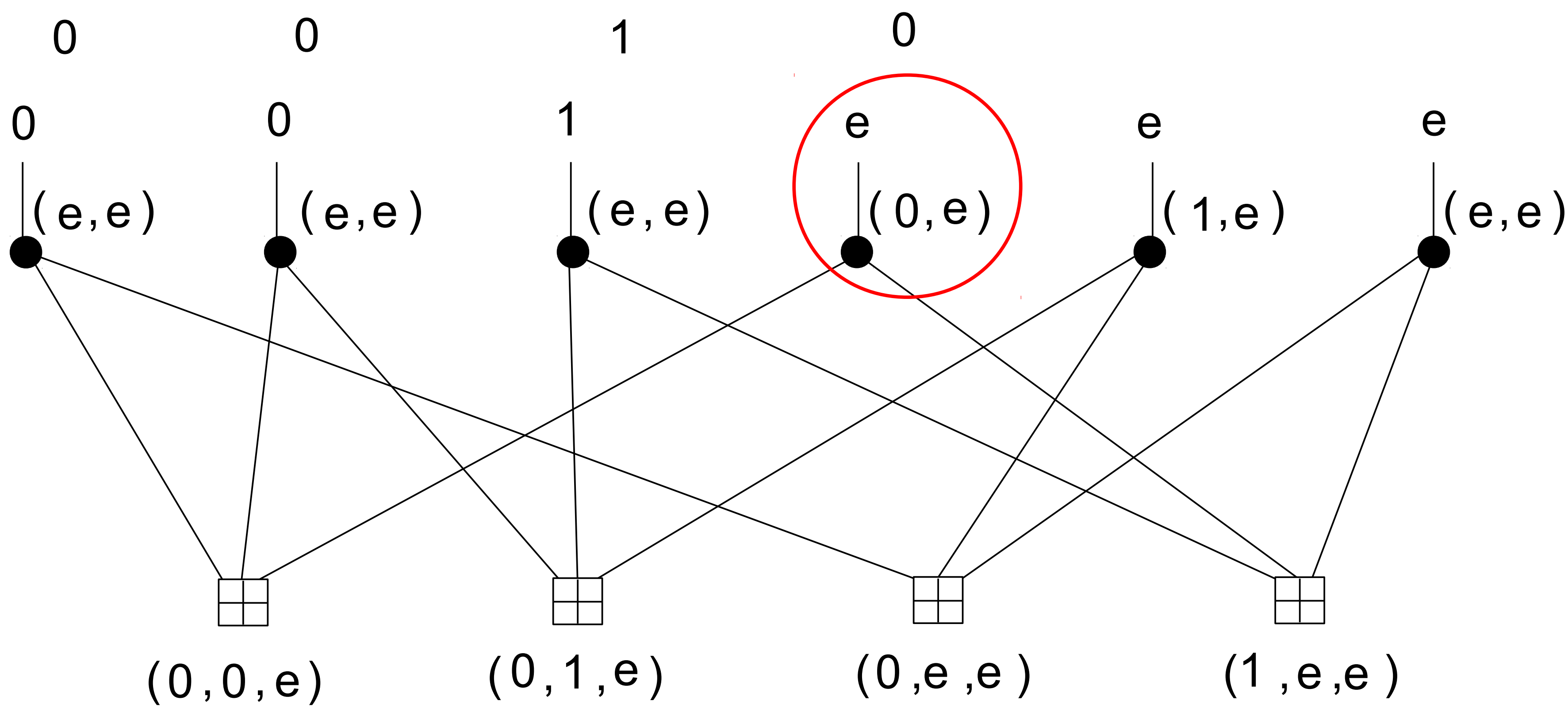

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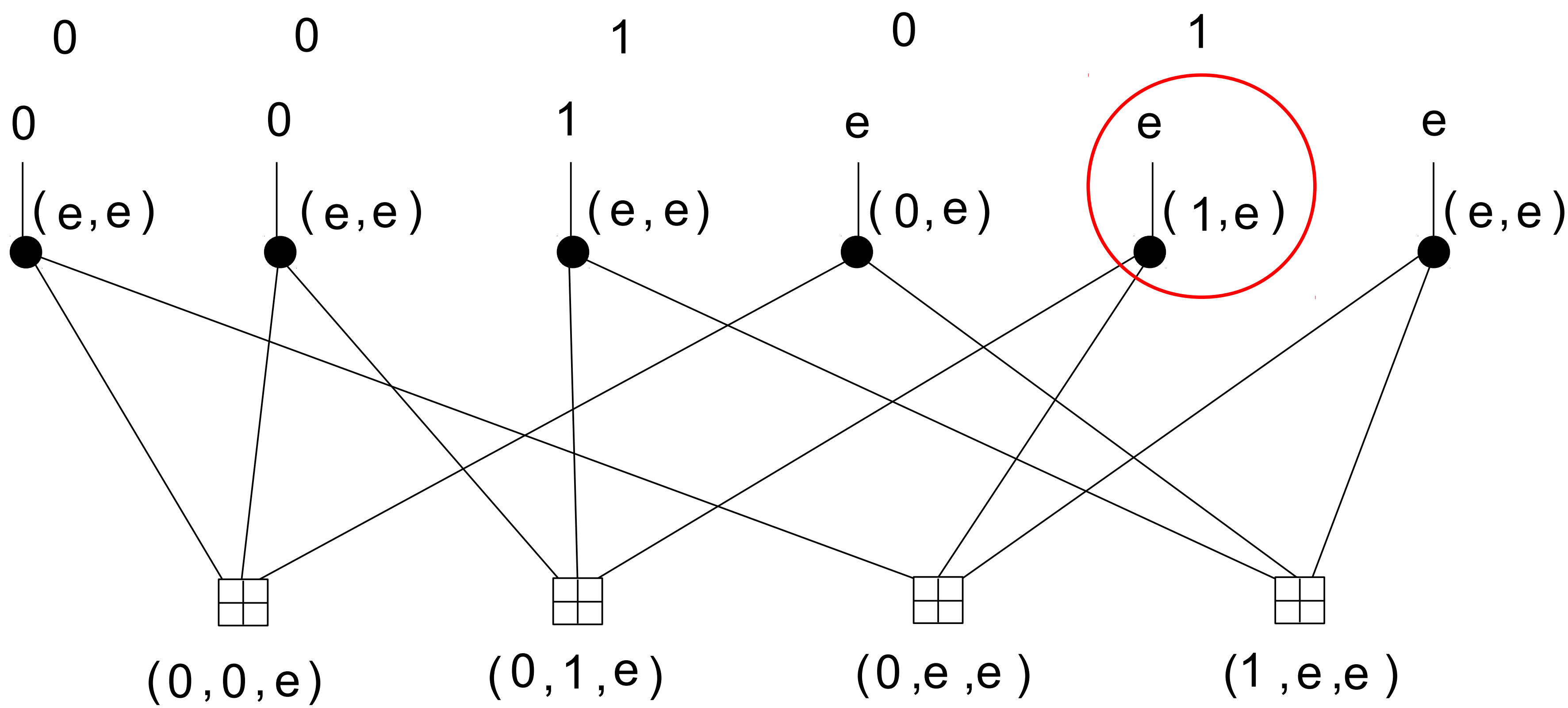

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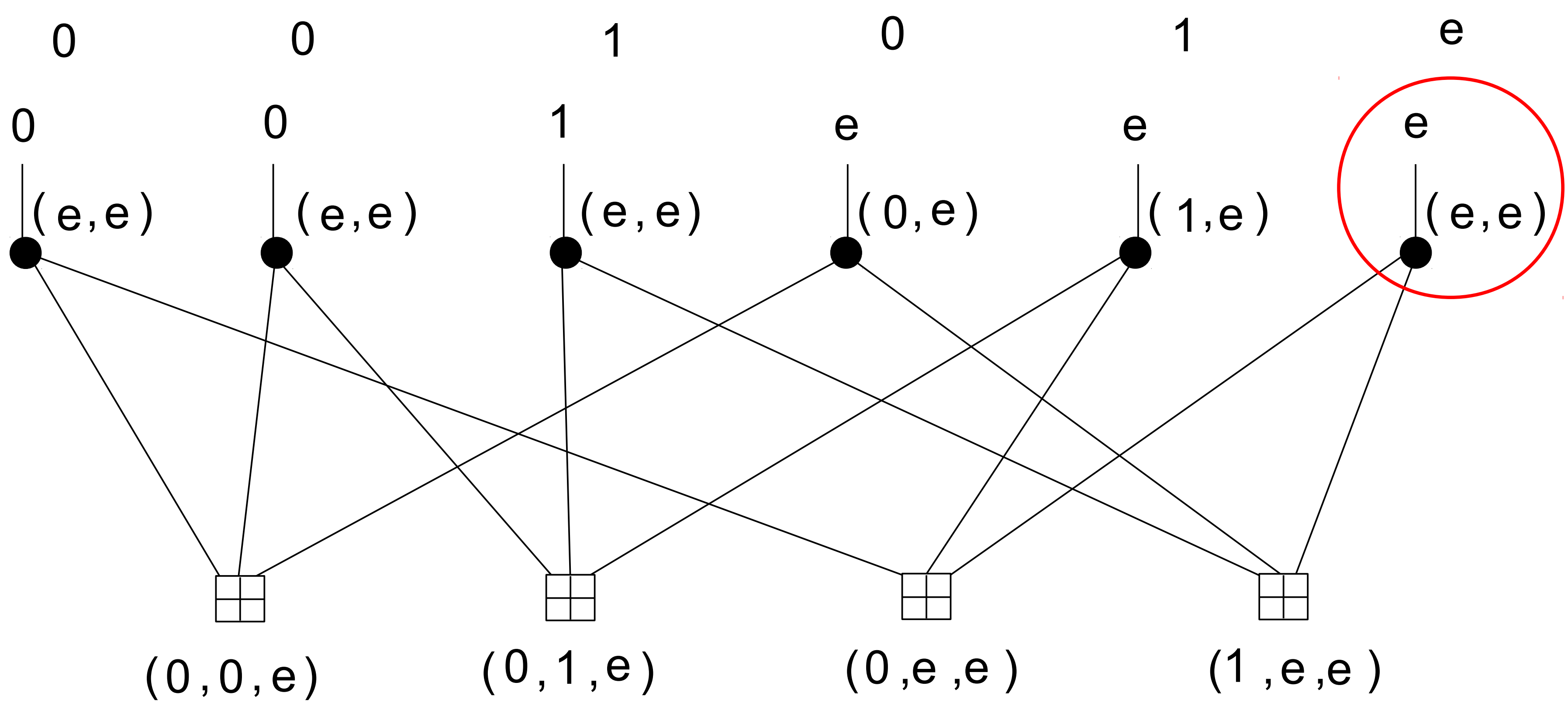

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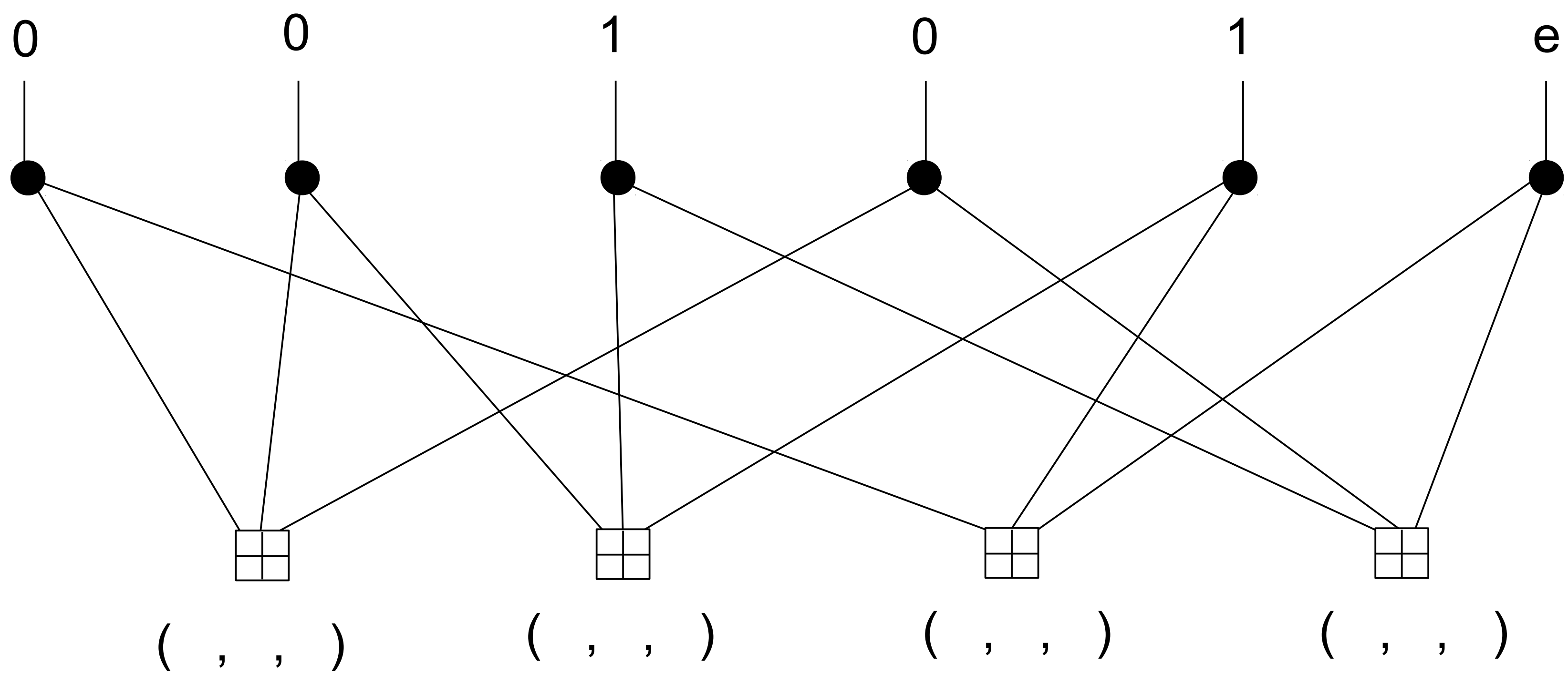

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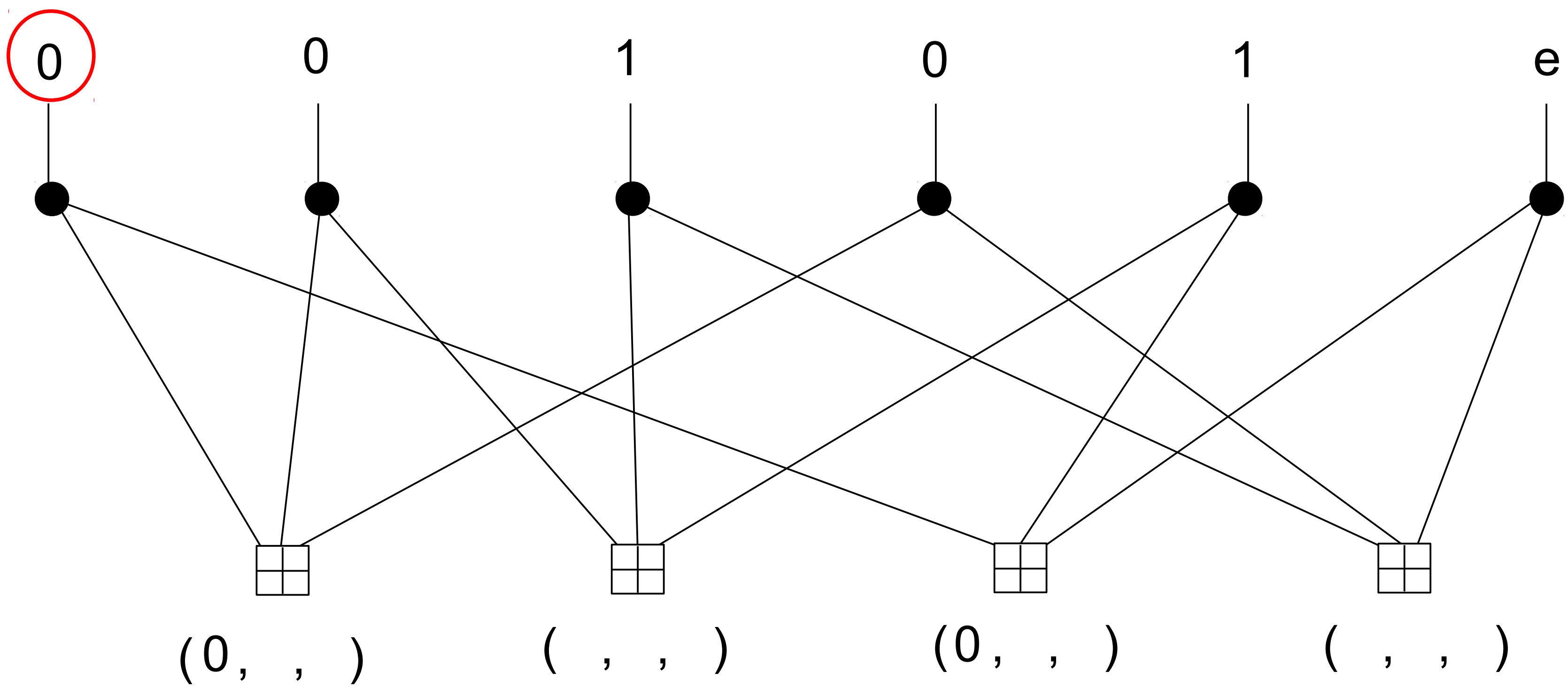

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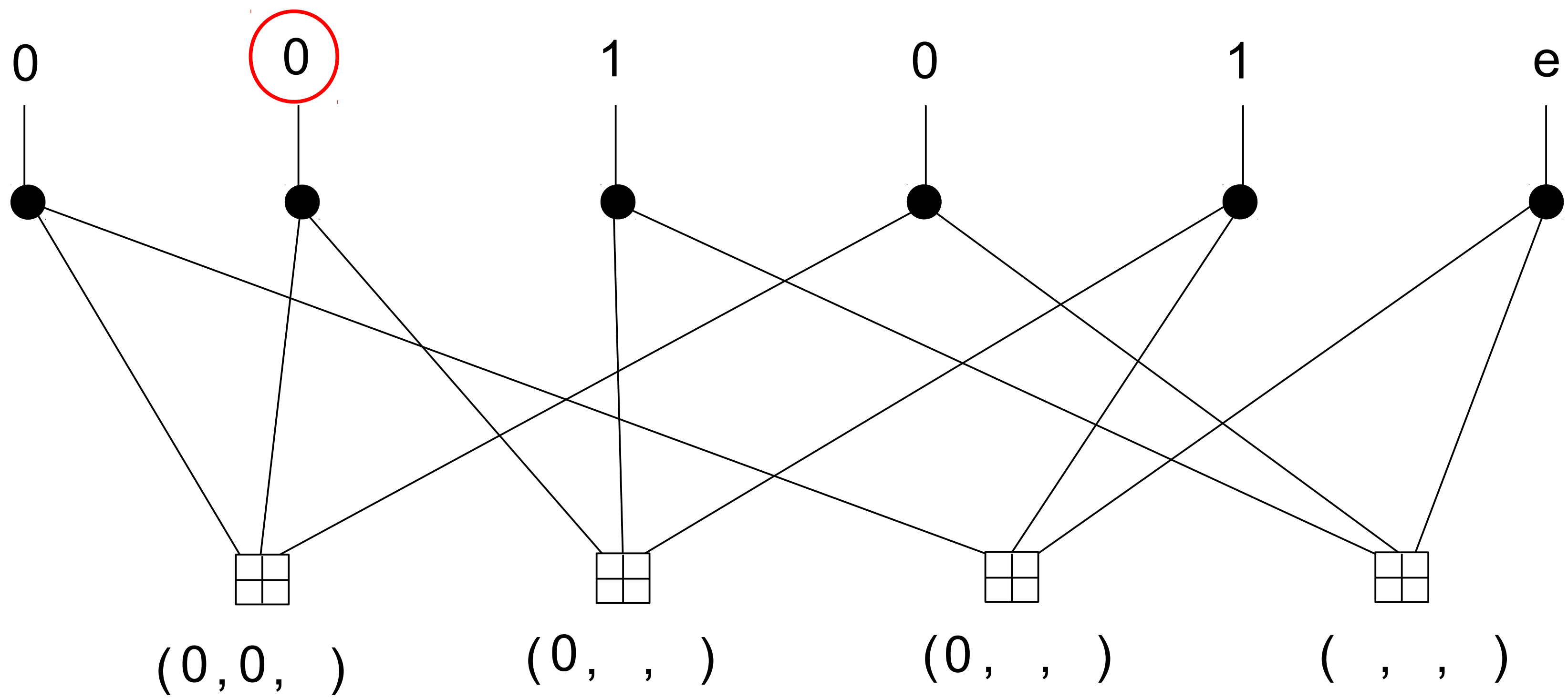

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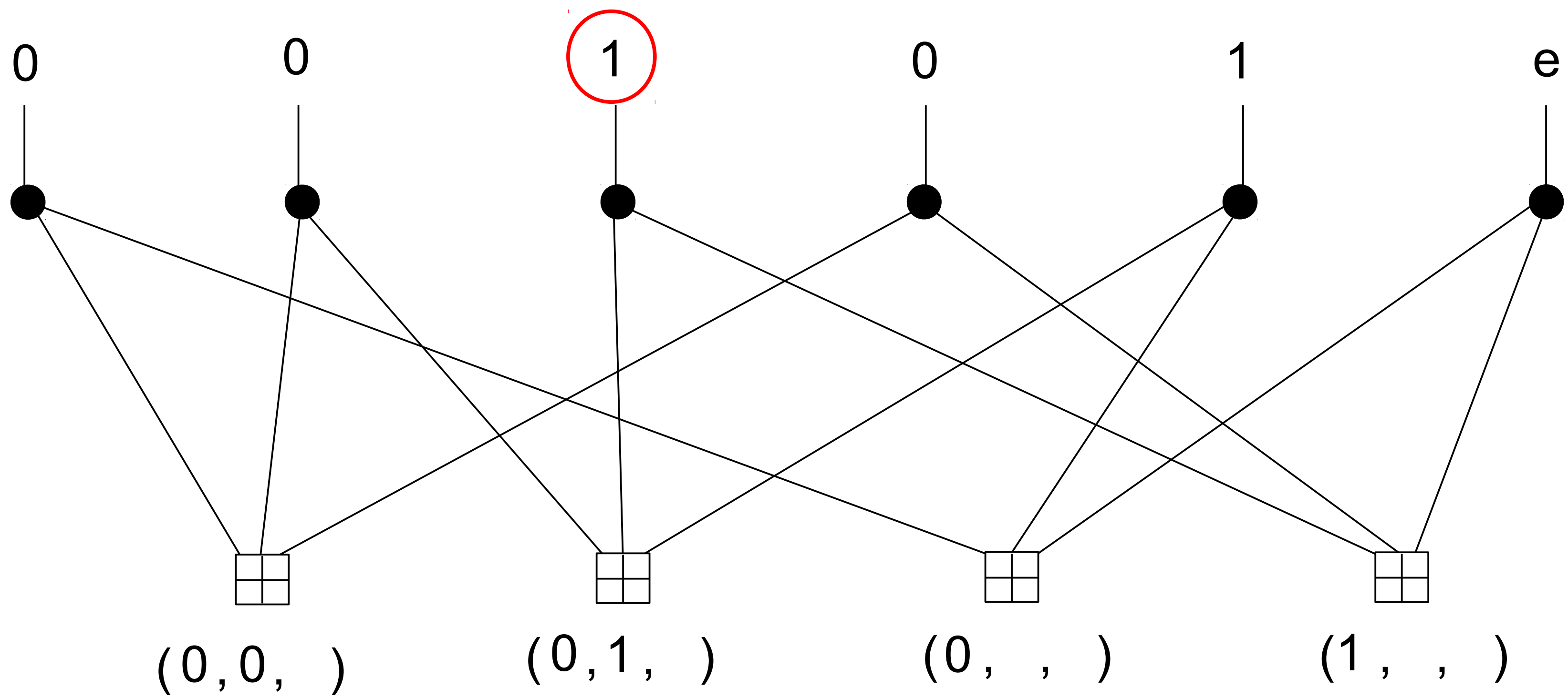

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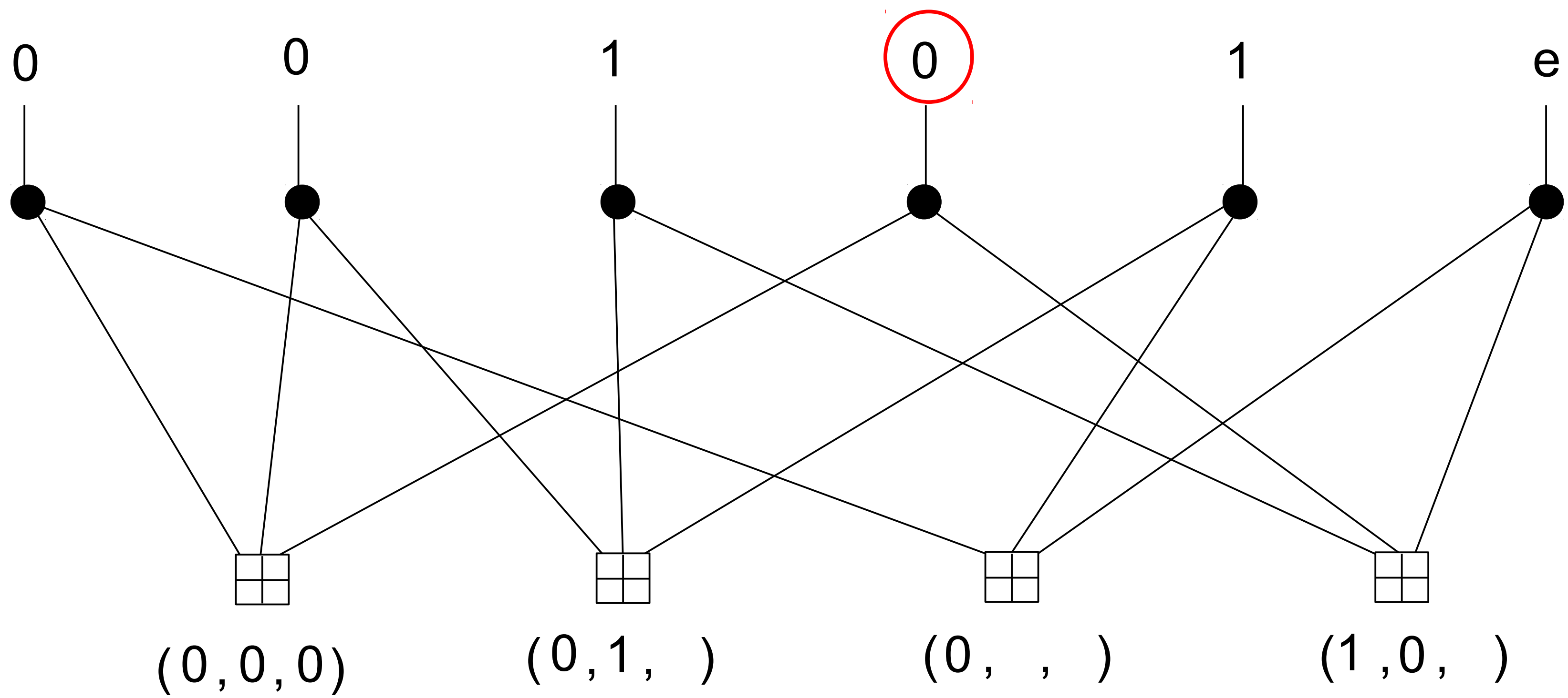

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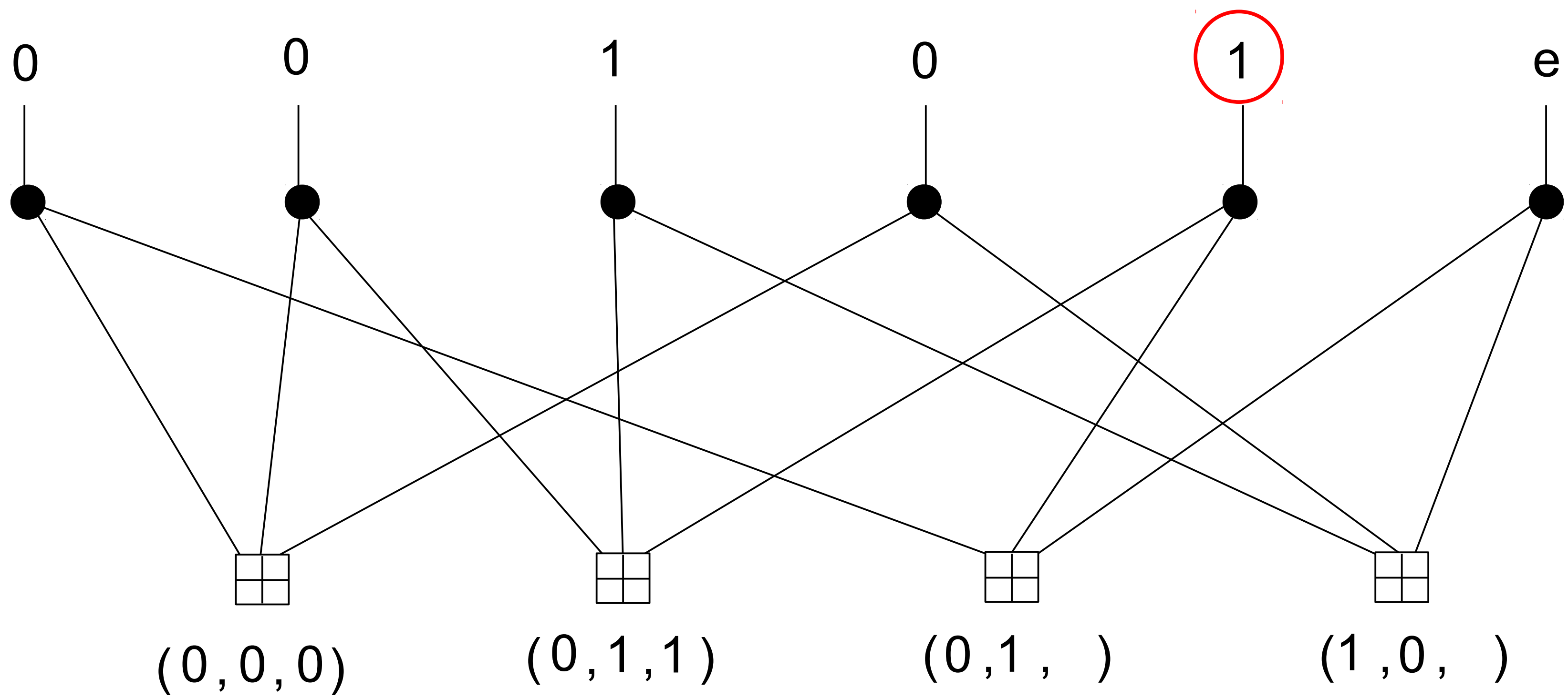

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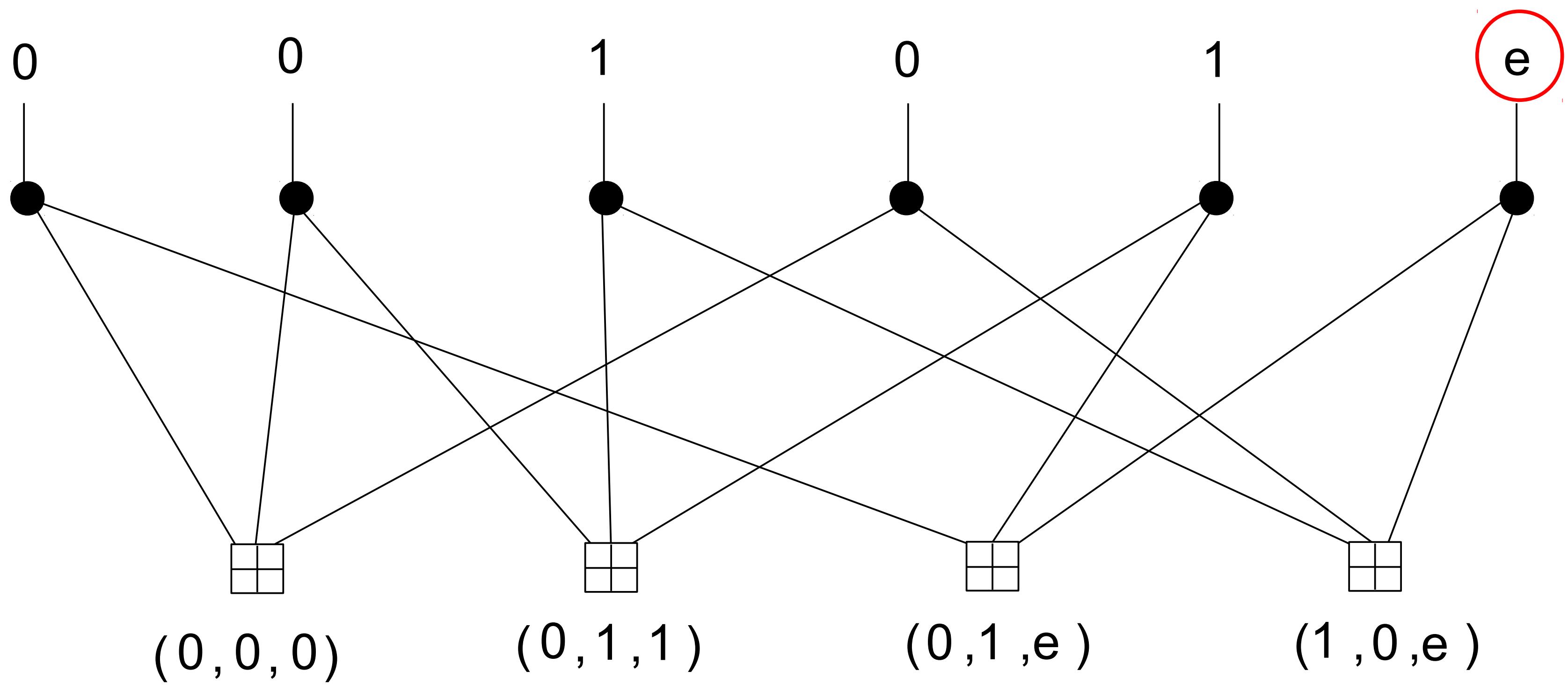

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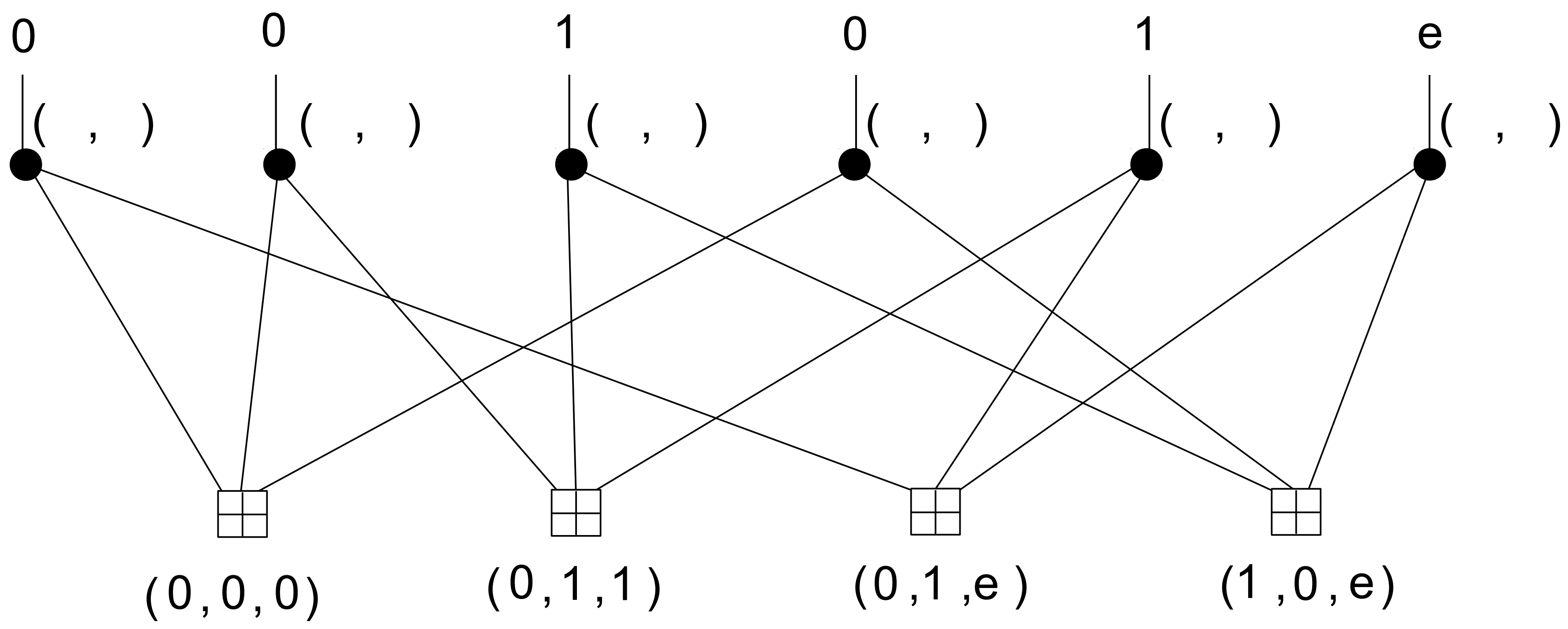

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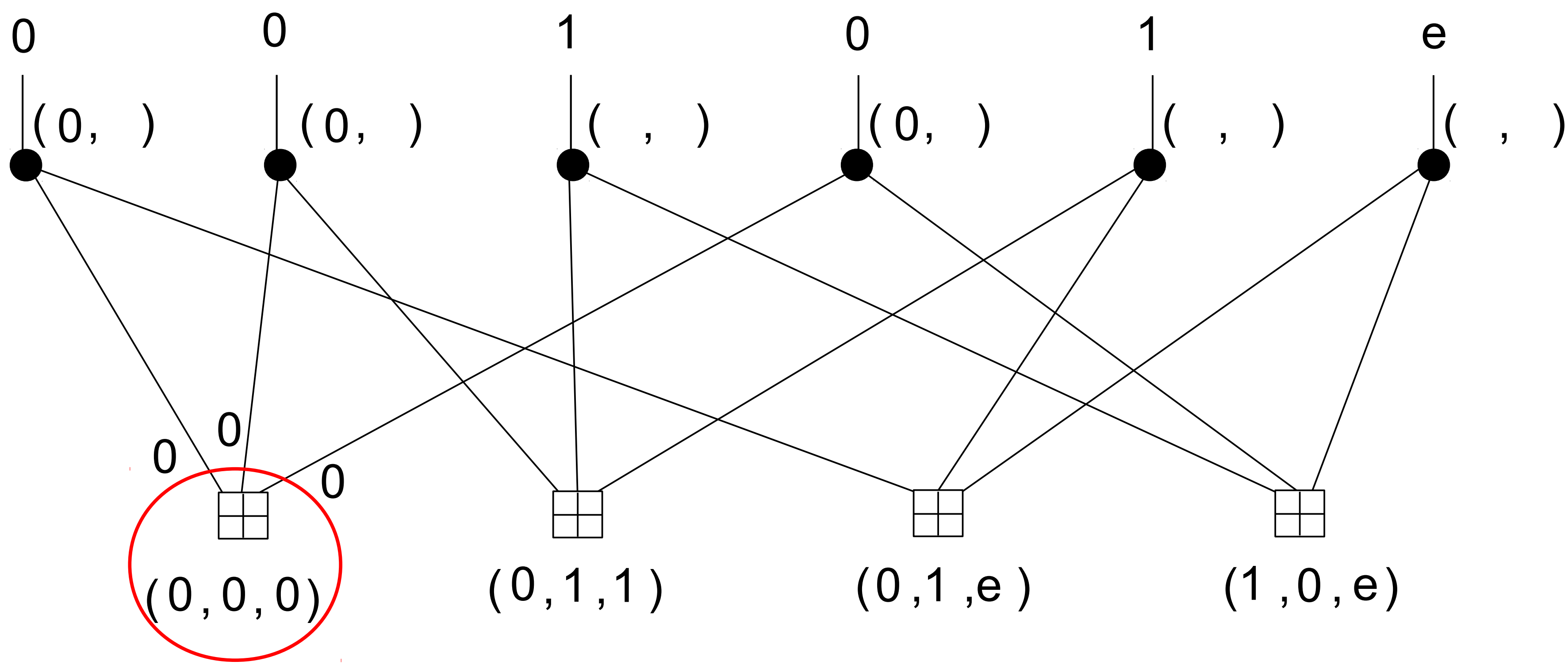

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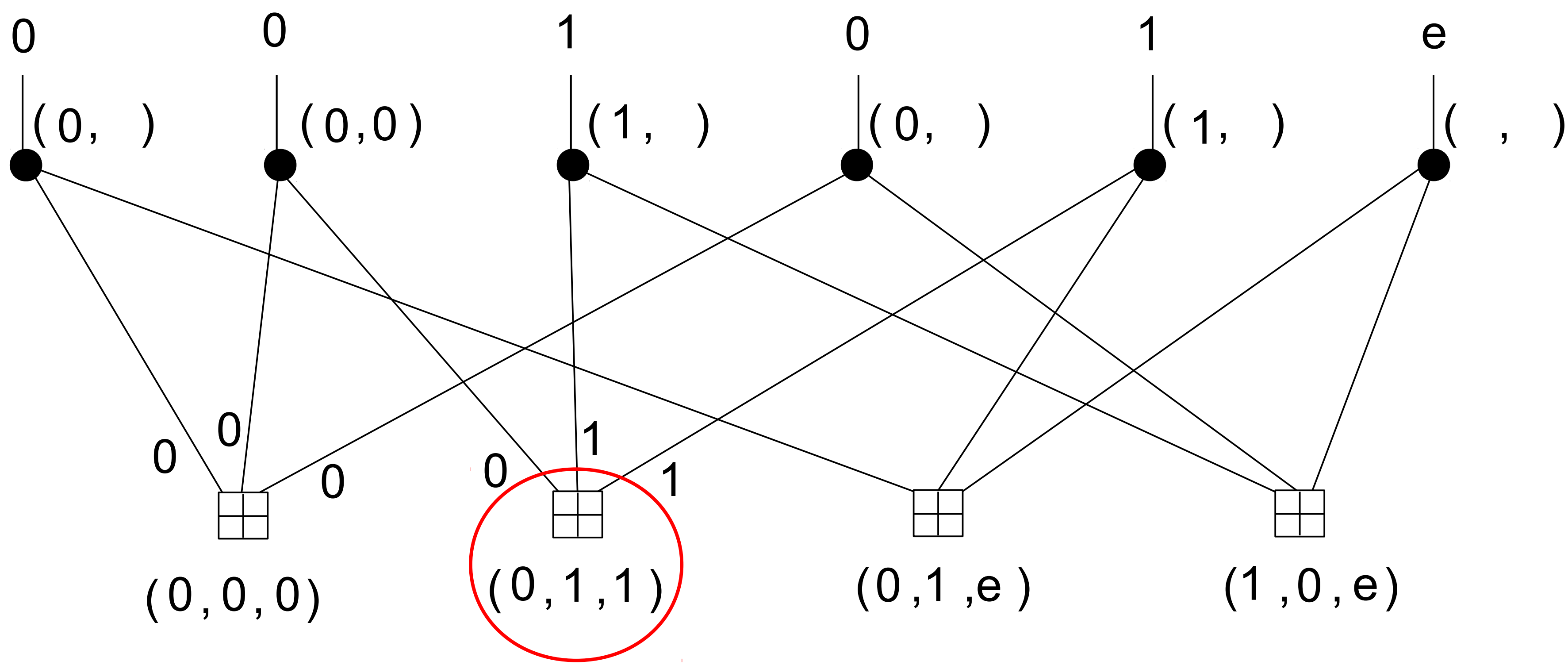

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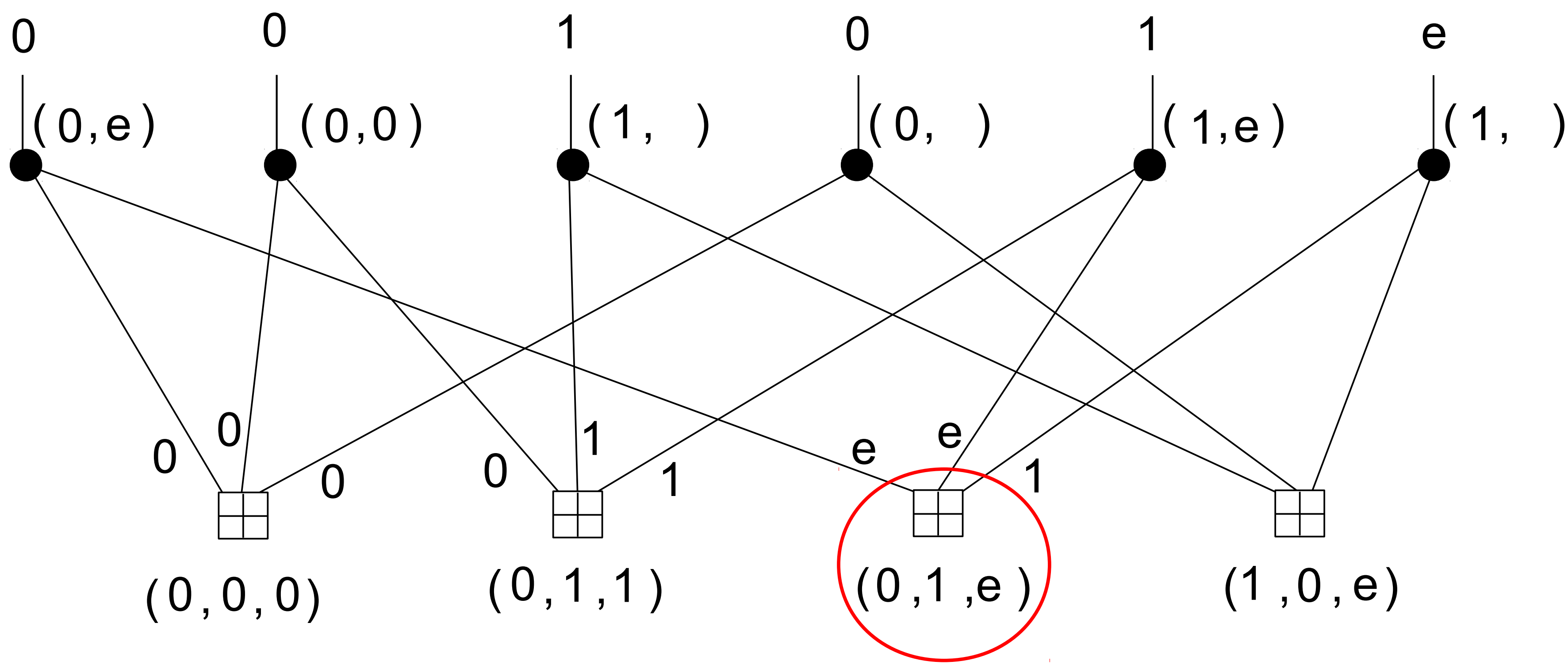

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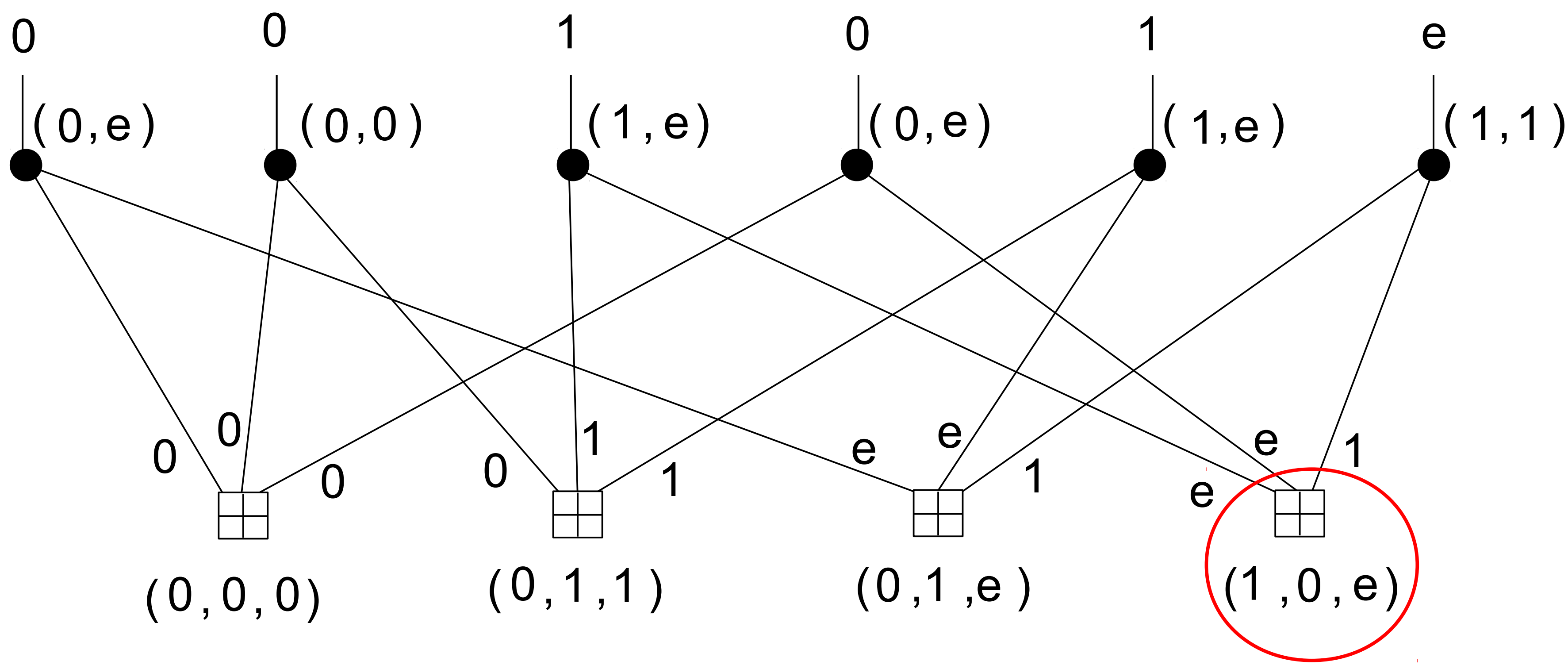

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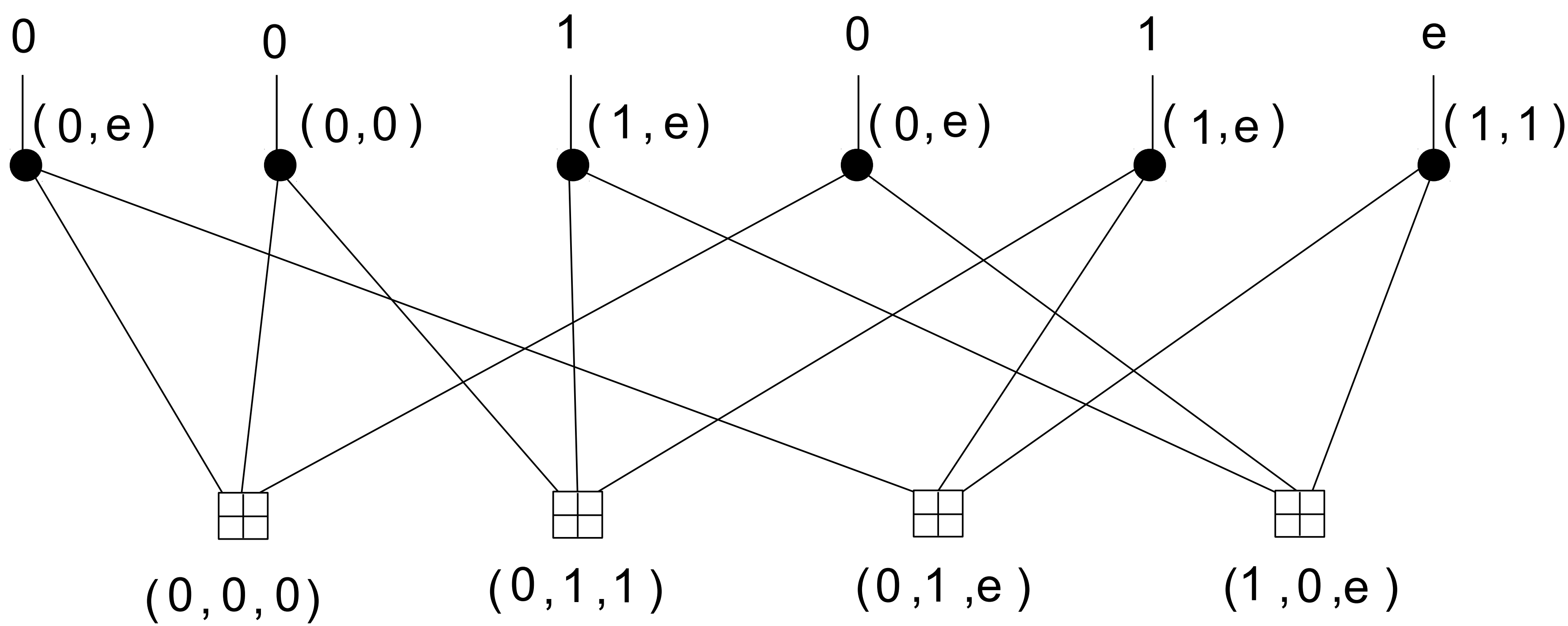

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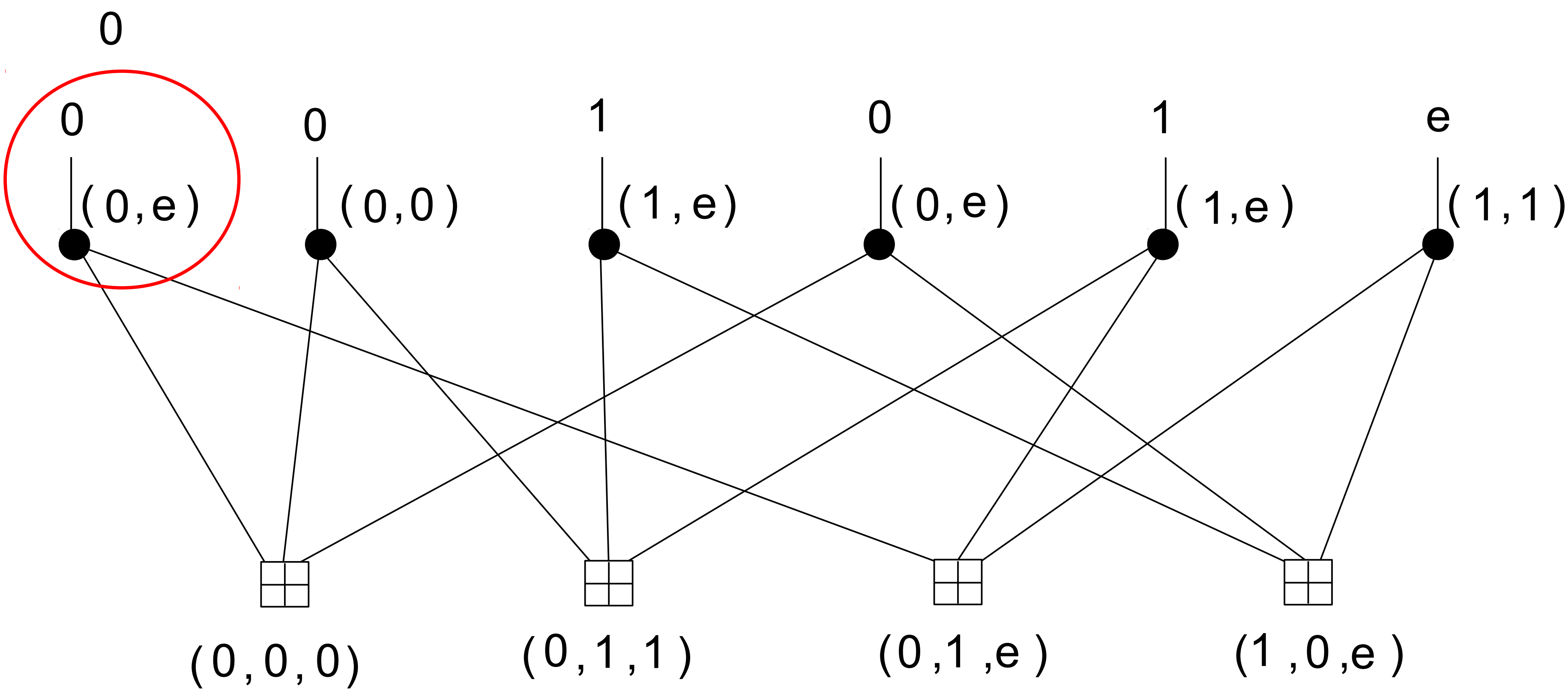

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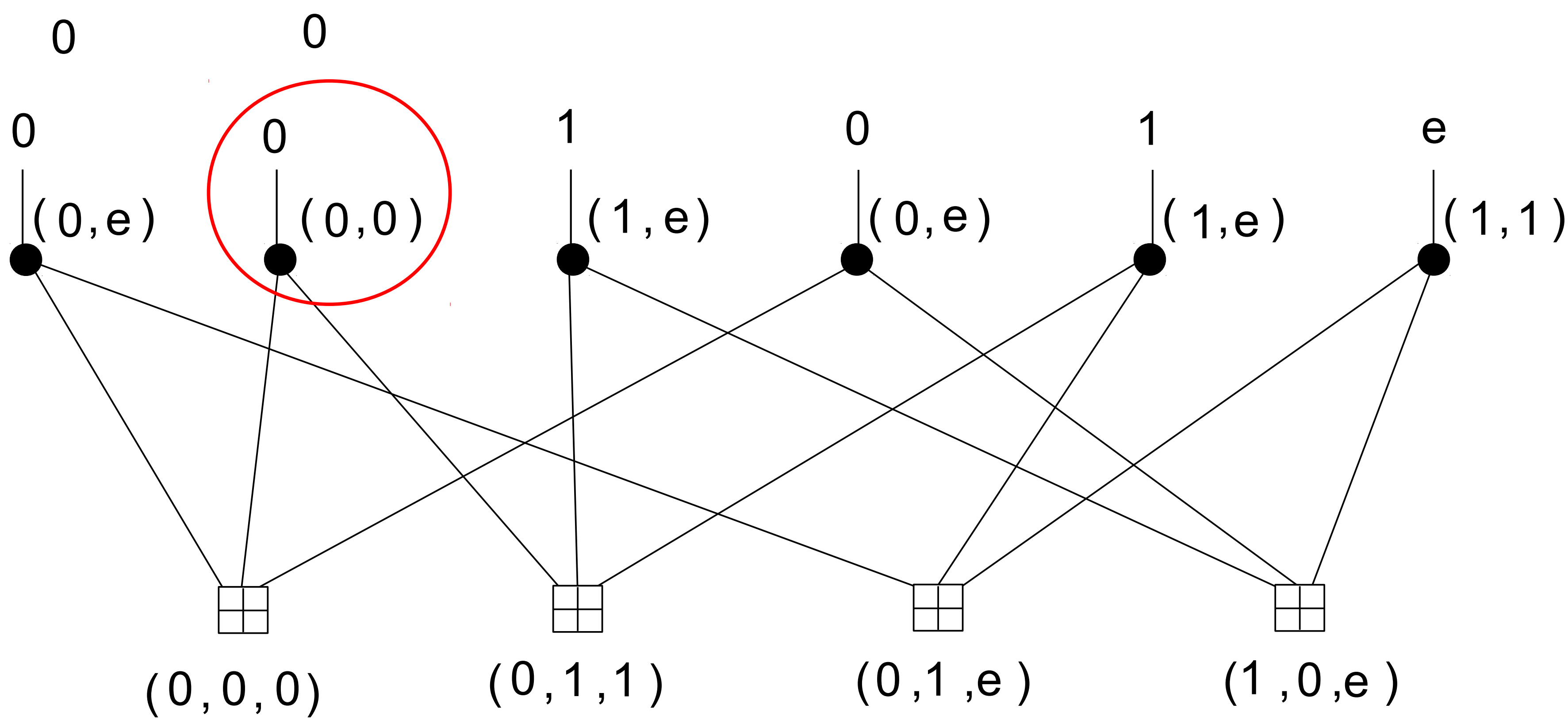

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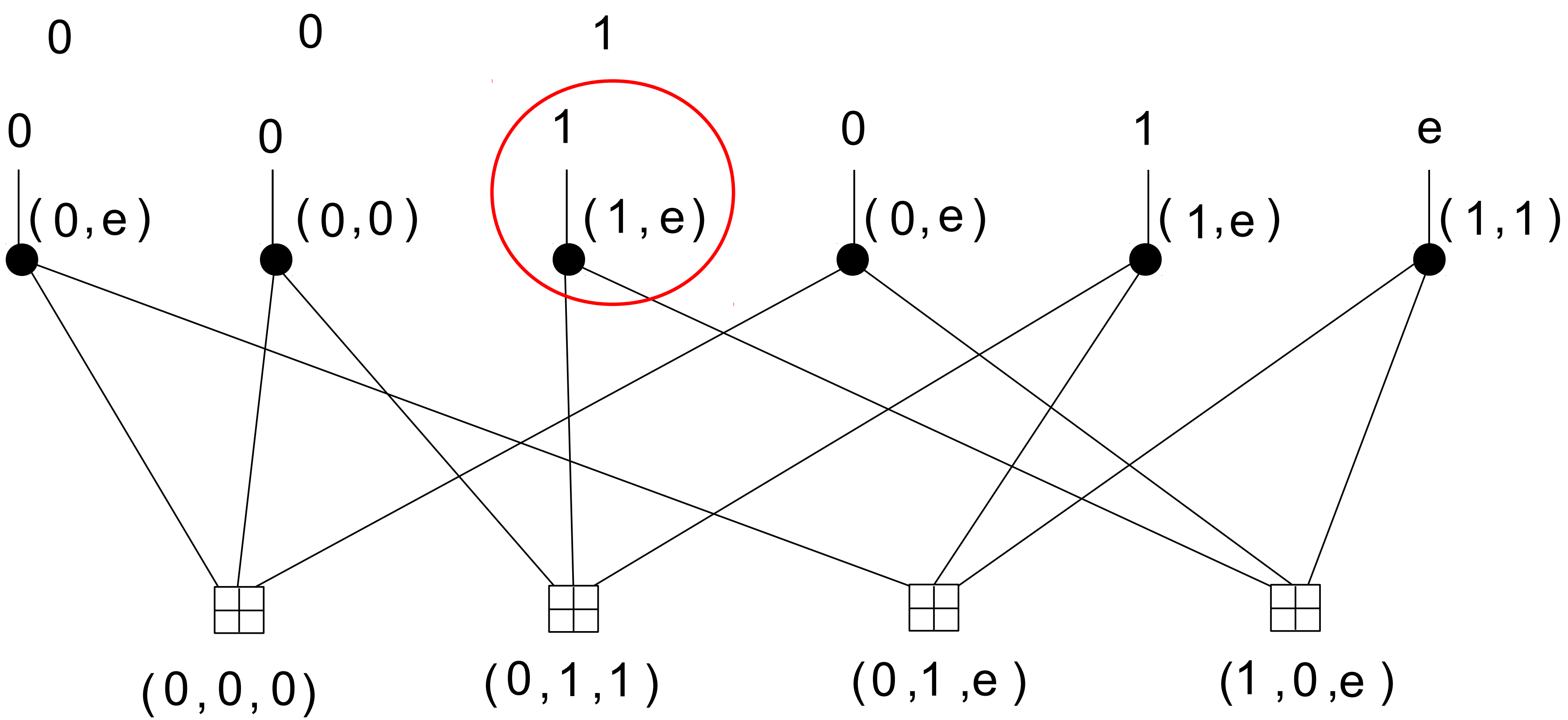

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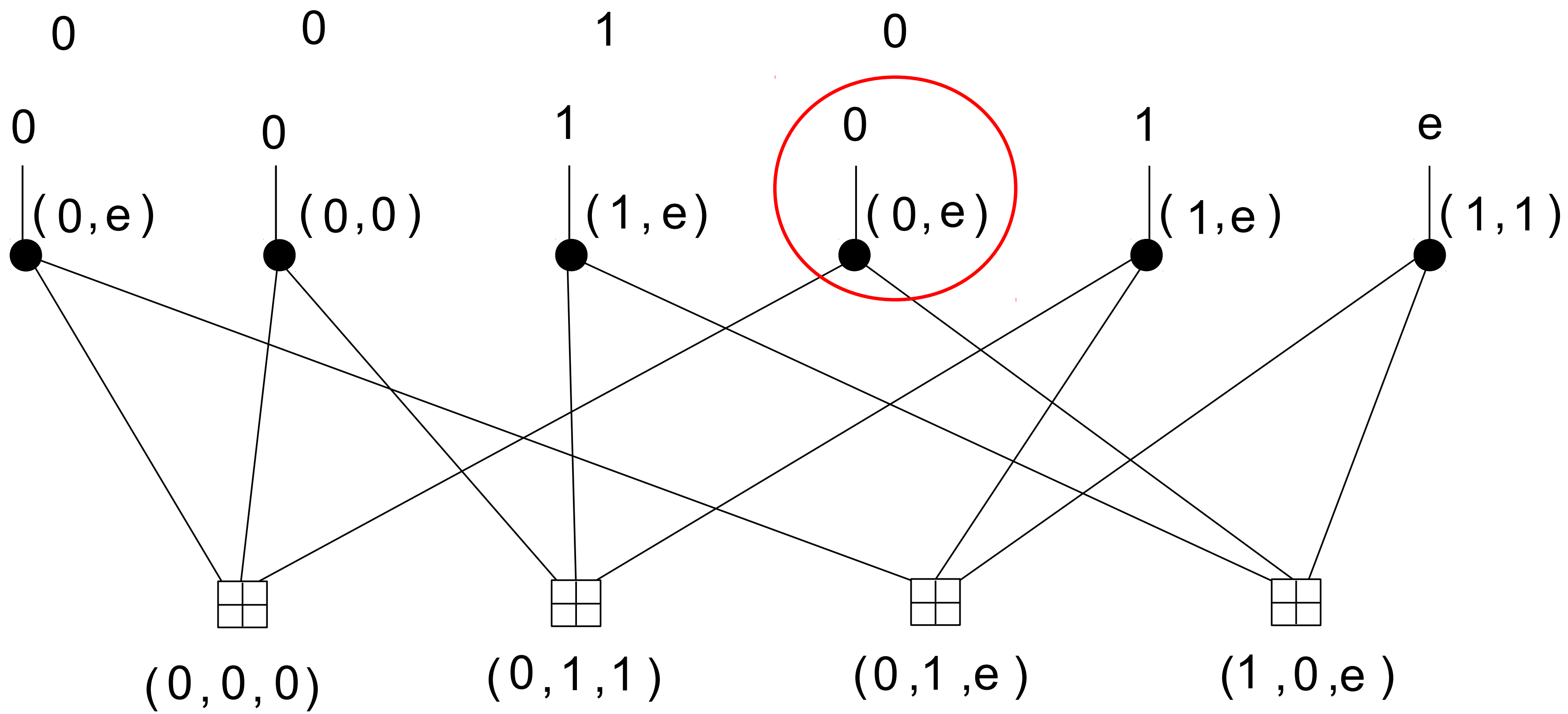

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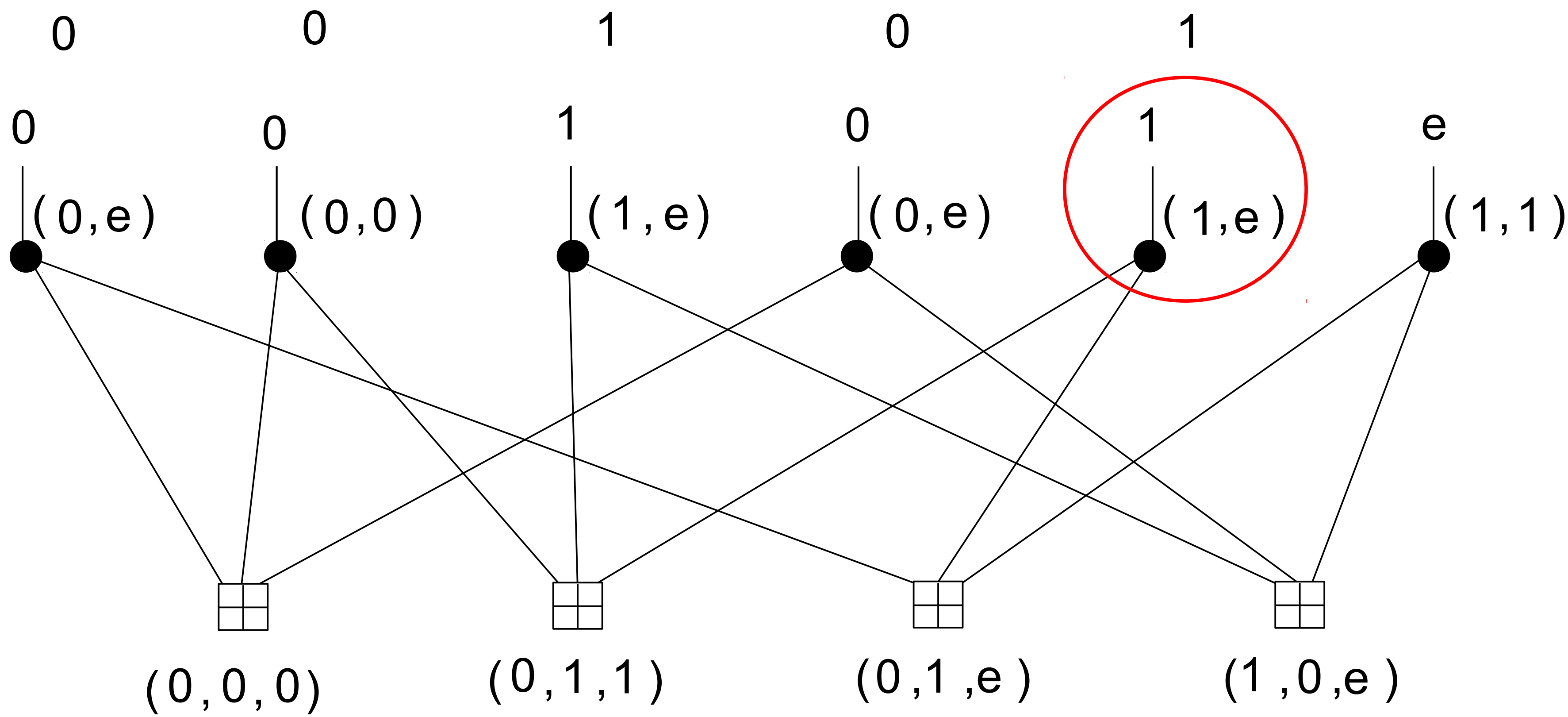

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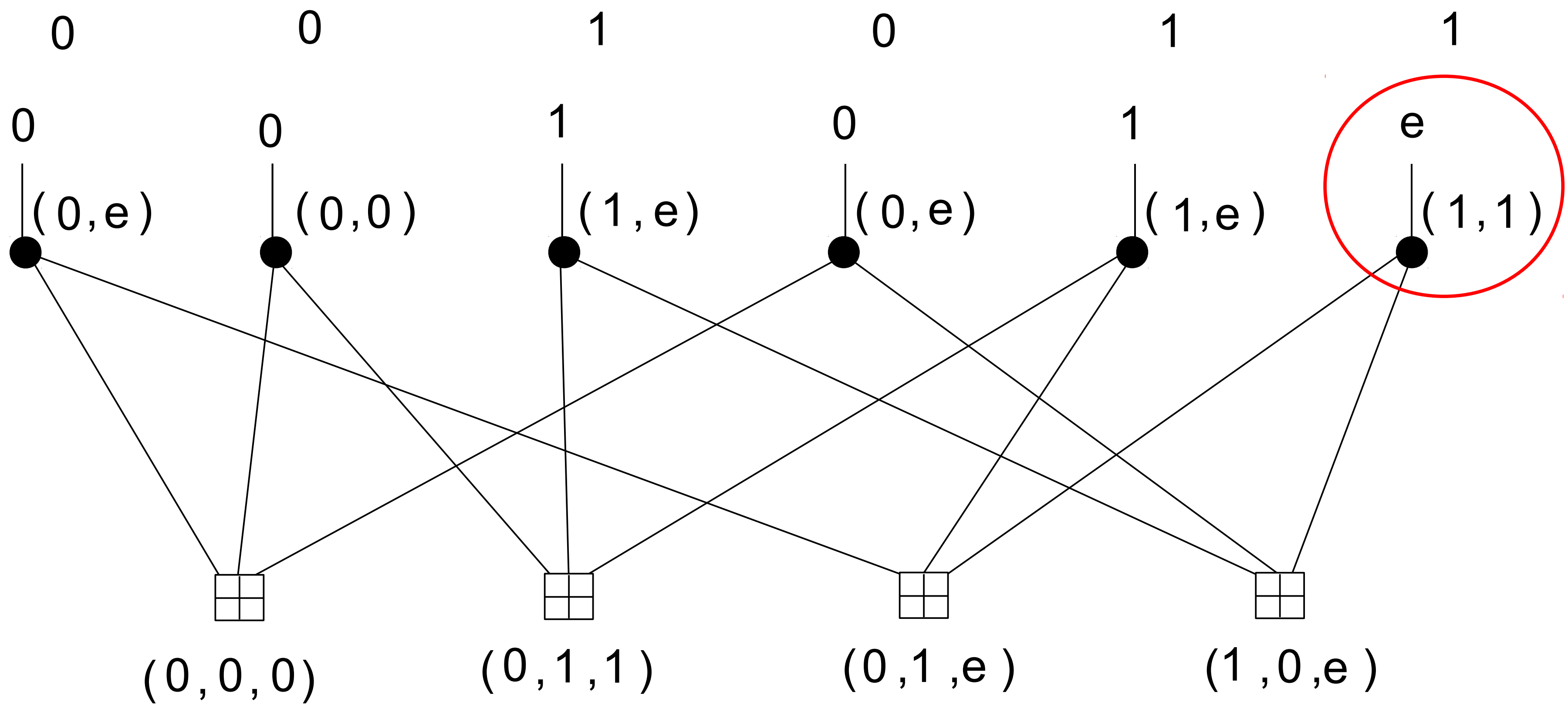

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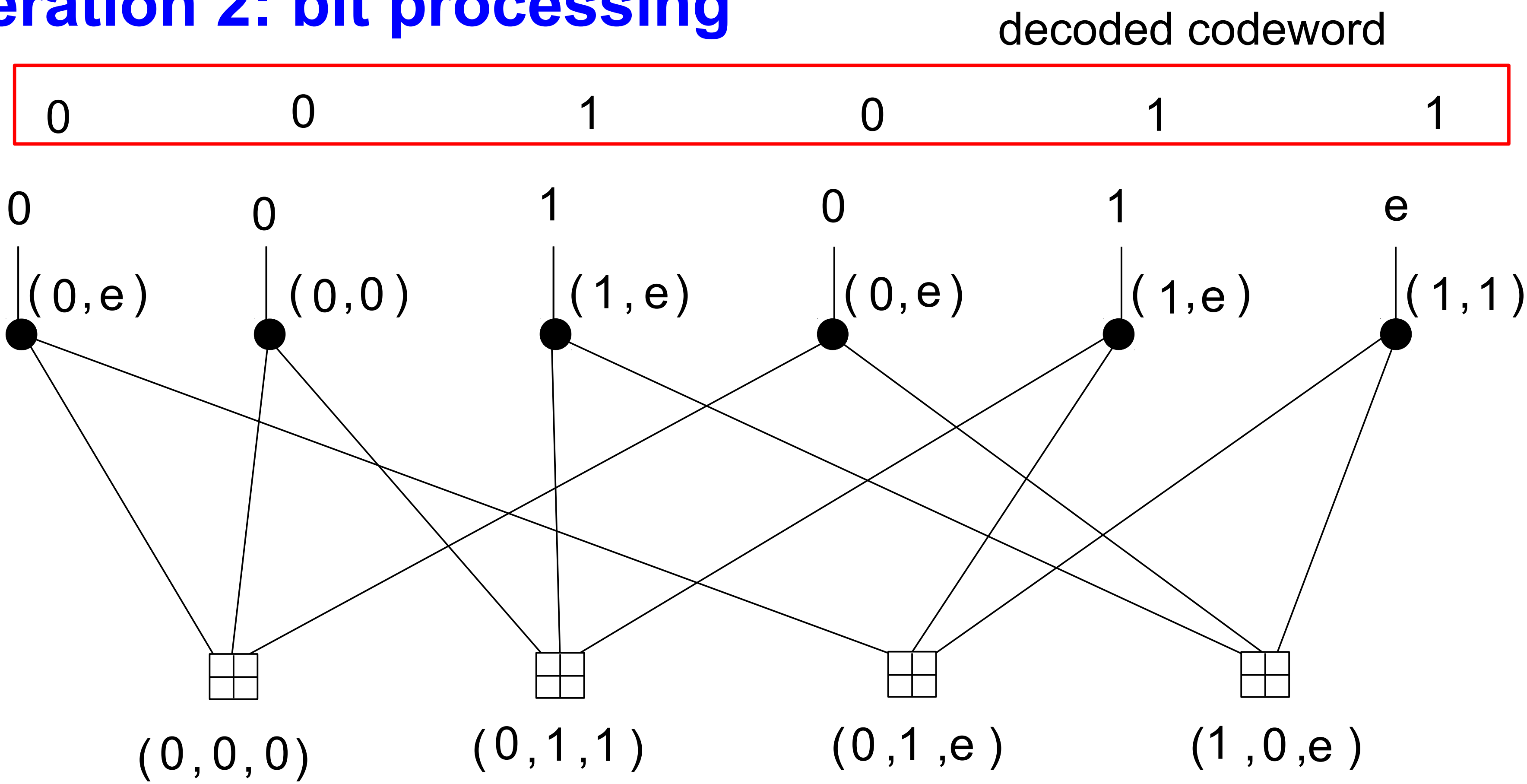
Iteration 2: bit processing



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- The corresponding (J,K)-regular LDPC code ensemble is the set of all such parity-check matrices for a fixed J,K, and n.

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0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
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- Gallager proved that for $J > 2$ and large block length n, the minimum distance typical of most members of a (J,K)-regular LDPC code ensemble is bounded below as $\delta_{min} n$, where $\delta_{min} > 0$ is a constant that depends on J and K (not n), i.e., $d_{min} \geq \delta_{min} n$.

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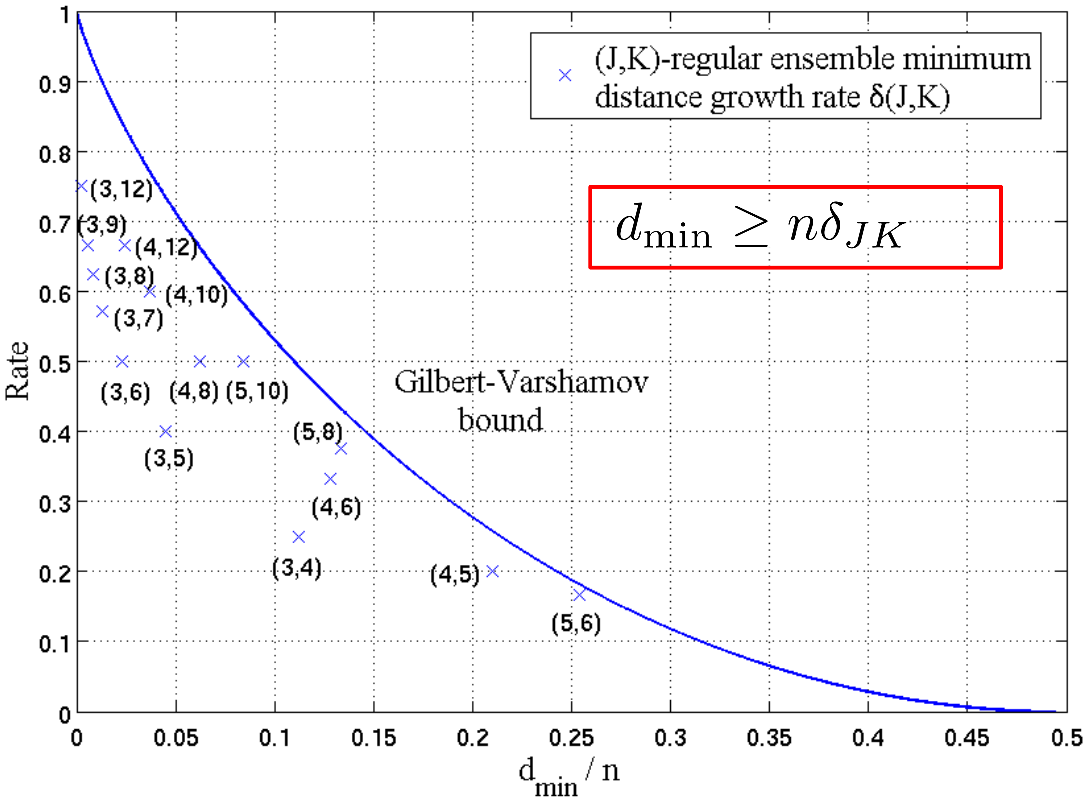
- Such code ensembles are called **asymptotically good**

→ Under ML decoding, we will achieve excellent performance

Code Ensembles – Minimum Distance Growth Rates

- δ_{JK} is called the **typical minimum distance ratio**, or **minimum distance growth rate**, of a (J,K)-regular code ensemble $(R \geq 1 - J/K)$.



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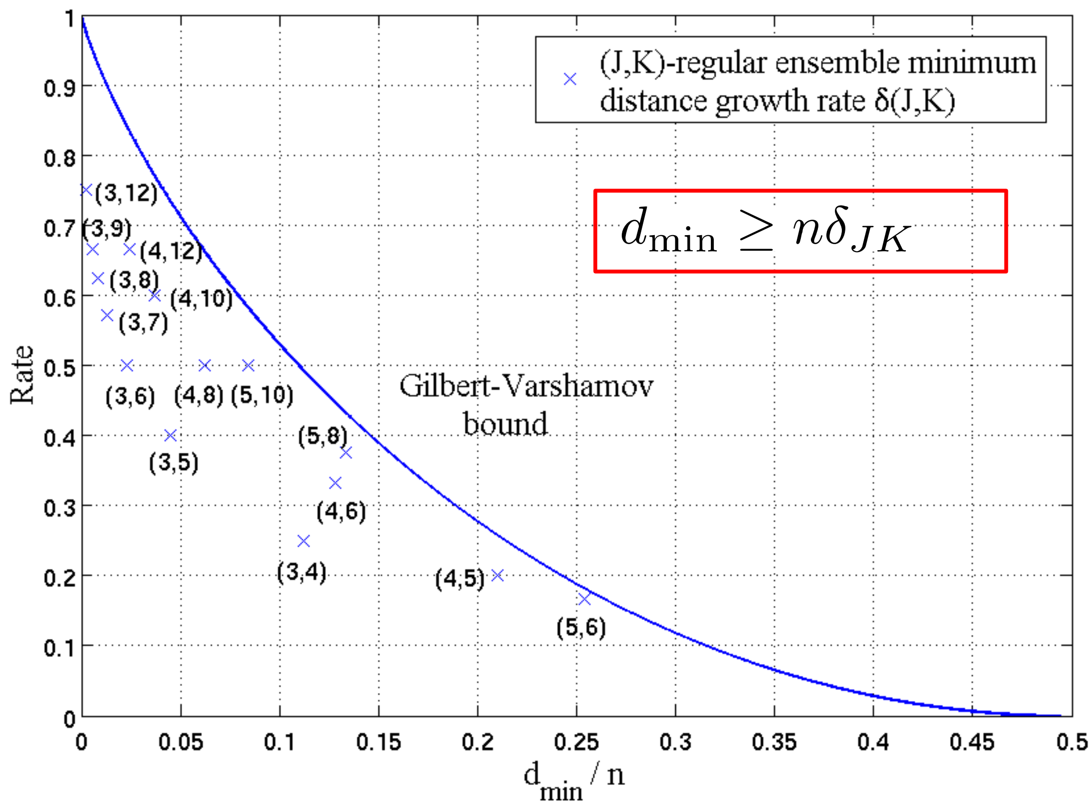

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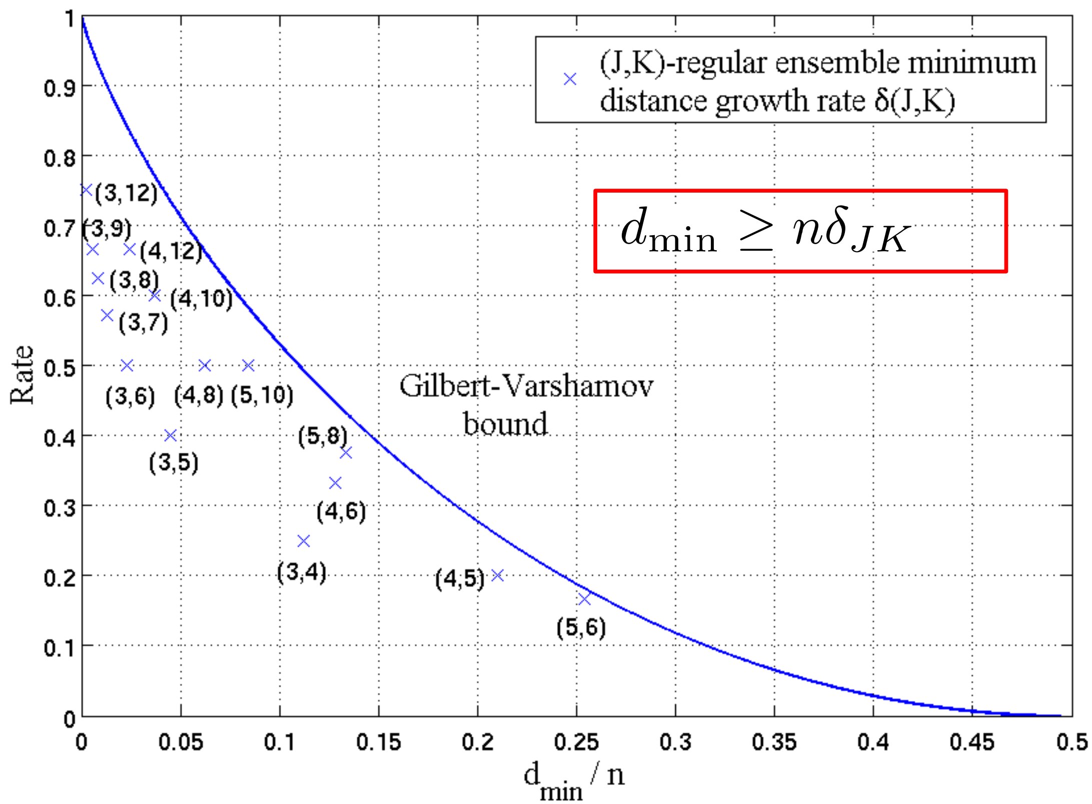

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- As the **density** (J and K) of (J,K)-regular ensembles **increases**, the minimum distance growth rate **approaches the Gilbert-Varshamov bound** for the ensemble of all block codes.

Density Evolution

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- We use **density evolution** (DE) to determine for which channel noise levels the message passing decoder can correct errors and for which levels it cannot.

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(2) The channel is memoryless

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- This is achieved by tracking probability density functions over iterations.

We assume:

(1) An ensemble of Tanner graphs

(2) The channel is memoryless

(3) The graphs are cycle free

- For a given code ensemble we determine an **iterative decoding threshold** – the maximum level of noise that can possibly be corrected by a **particular code ensemble**

DE on the BEC

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DE for (J,K)-regular ensembles:

- Messages are '1', '0', or 'e'
- Let $q^{(l)}$ be the probability that a check-to-bit message is an 'e' at iteration l
- Let $p^{(l)}$ be the probability that a bit-to-check message is an 'e' at iteration l

Check-to-bit probabilities

Recall: At iteration l, a message from check-to-bit is an 'e' iff any of the $K-1$ other incoming messages are an 'e'

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- This implies that the probability that **at least one** incoming message is and 'e' is

$$q^{(l)} = 1-(1-p^{(l)})^{K-1}$$

Bit-to-check probabilities

- At iteration l, a message from bit-to-check is an 'e' if
  (1) The original message from the channel is an 'e', **and**
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- We can use this method to determine for what values of ε the message passing decoder is likely to correct erasures and how many iterations are necessary!

Density Evolution: Example

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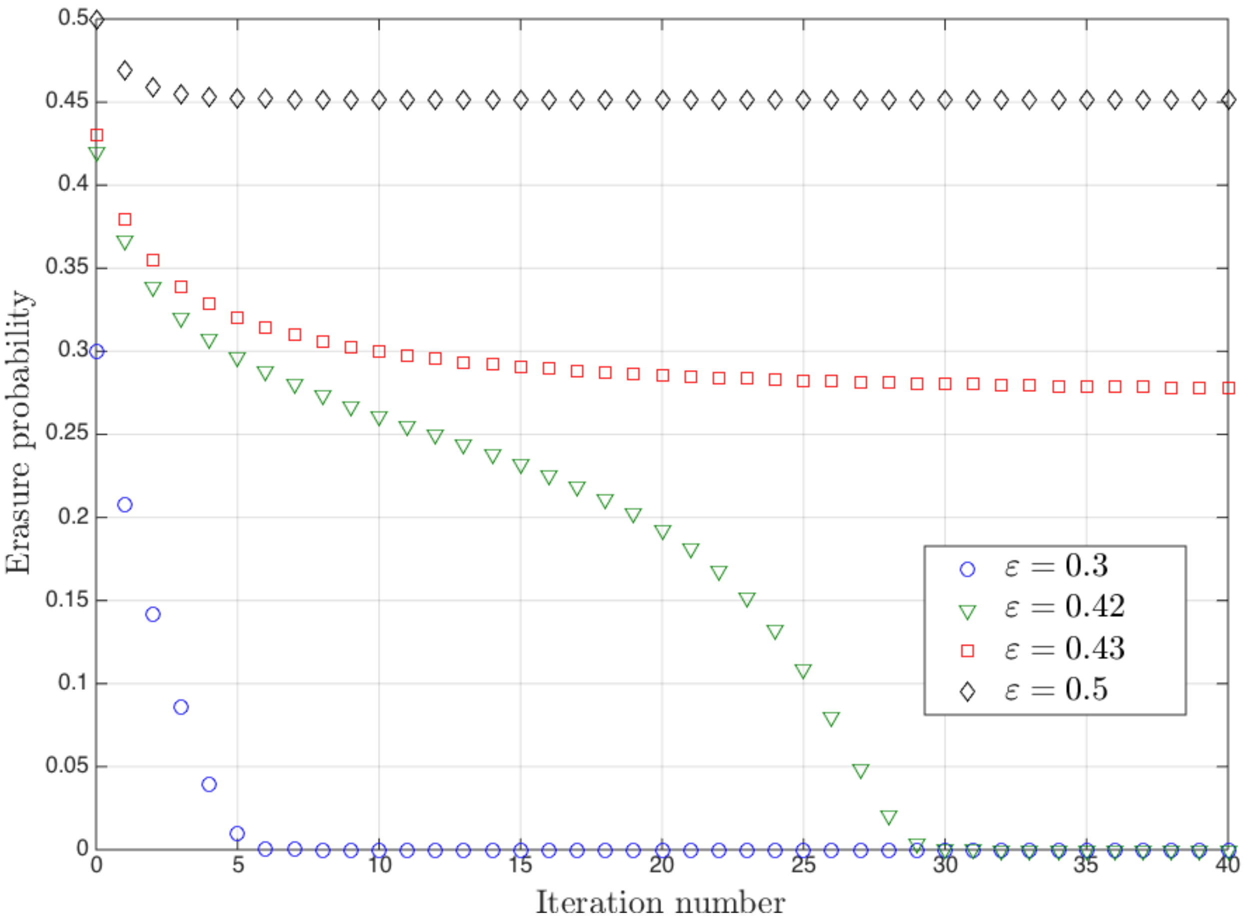
→ After **7 iterations**, the erasure probability in a codeword will go to 0 for a (3,6)-regular LDPC code (with no 14-cycles or less) transmitted over a BEC with $\varepsilon = 0.3$

Example (cont.)



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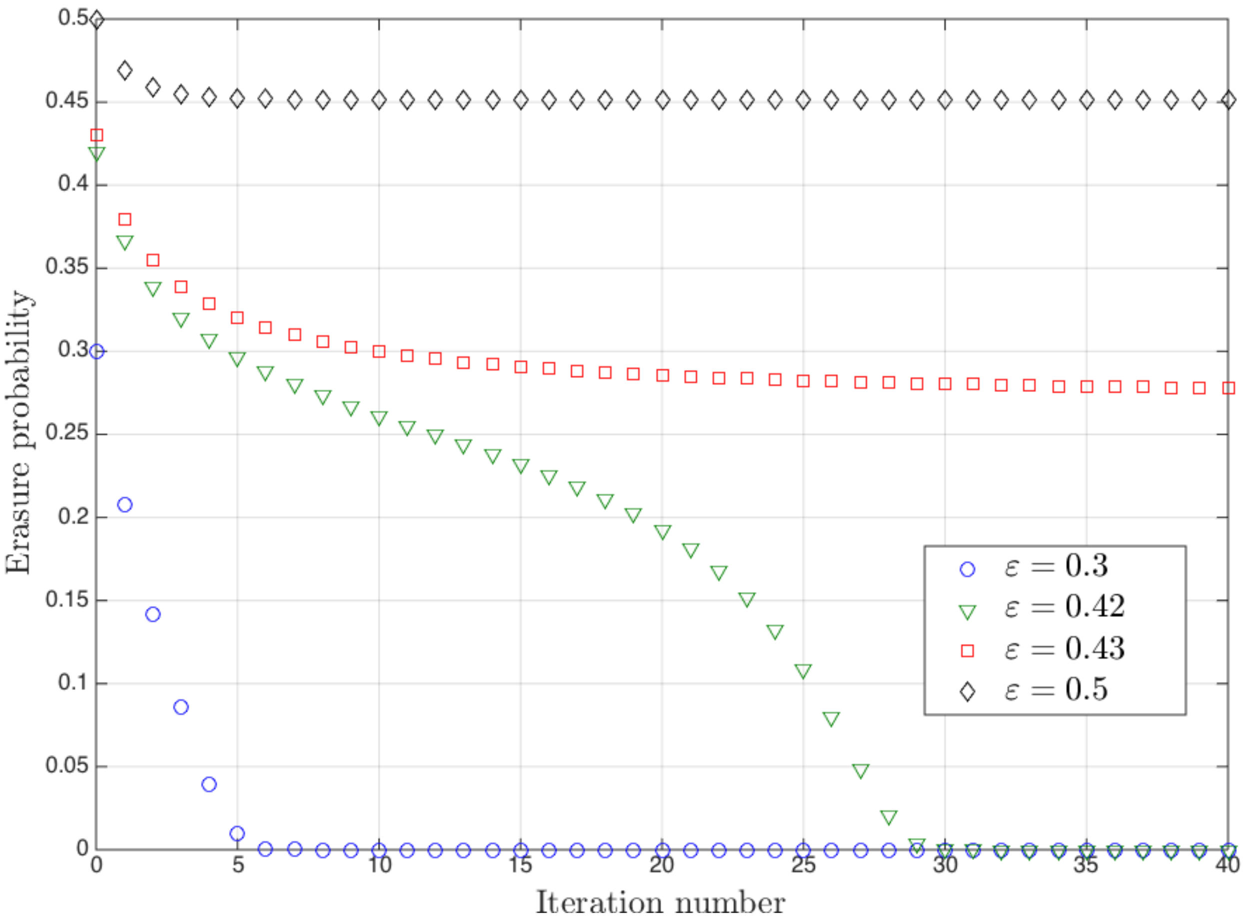

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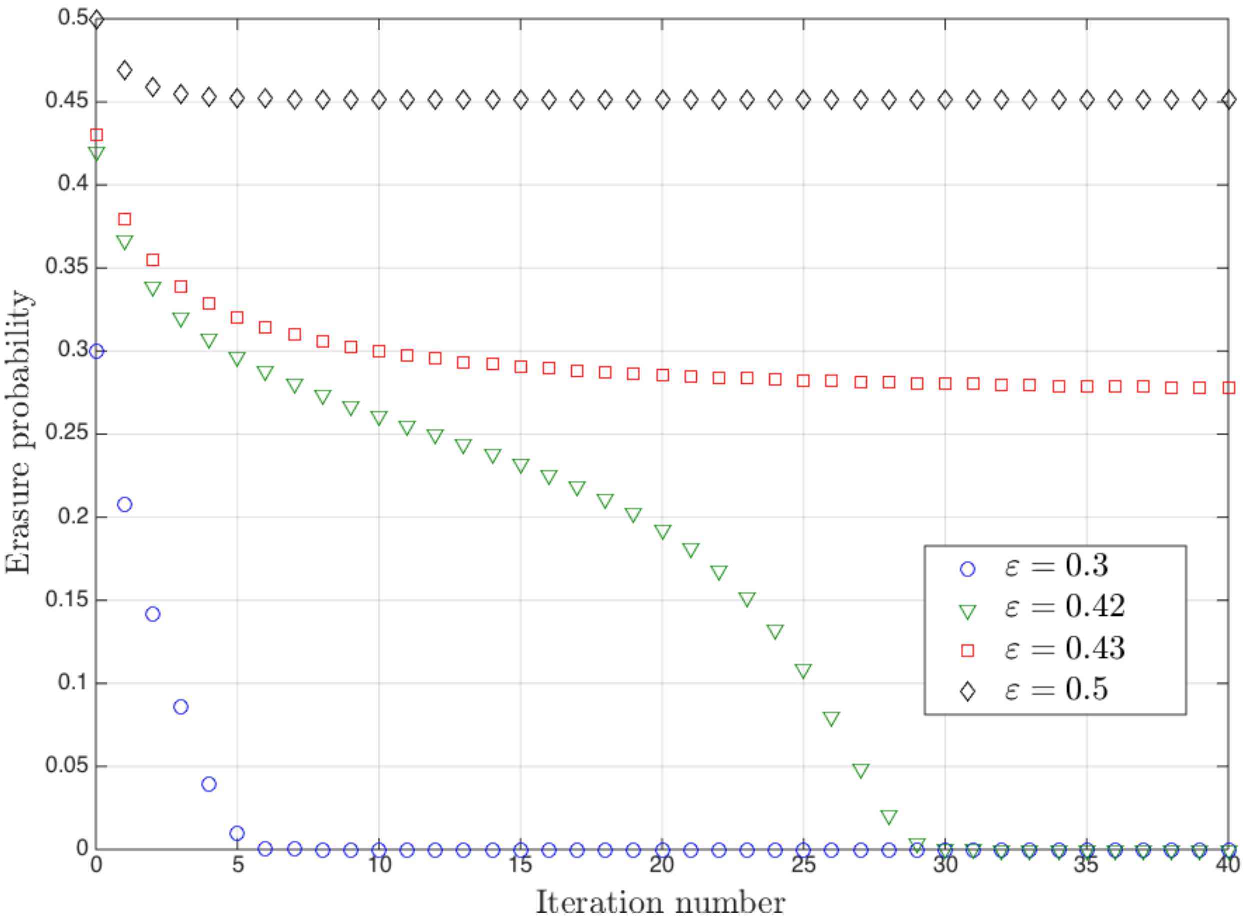

- As ε increases, it **takes more iterations** for the erasure probability to converge to 0.
- For a large enough ε (a bad channel) the erasure probability **does not go to 0.**
- For (3,6)-regular LDPC codes, this value is $\varepsilon^* = 0.429$. ε^* is called the **iterative decoding threshold**.

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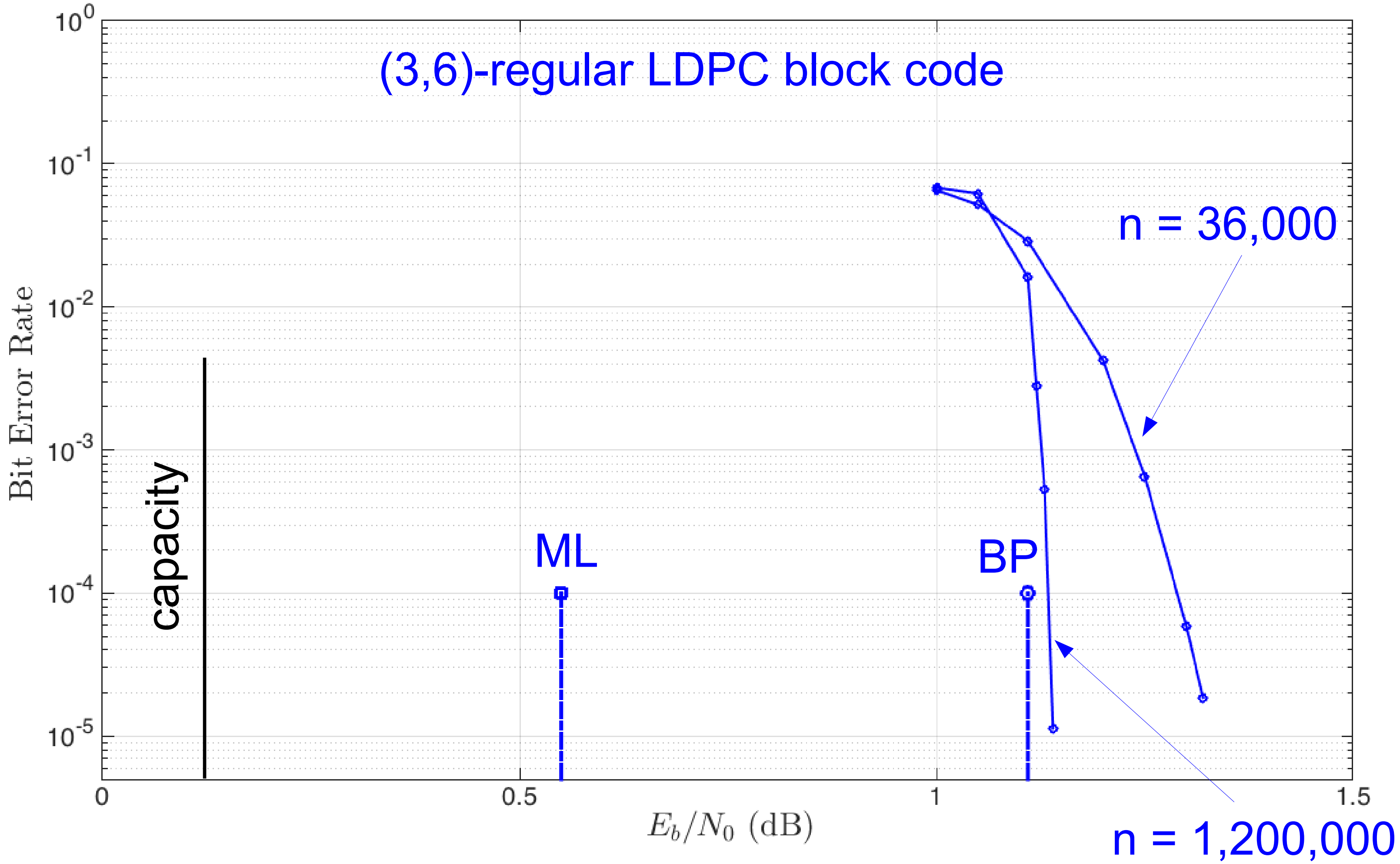
➡ We track the PDFs over iterations and determine if the probability of error goes to zero as the number of iterations goes to infinity for a particular channel SNR.

Example: AWGNC

BP = iterative (suboptimal) decoding threshold
ML = maximum likelihood (optimal) decoding threshold



Thresholds of (J,K)-regular LDPC Code Ensembles

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- The iterative decoding thresholds can be calculated for a variety of (J,K)-regular LDPC code ensembles using DE.

BEC thresholds

J	K	Rate	ε^*	ε_{Sh}
3	6	0.5	0.429	0.5
4	8	0.5	0.383	0.5
5	10	0.5	0.341	0.5
3	5	0.4	0.517	0.6
4	6	0.333	0.506	0.667
3	4	0.25	0.647	0.75

AWGNC thresholds

J	K	Rate	$(E_b/N_0)^*$	$(E_b/N_0)_{\text{Sh}}$
3	6	0.5	1.11	0.184
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5	10	0.5	2.04	0.184
3	5	0.4	0.96	-0.229
4	6	0.333	1.67	-0.480
3	4	0.25	1.00	-0.790

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- There exists a relatively **large gap to capacity**.
- Iterative decoding thresholds get **further from capacity** as the graph **density increases**. (Minimum distance growth rates improve.)

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Choosing the degree distribution

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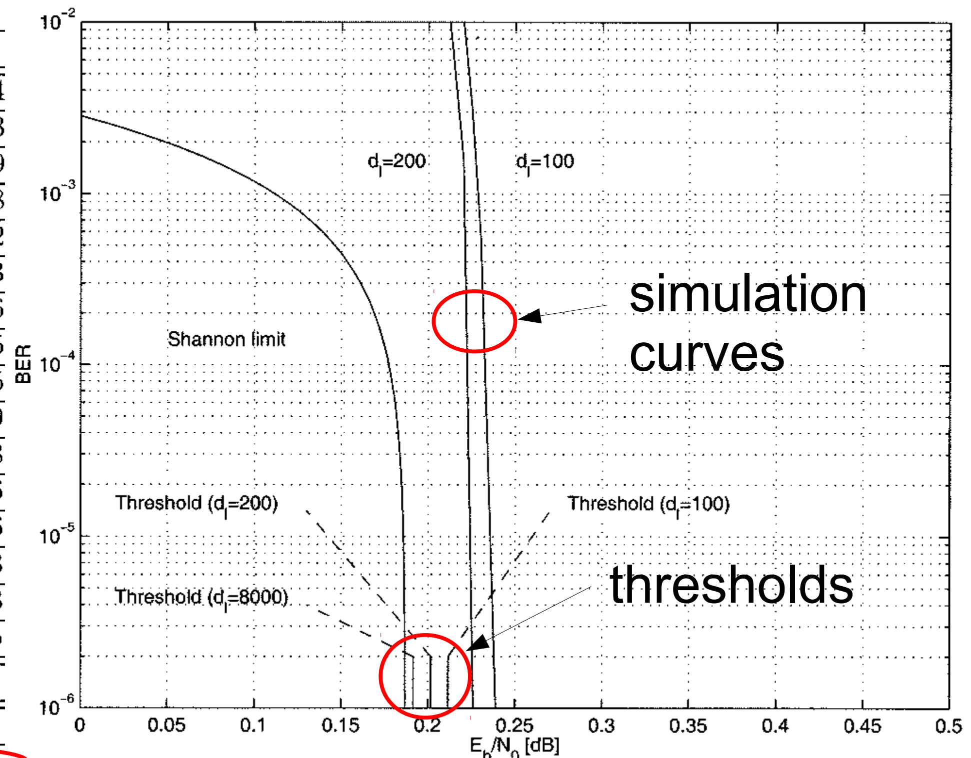
- Check node irregularity is not essential to improve thresholds. Typically good code designs have one or two different check node degrees.
- Trying all possible degree distributions is not practical. Optimization techniques such as iterative linear programming and differential evolution have been applied to find good degree distributions.

Random Irregular LDPC Codes

- Good rate $R = 1/2$ irregular LDPC codes: [CFRU01]

d_l	100		200		8000	
	x	λ_x	x	λ_x	x	λ_x
	2	0.170031	2	0.153425	2	0.096294
	3	0.160460	3	0.147526	3	0.095393
	6	0.112837	6	0.041539	6	0.033599
	7	0.047489	7	0.147551	7	0.091918
	10	0.011481	18	0.047938	15	0.031642
	11	0.091537	19	0.119555	20	0.086563
	26	0.152978	55	0.036379	50	0.093896
	27	0.036131	56	0.126714	70	0.006035
	100	0.217056	200	0.179373	100	0.018375
					150	0.086919
					400	0.089018
					900	0.057176
					2000	0.085816
					3000	0.006163
					6000	0.003028
					8000	0.118165
ρ_{av}	10.9375		12.0000		18.5000	
σ	0.97592		0.97704		0.9781869	
SNR_{norm}	0.0247		0.0147		0.00450	



[CFRU01] S.-Y. Chung, G. D. Forney, Jr., T. J. Richardson, and R. Urbanke, “On the Design of Low-Density Parity-Check Codes within 0.0045 dB of the Shannon Limit”, *IEEE Comm. Letters*, vol. 5 no. 2, Feb. 2001.

Error Floors of LDPC Codes

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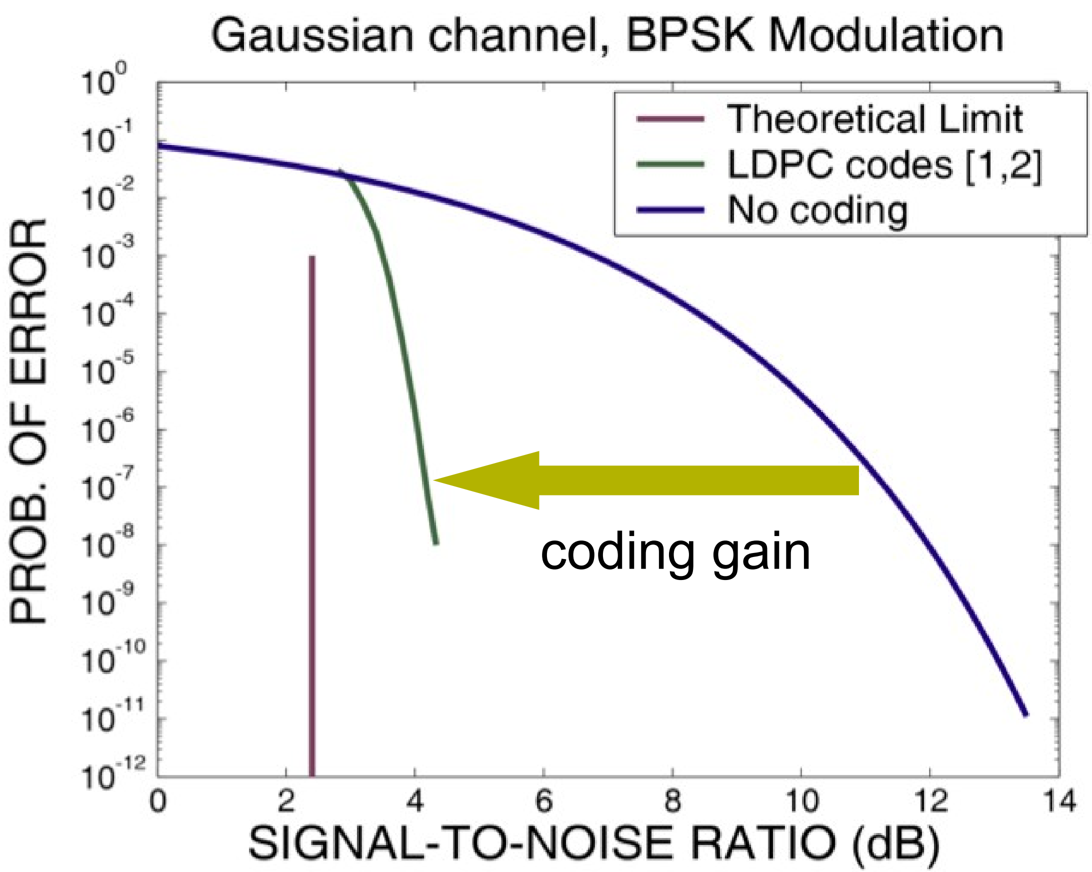

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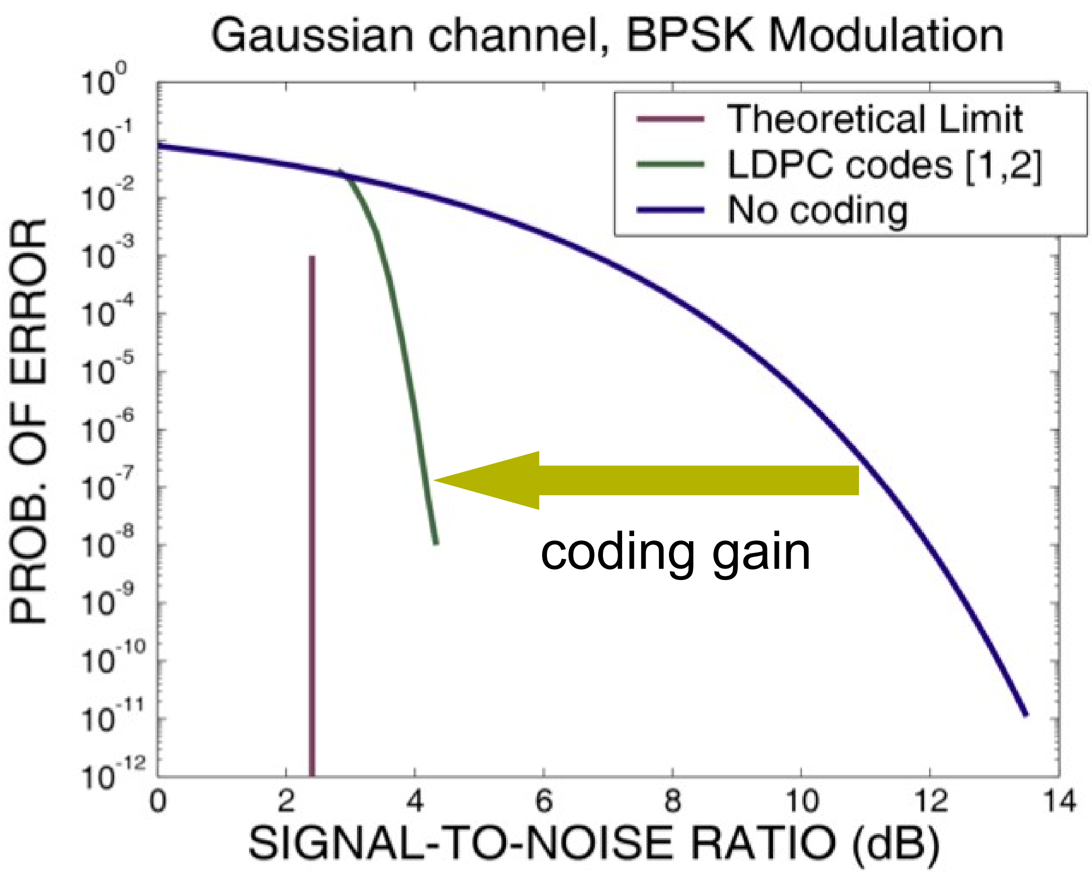

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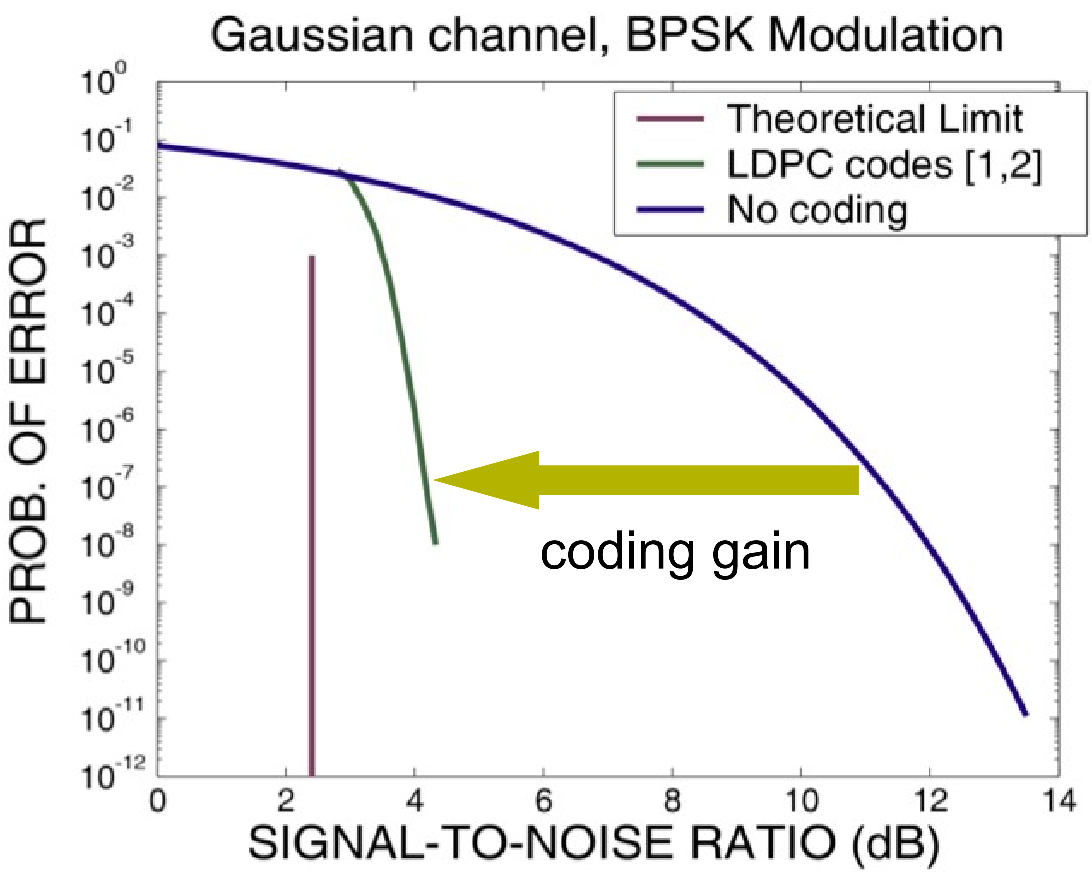

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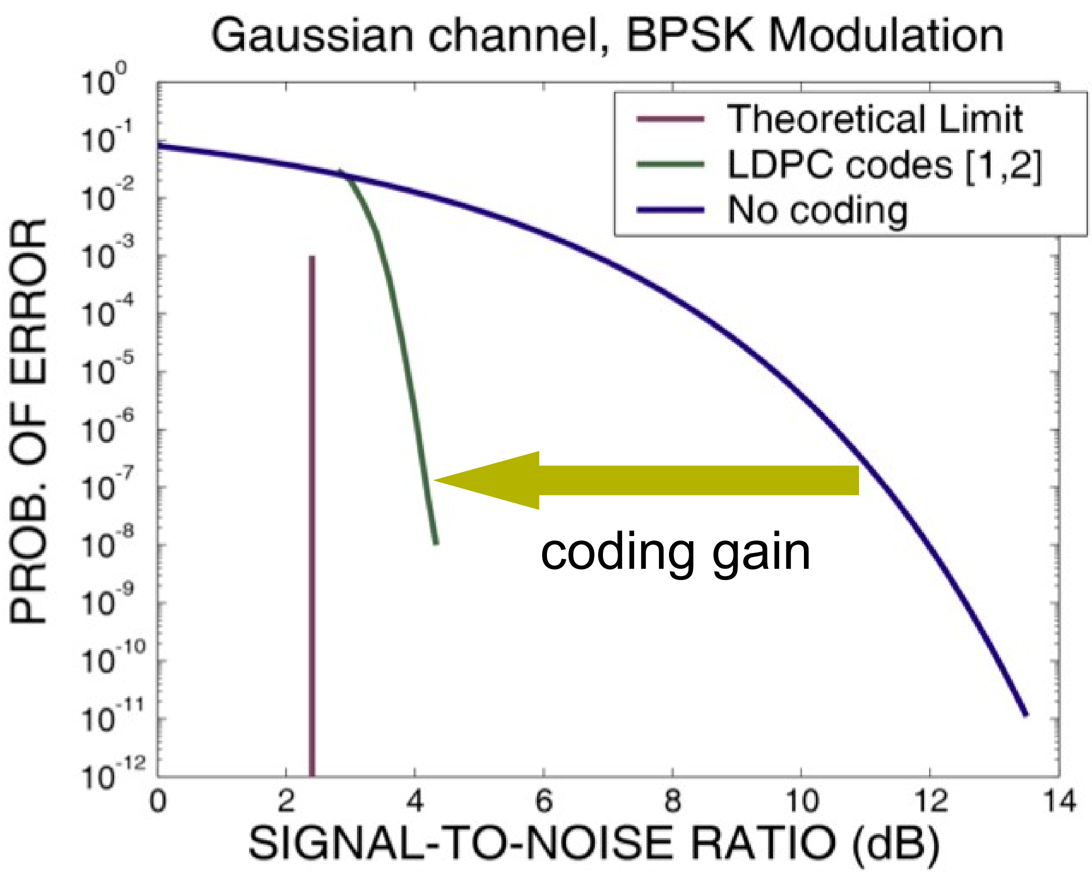

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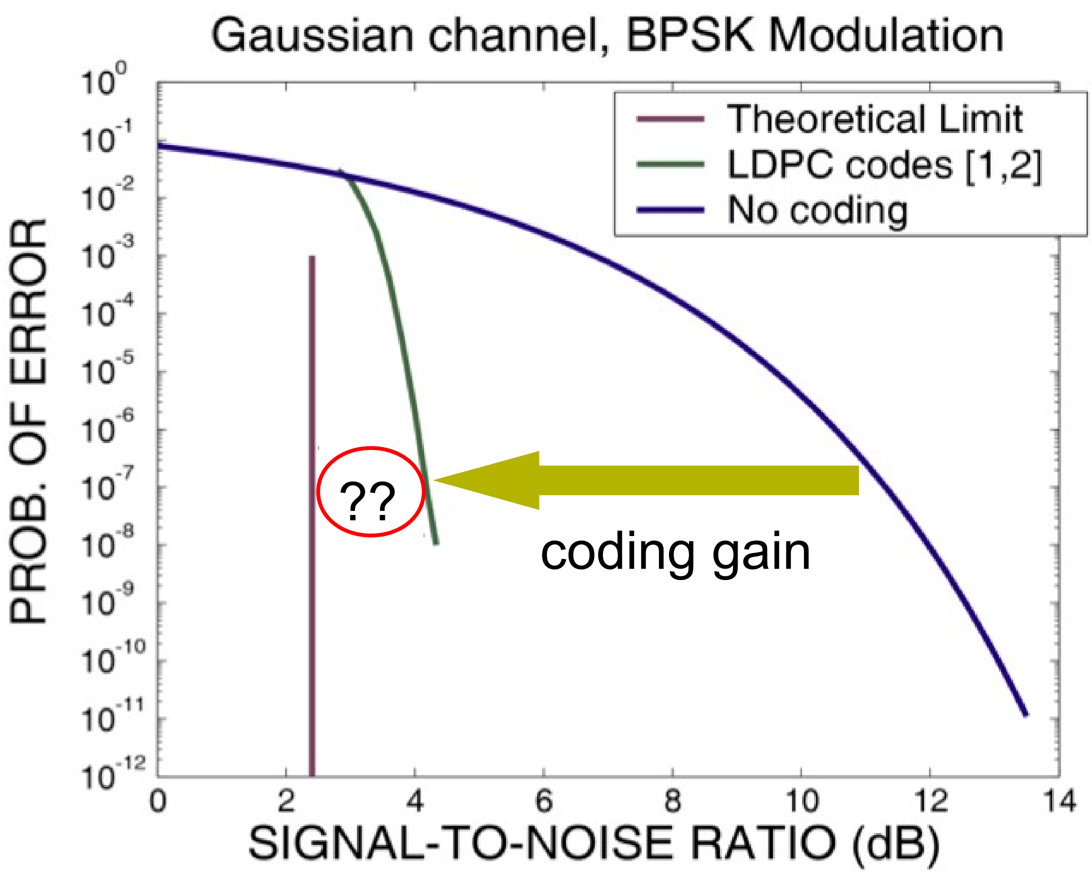

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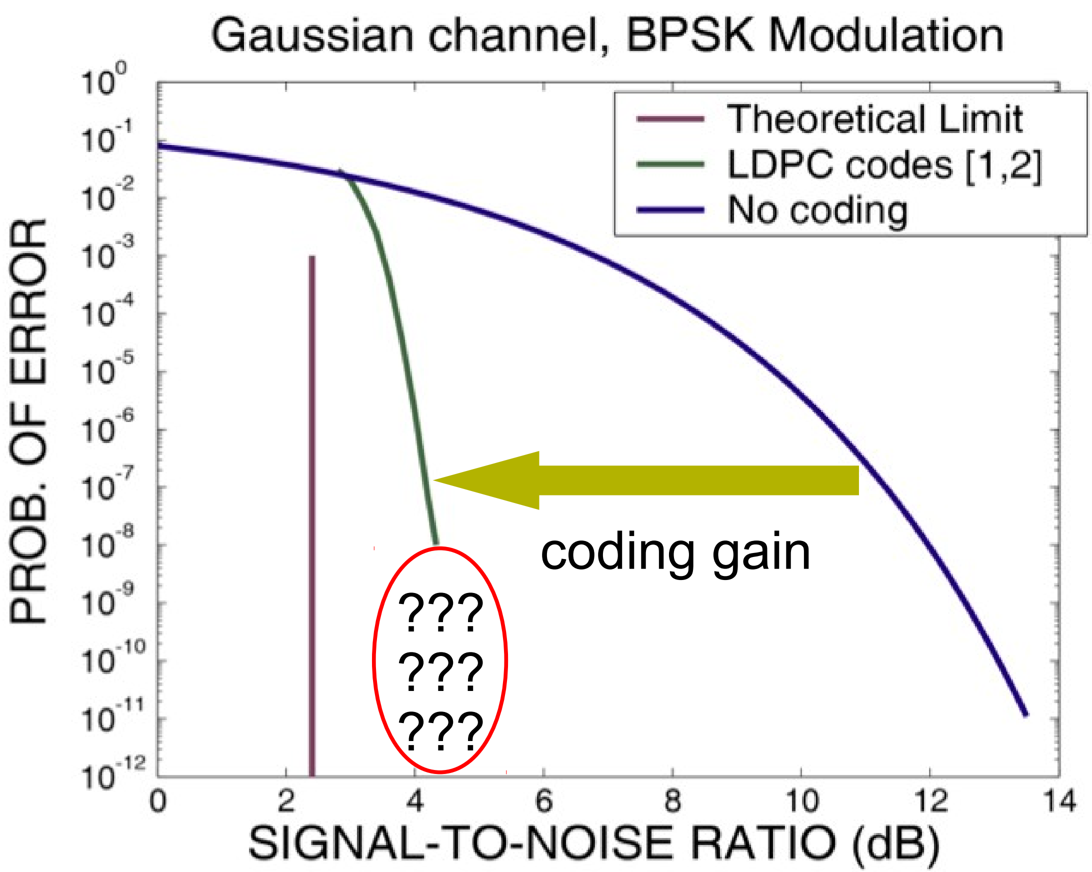

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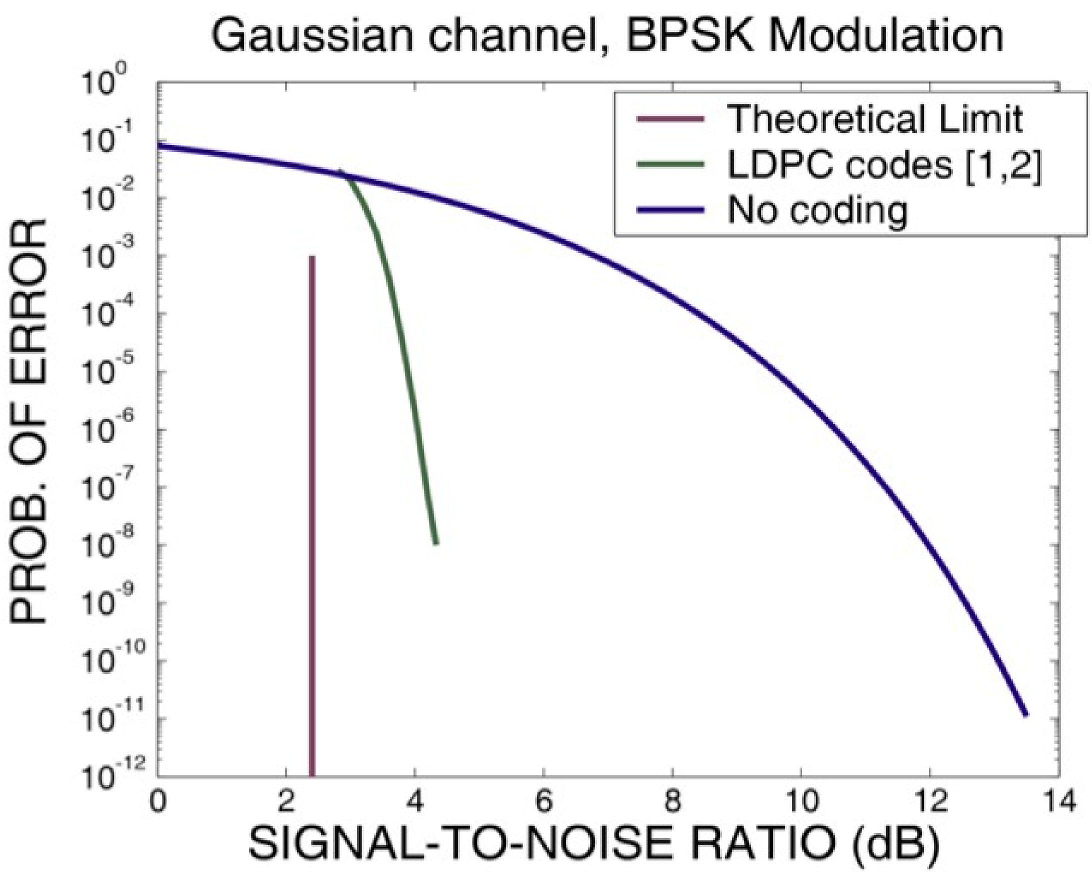

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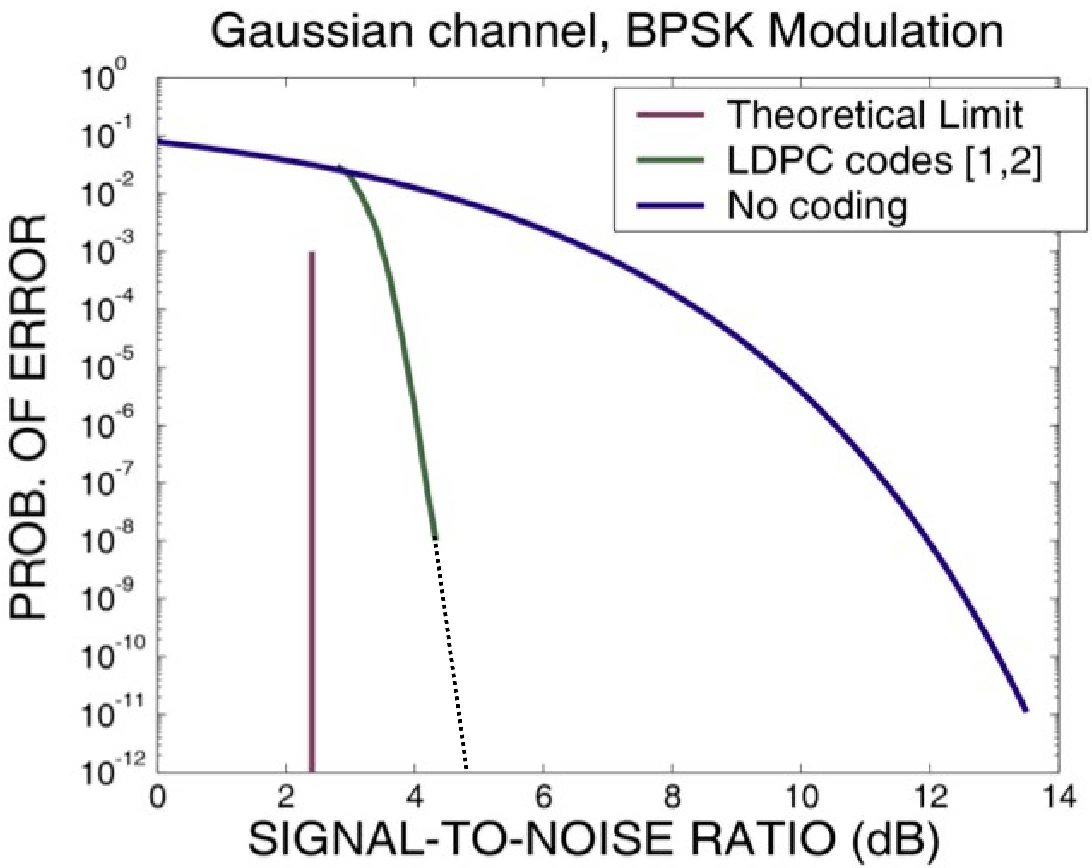

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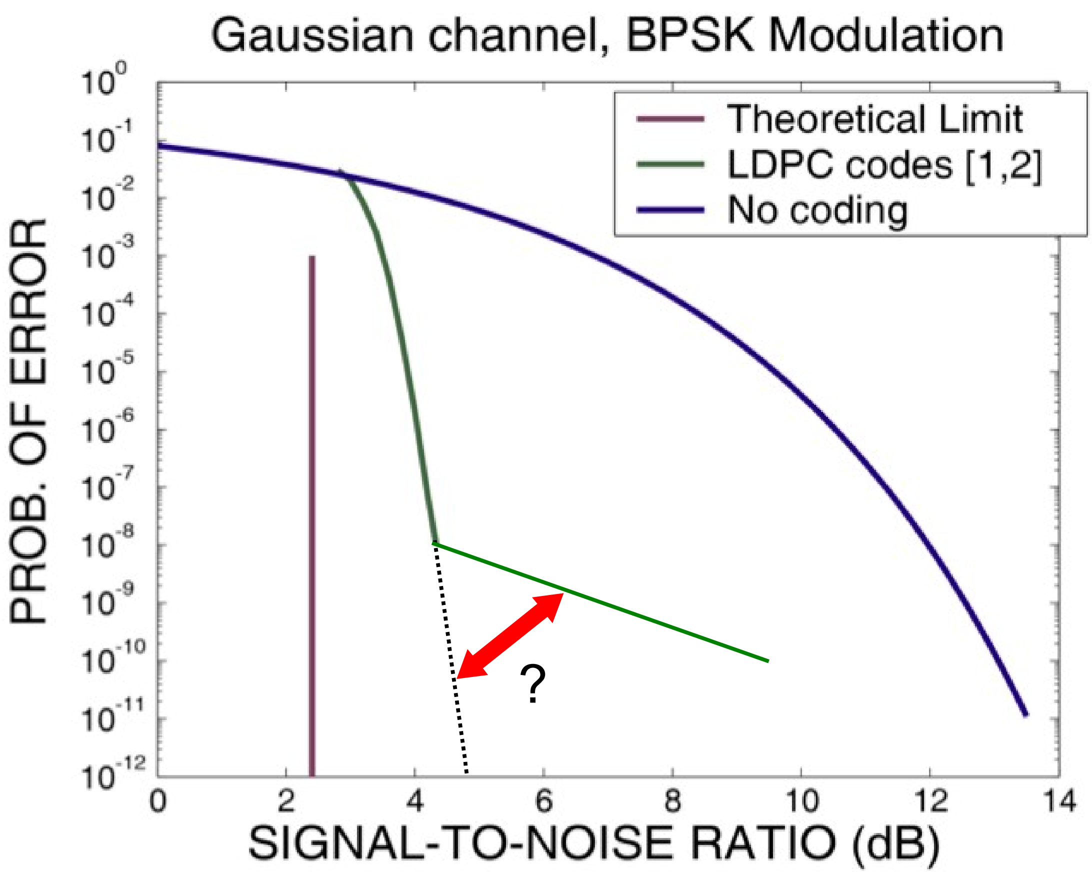

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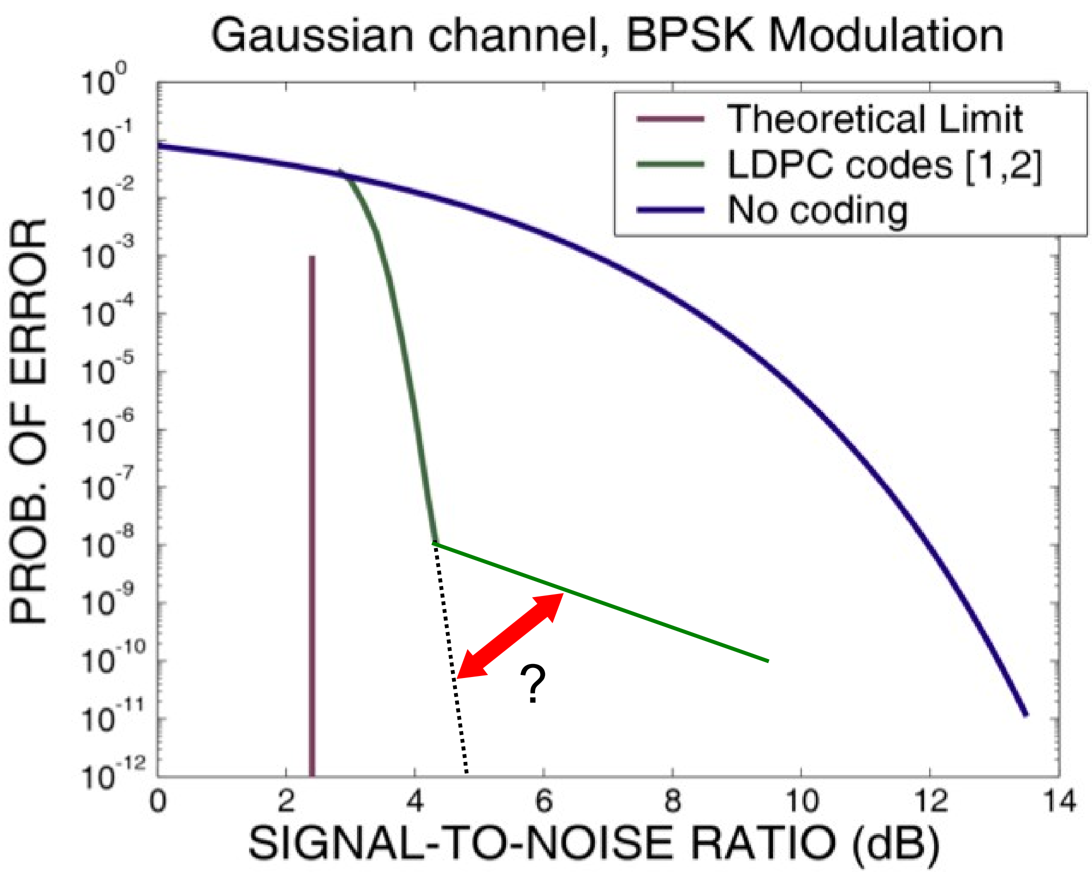

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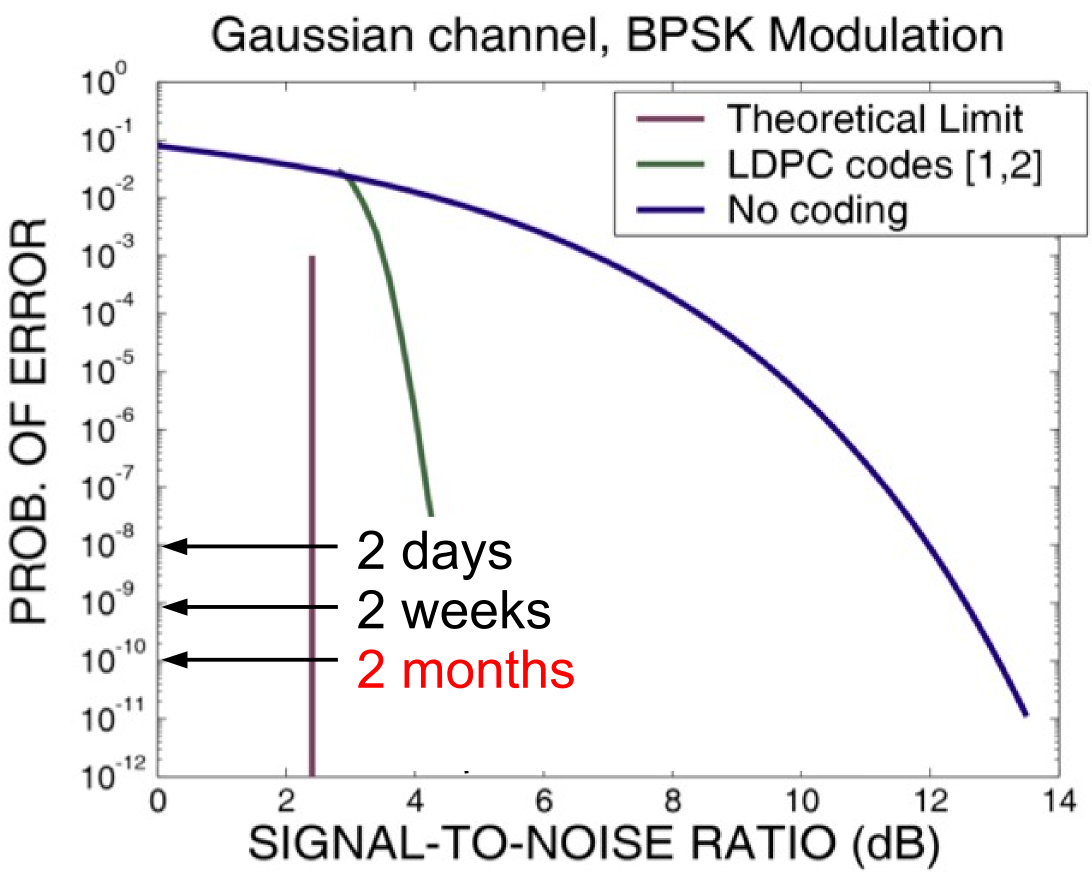

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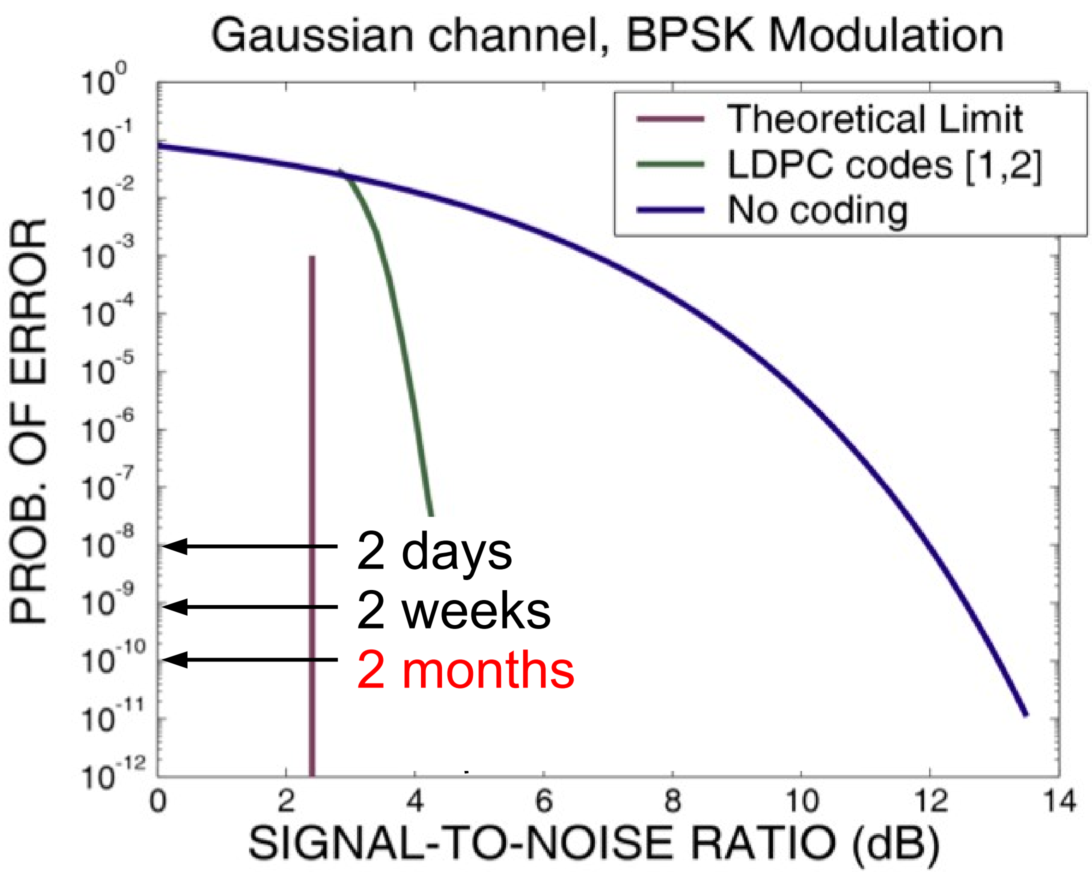

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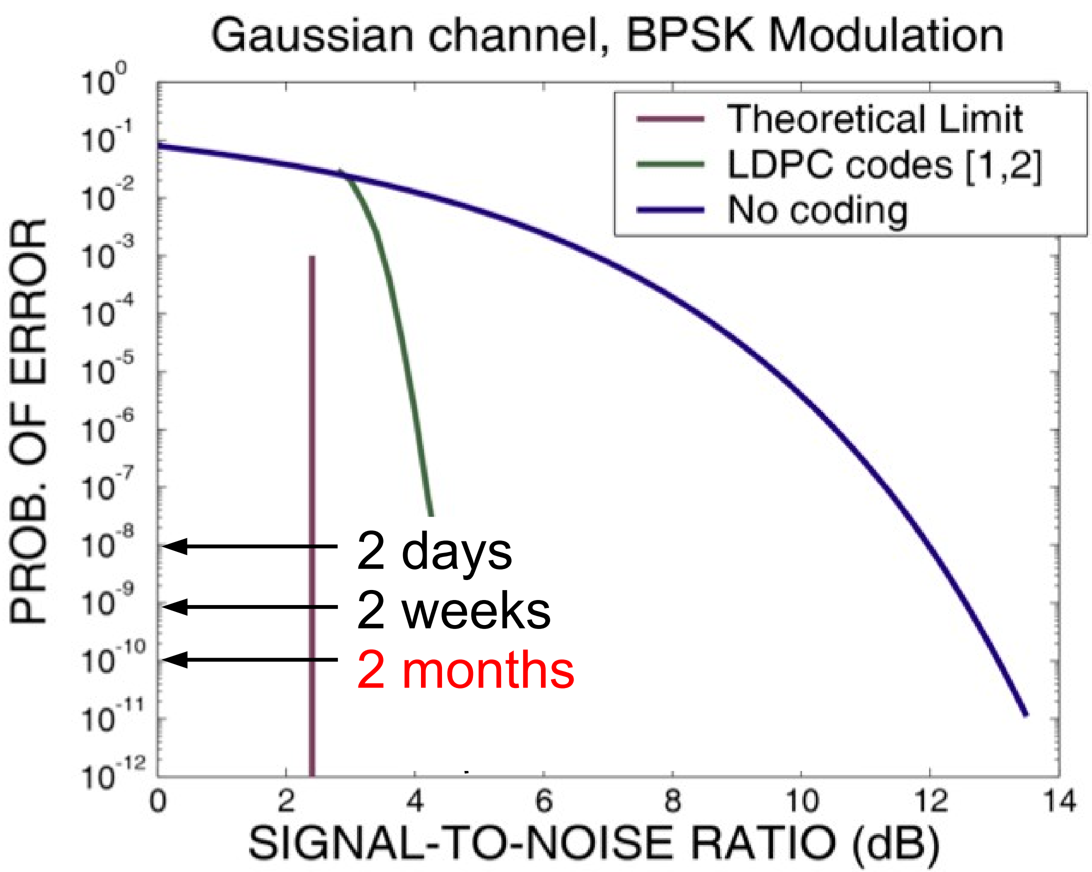

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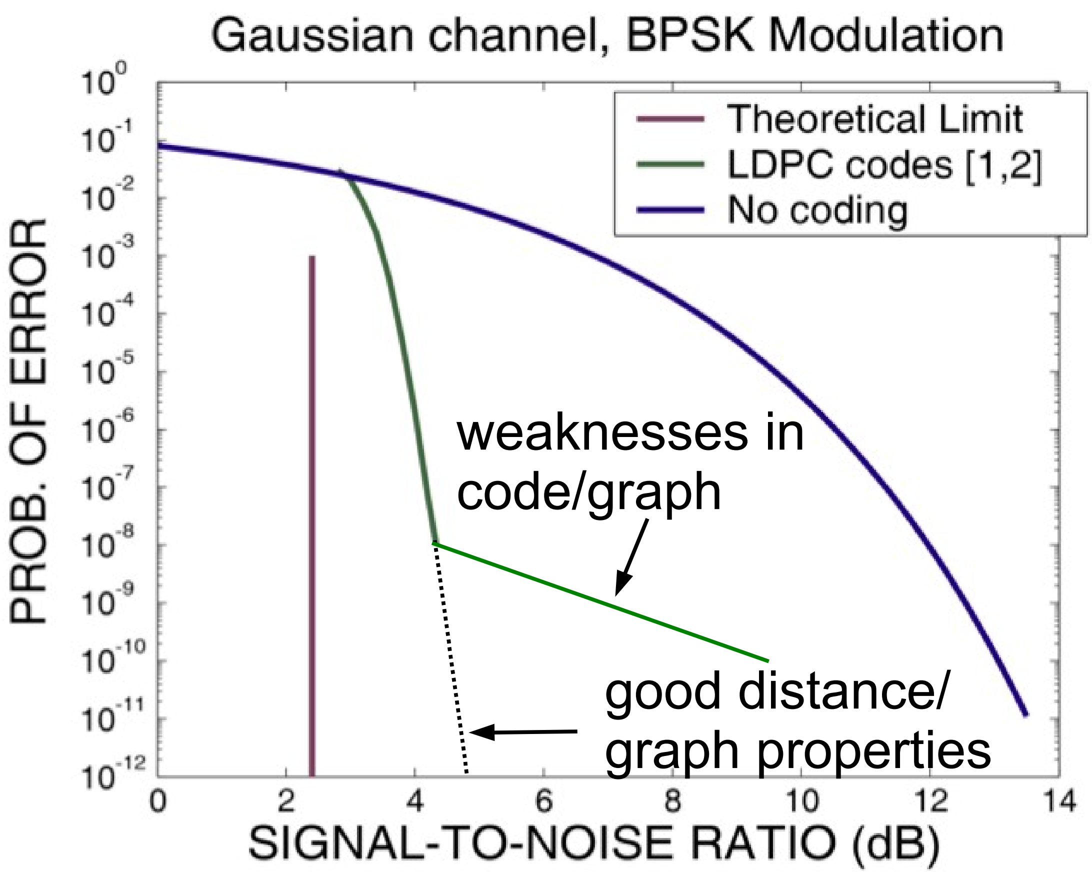

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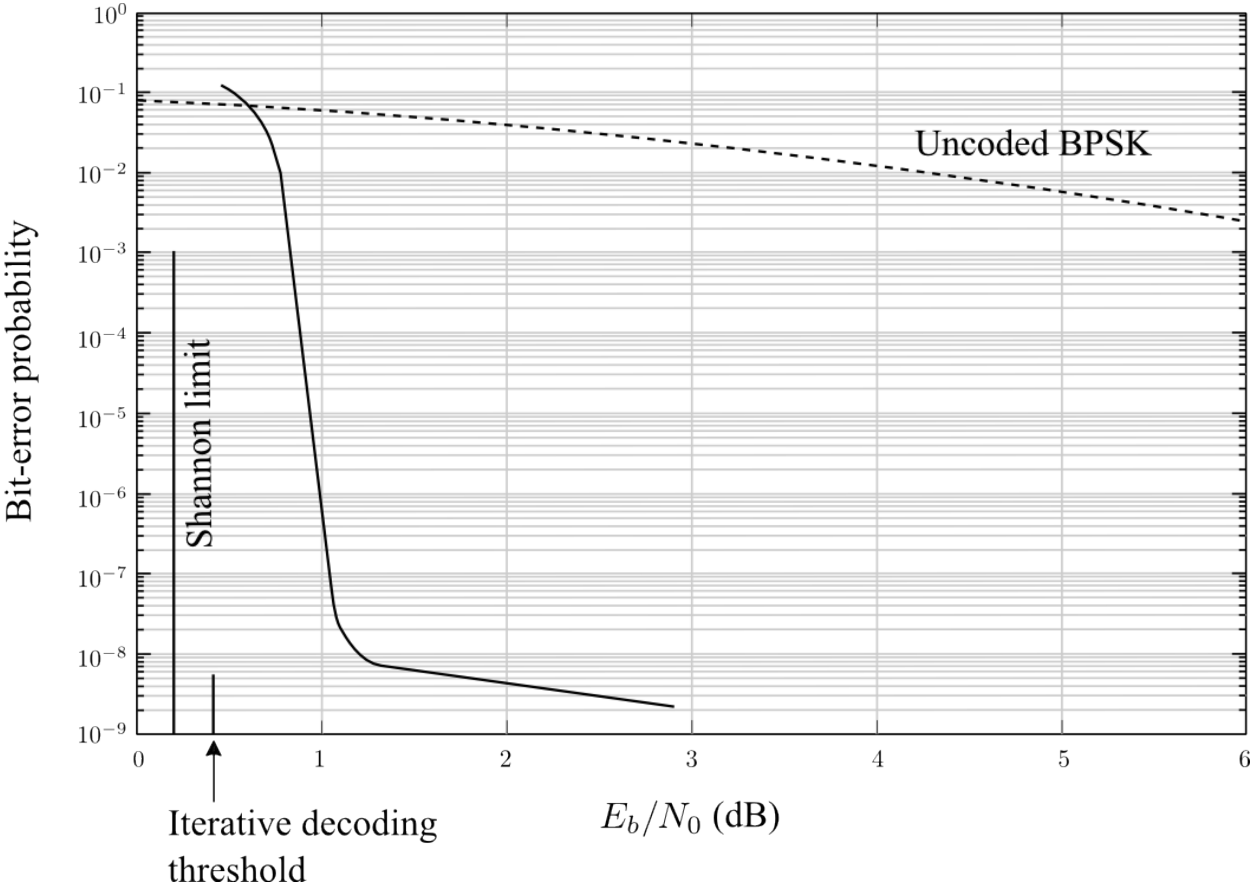
Typical LDPC Code Behavior



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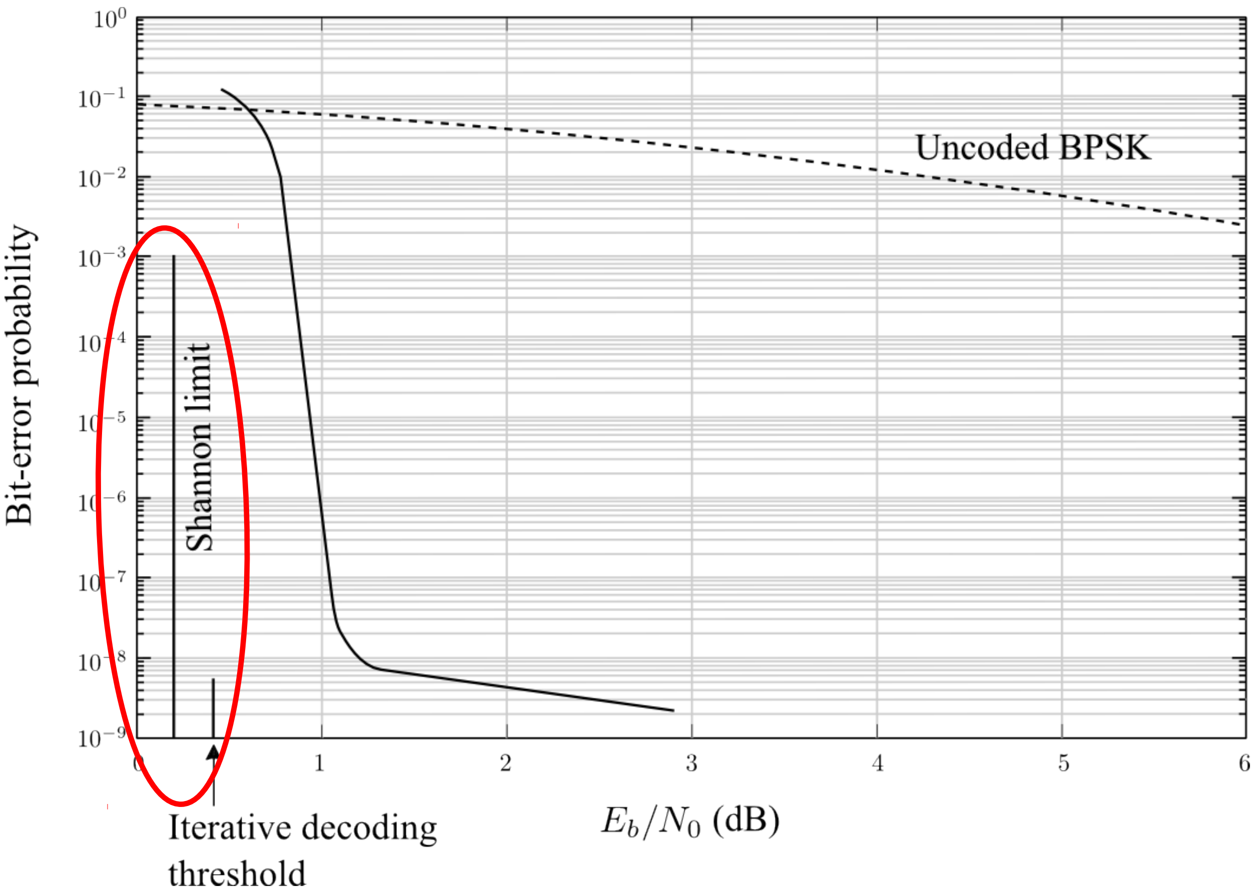

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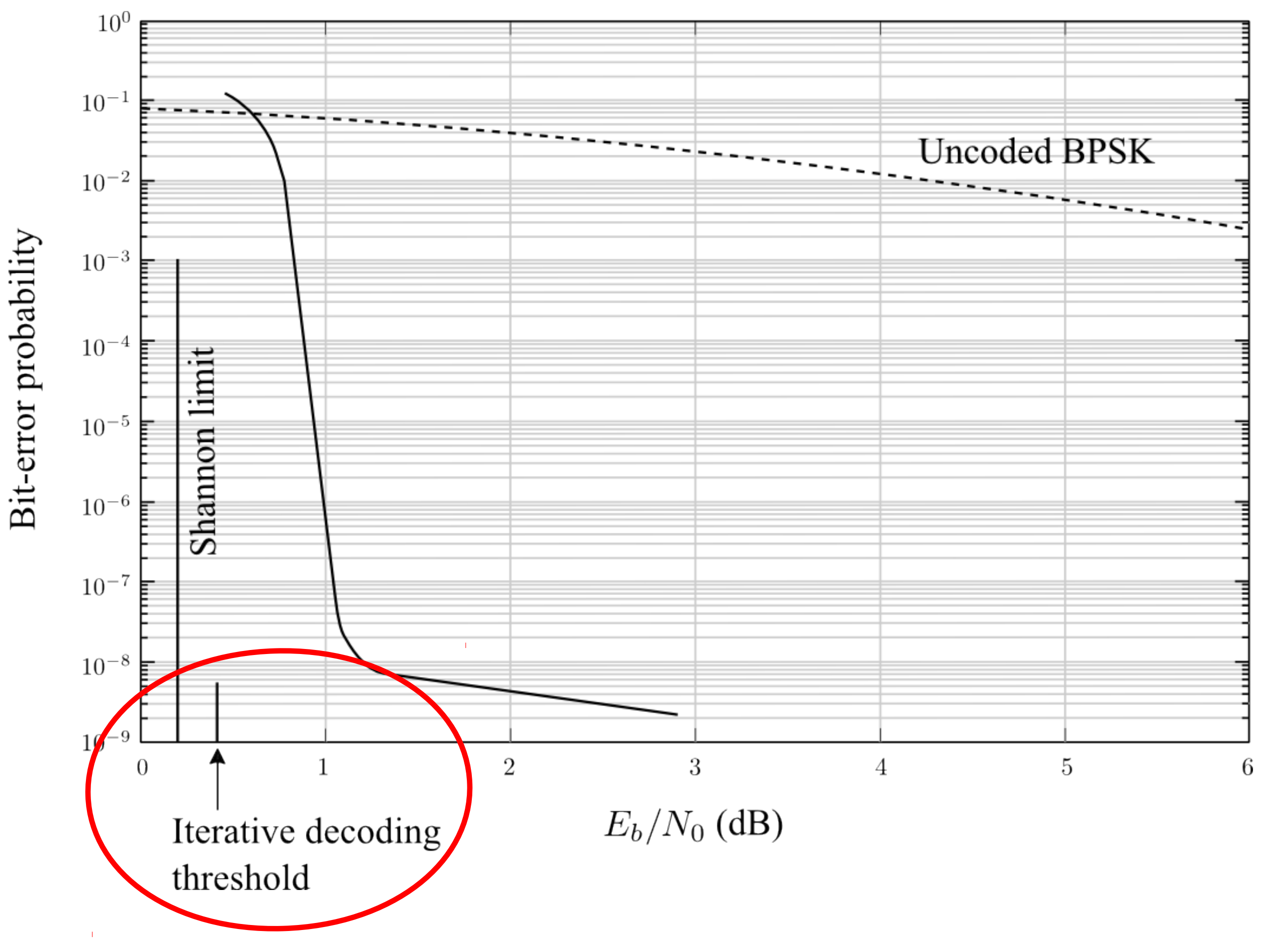

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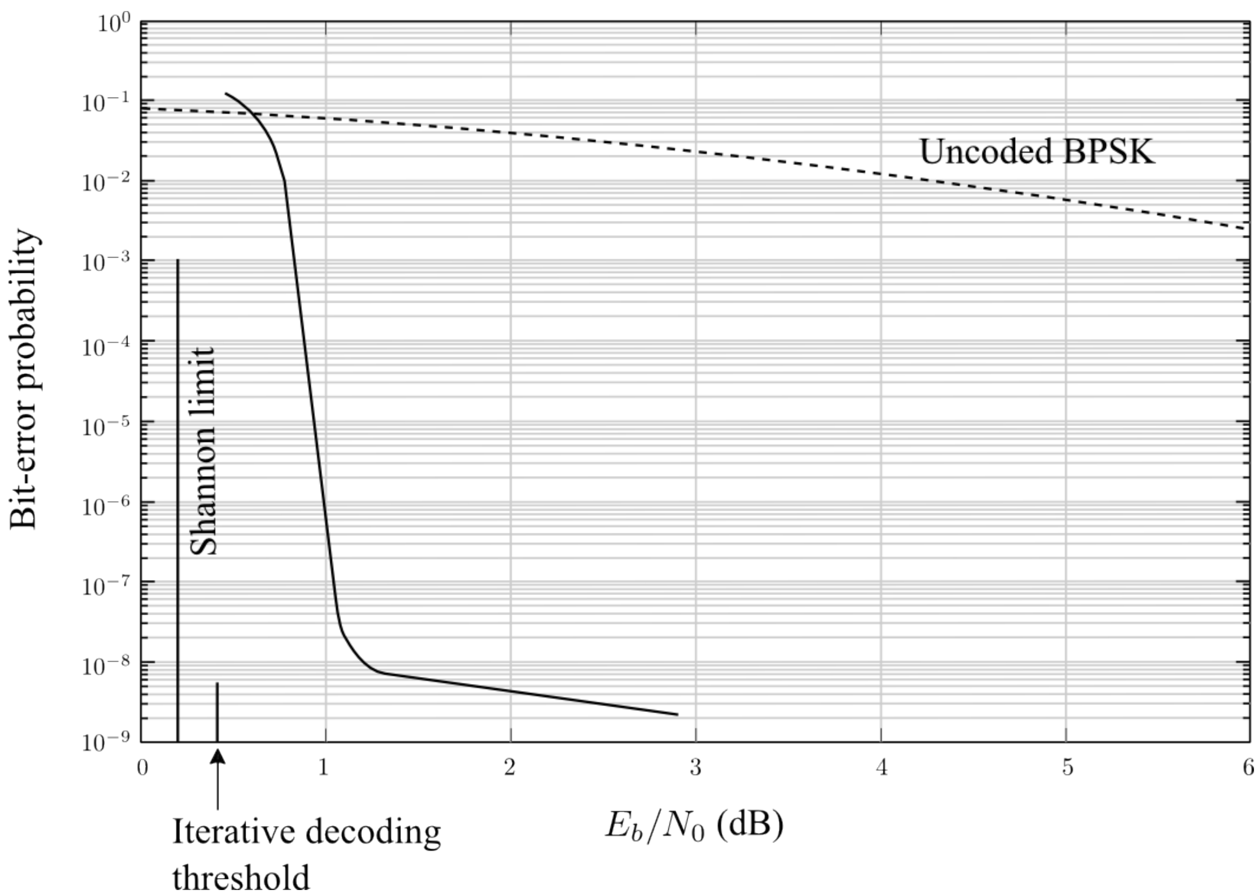

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- With iterative decoding, codes typically display:

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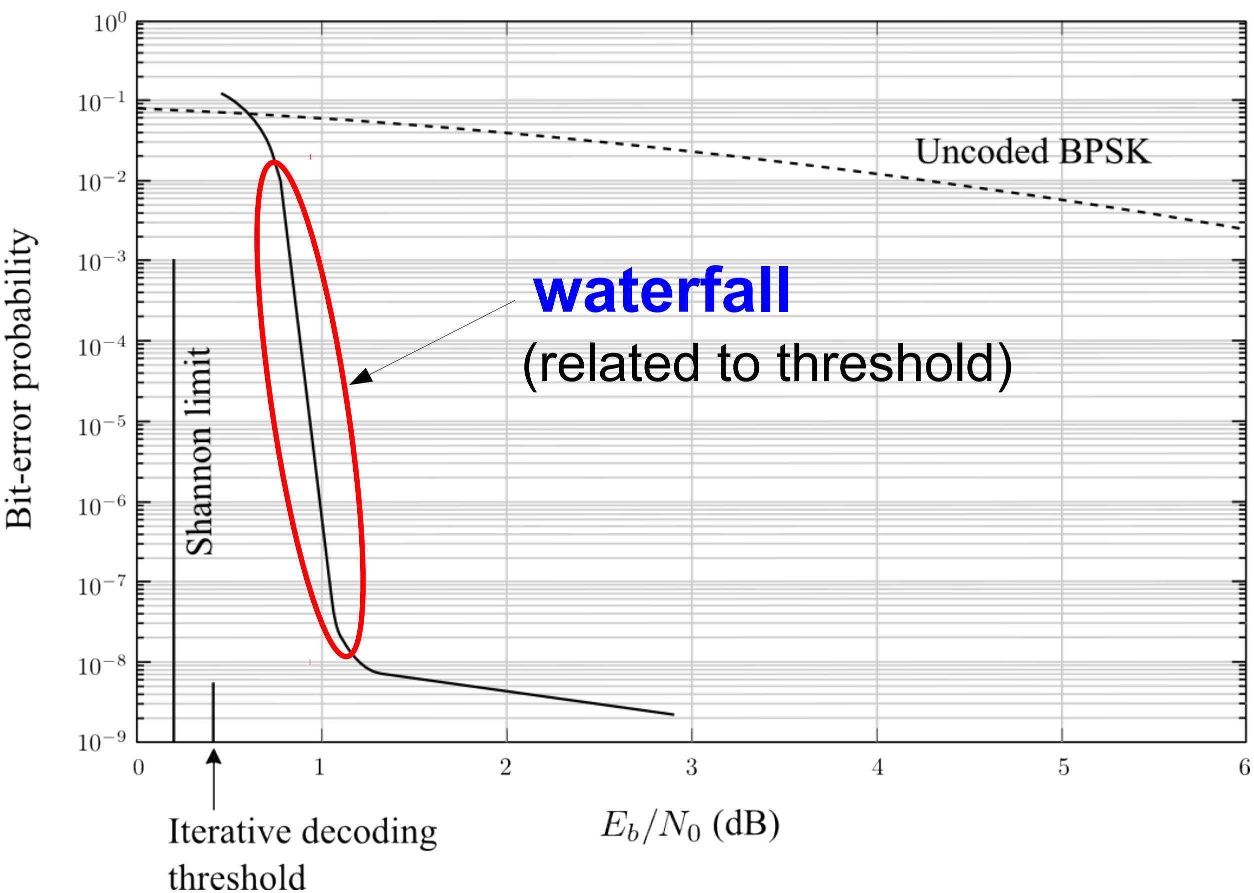

- The **iterative decoding threshold** depends on:
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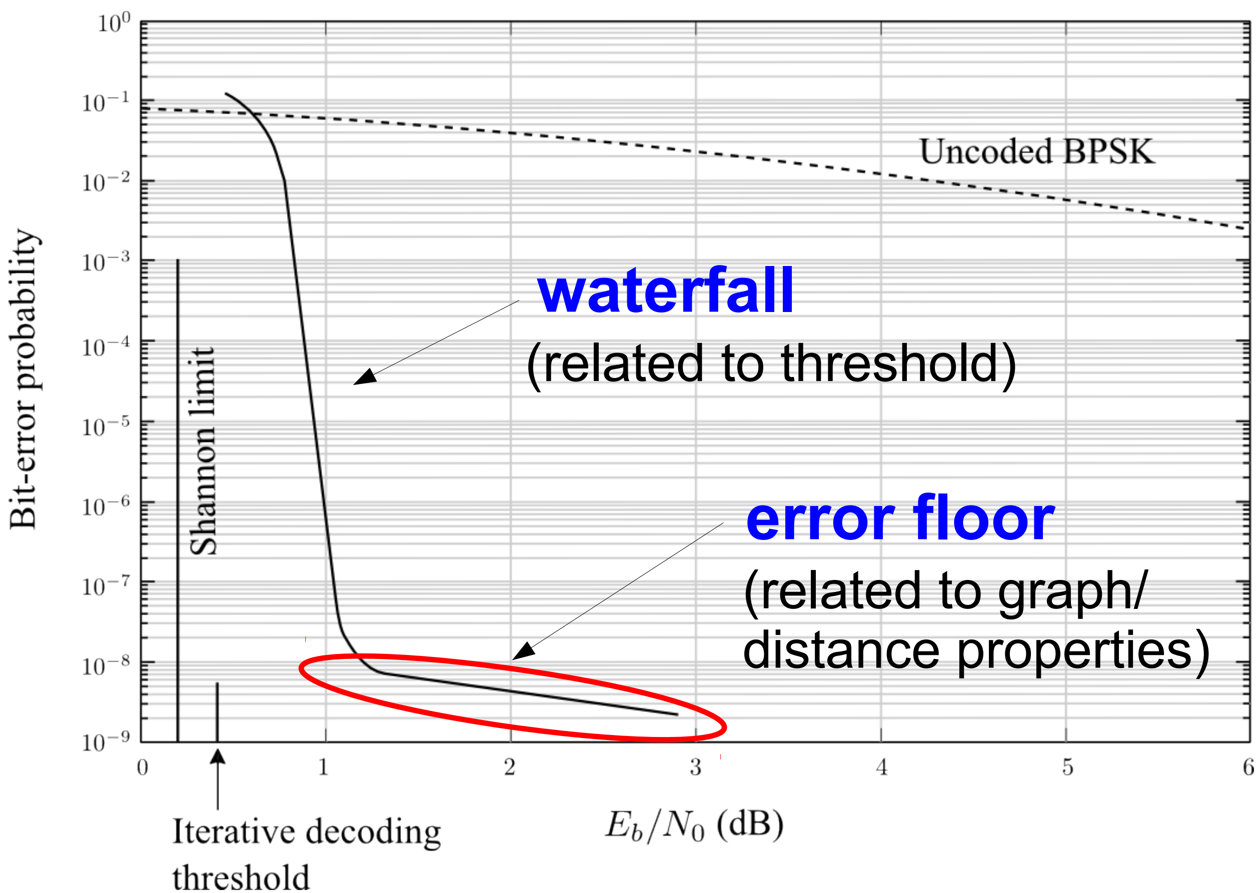

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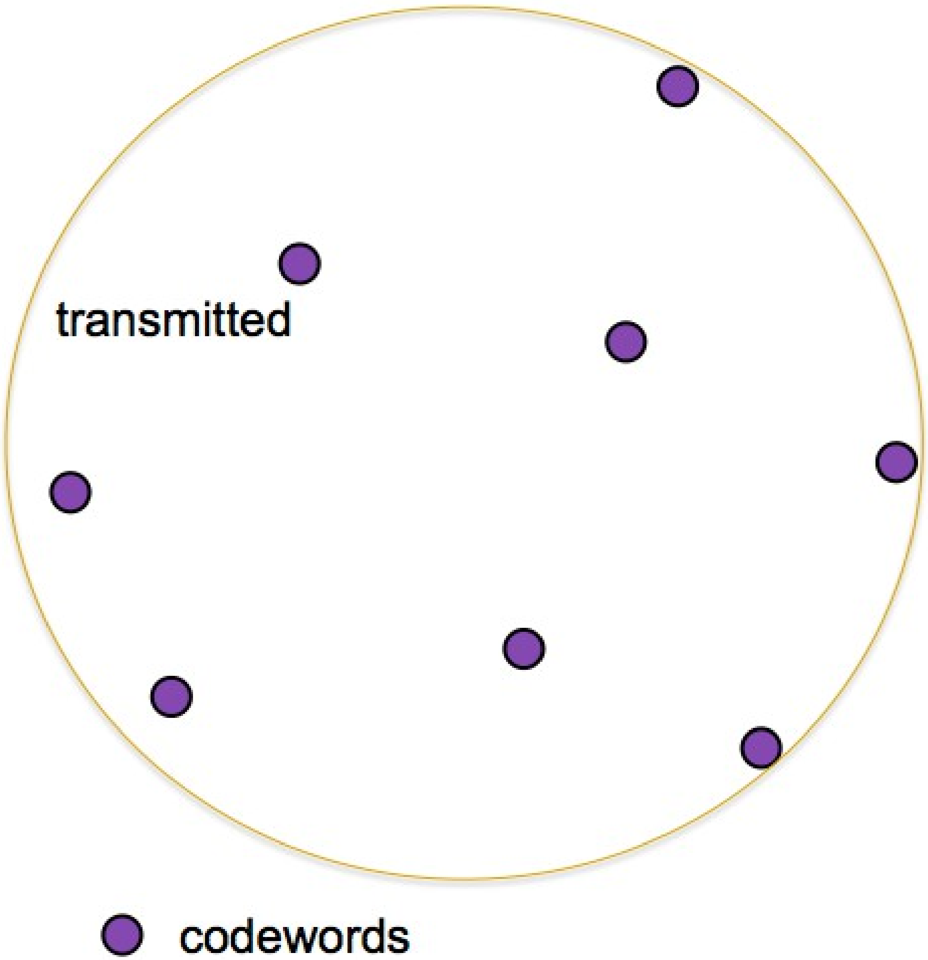
What causes decoding errors?



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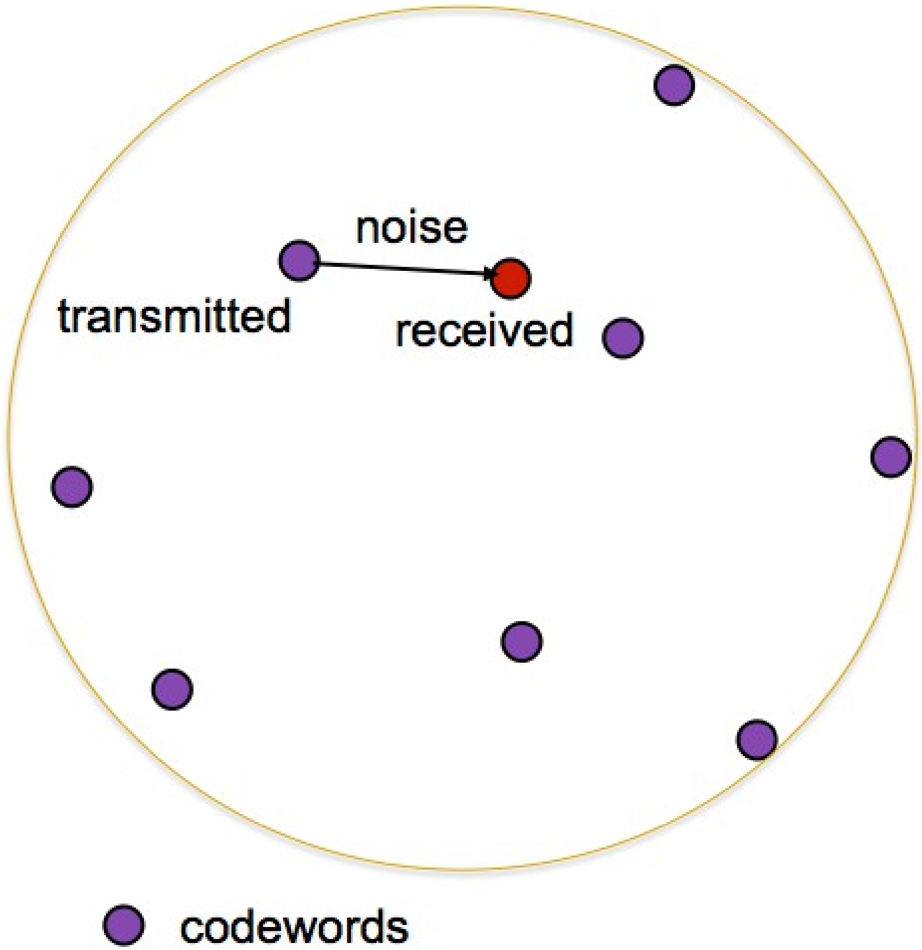

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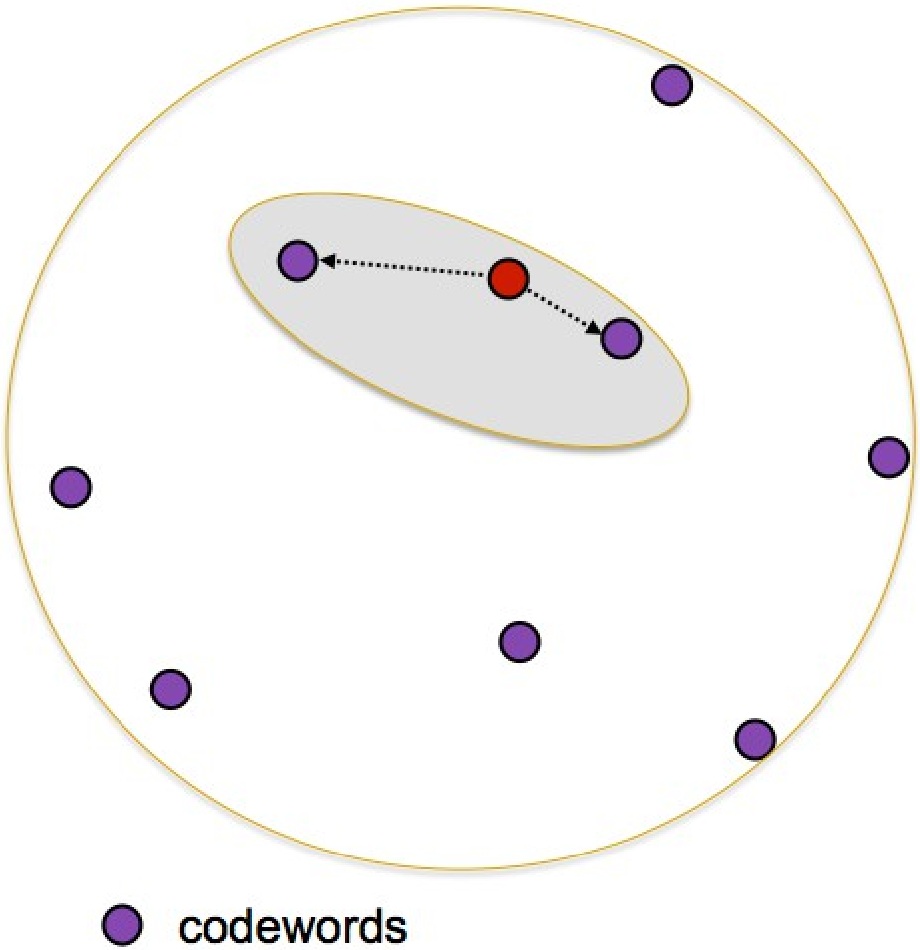

- Optimal choice:
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What causes decoding errors?

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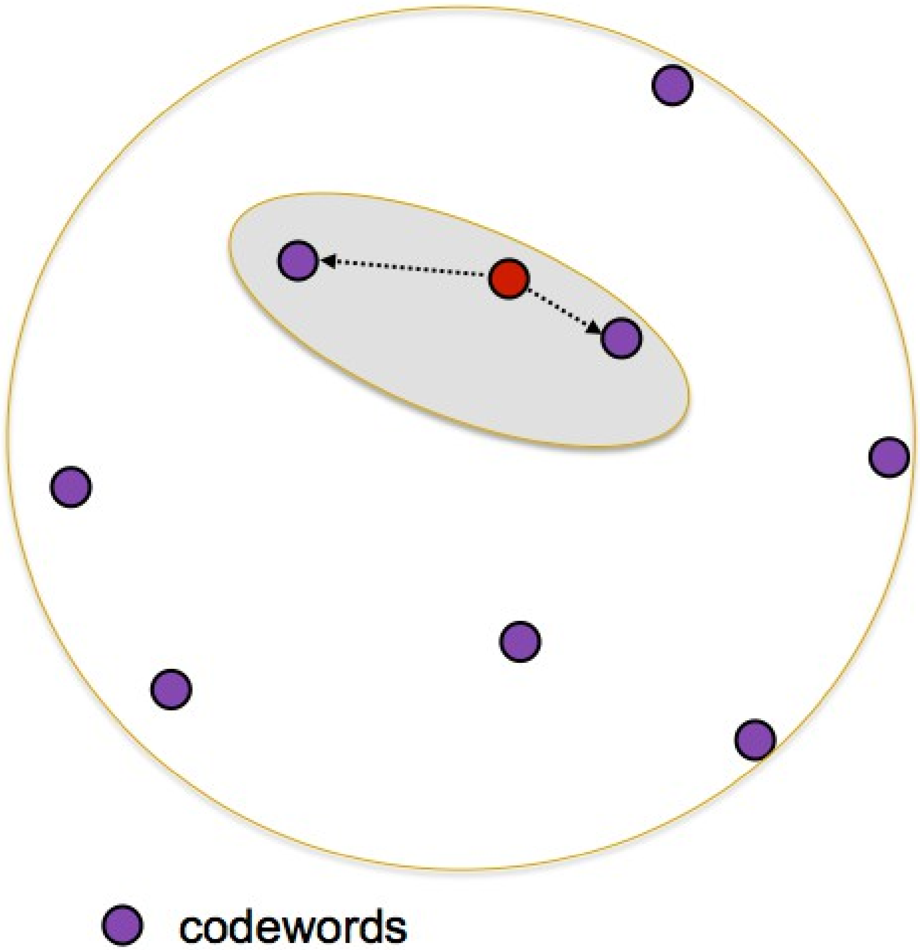

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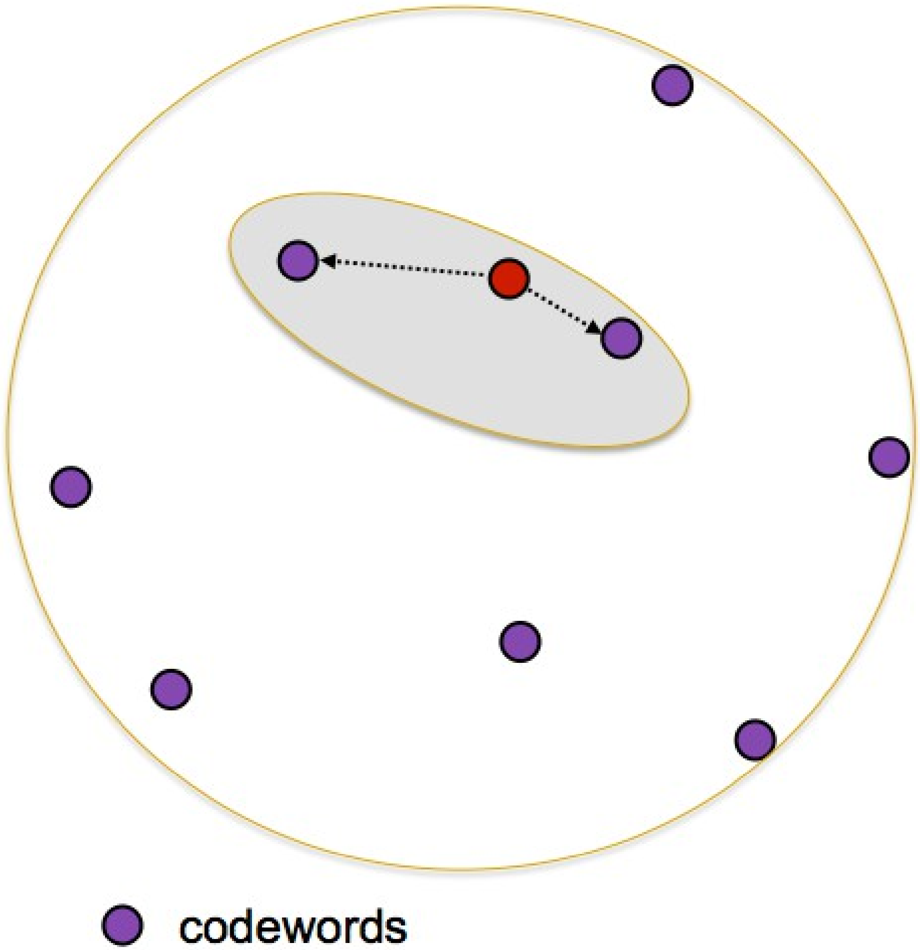

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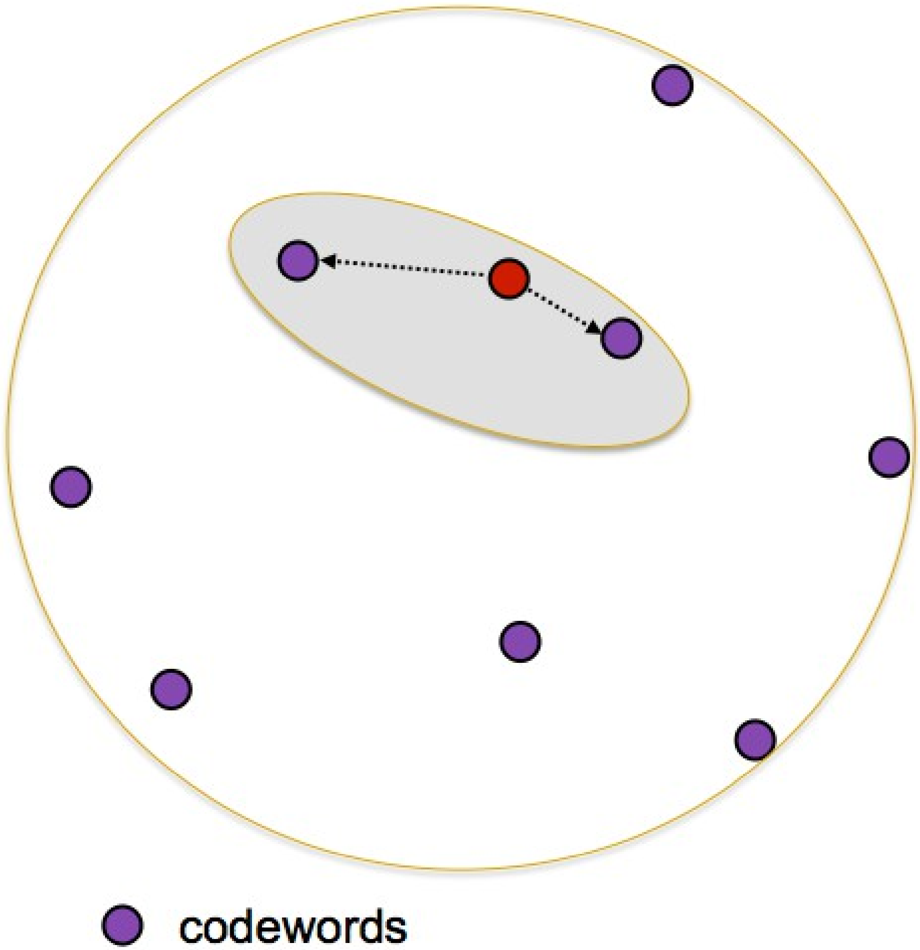

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What causes decoding errors?

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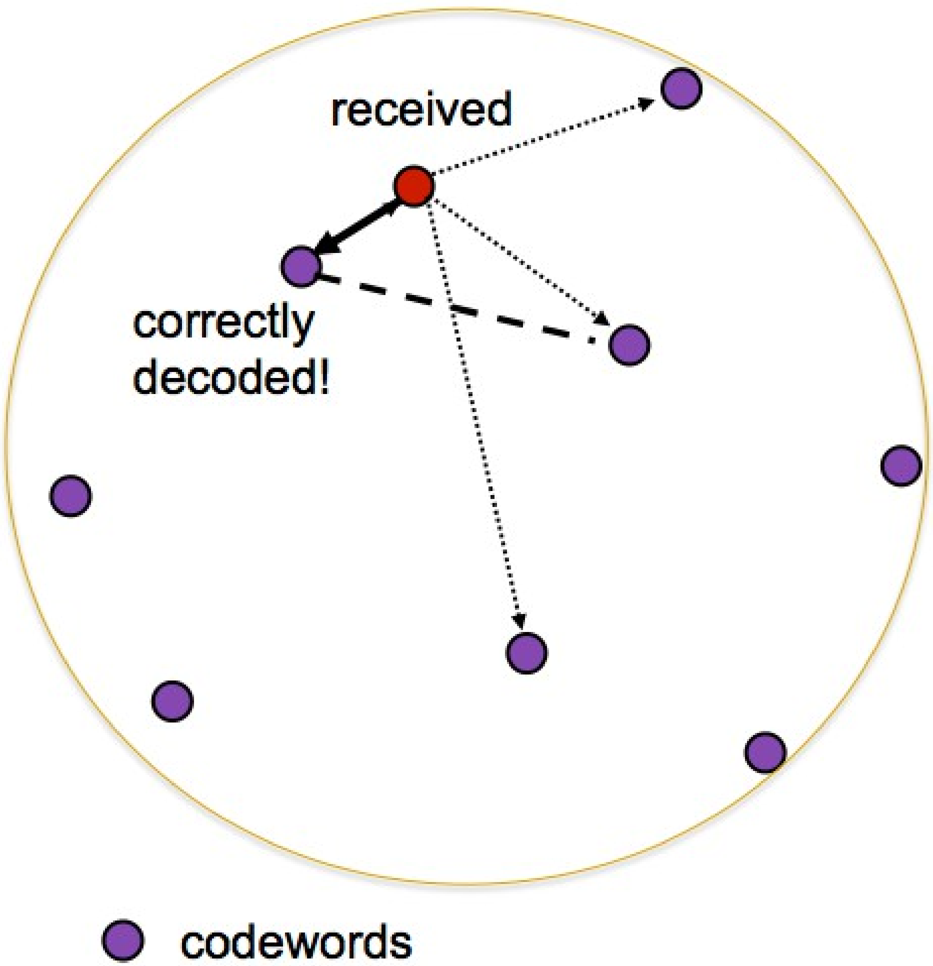

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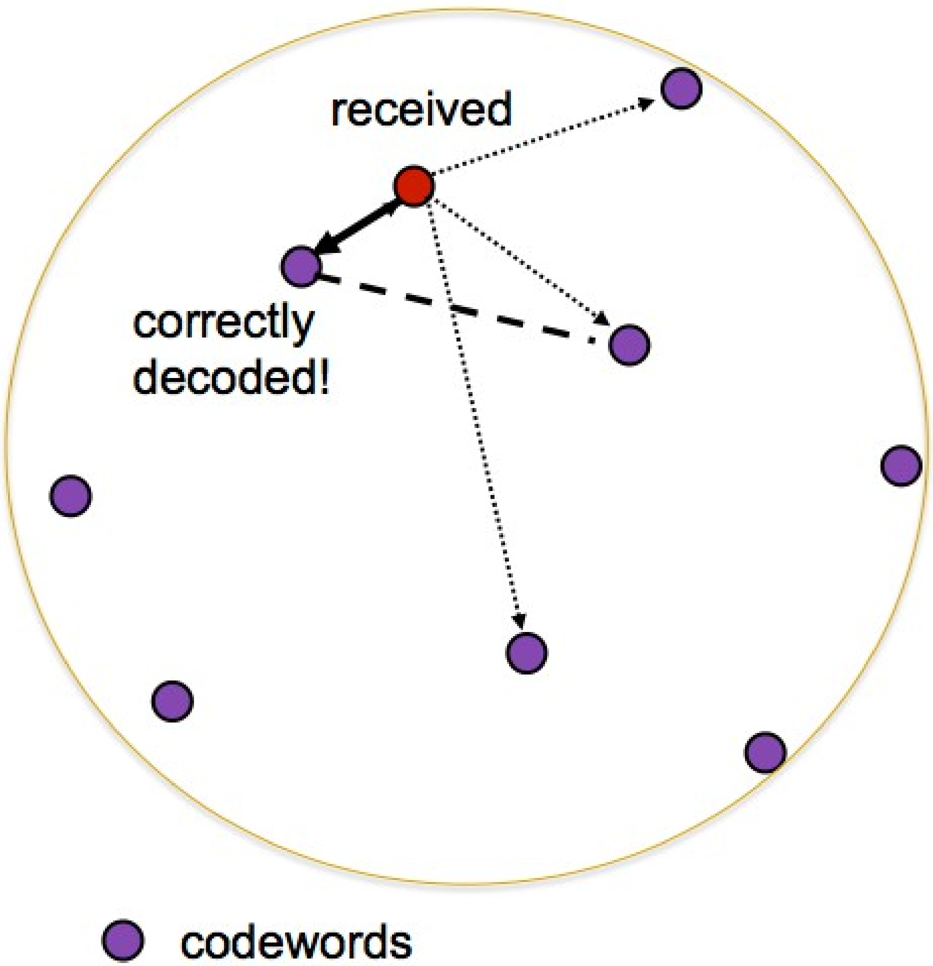
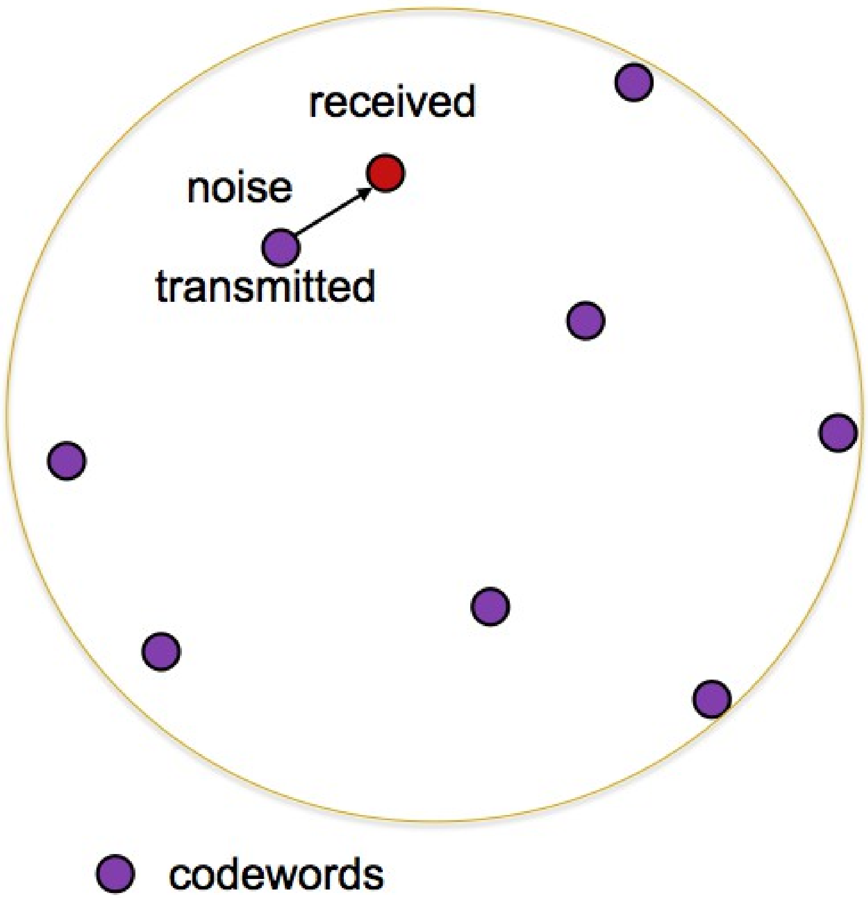


- The minimum distance is the minimum separation between any two codewords

→ Large minimum distance implies high probability of successful decoding!

Back to iterative decoding...



- Codewords are fixed points, but are there other fixed points?

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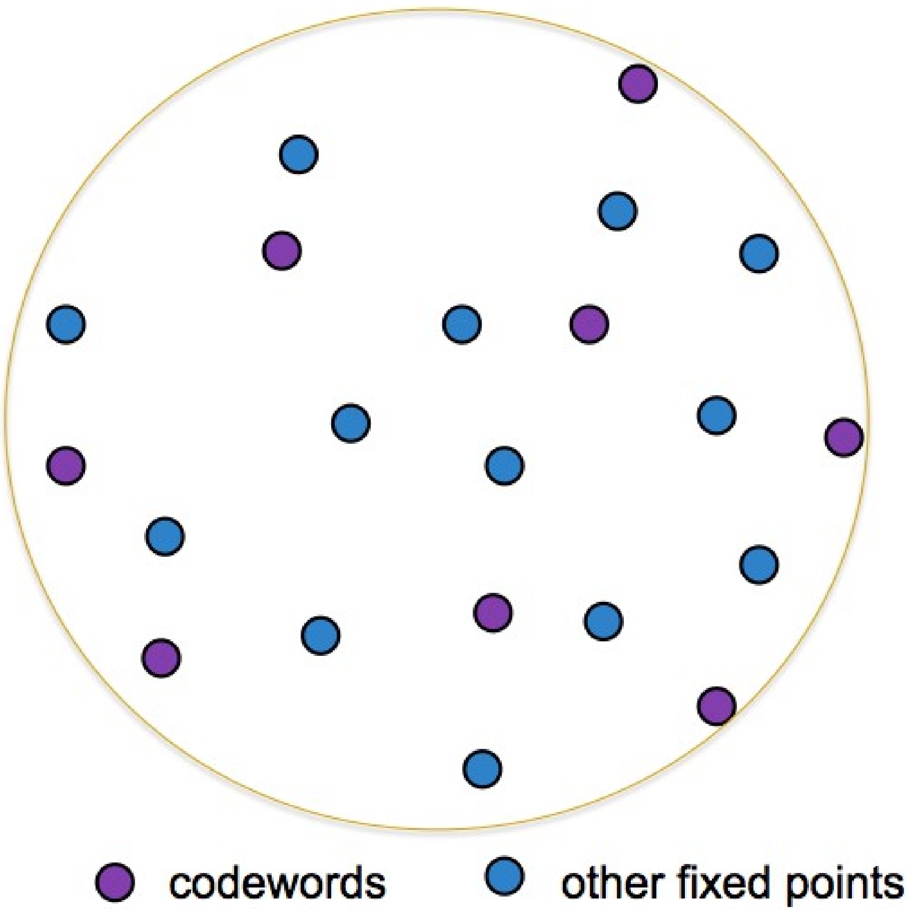

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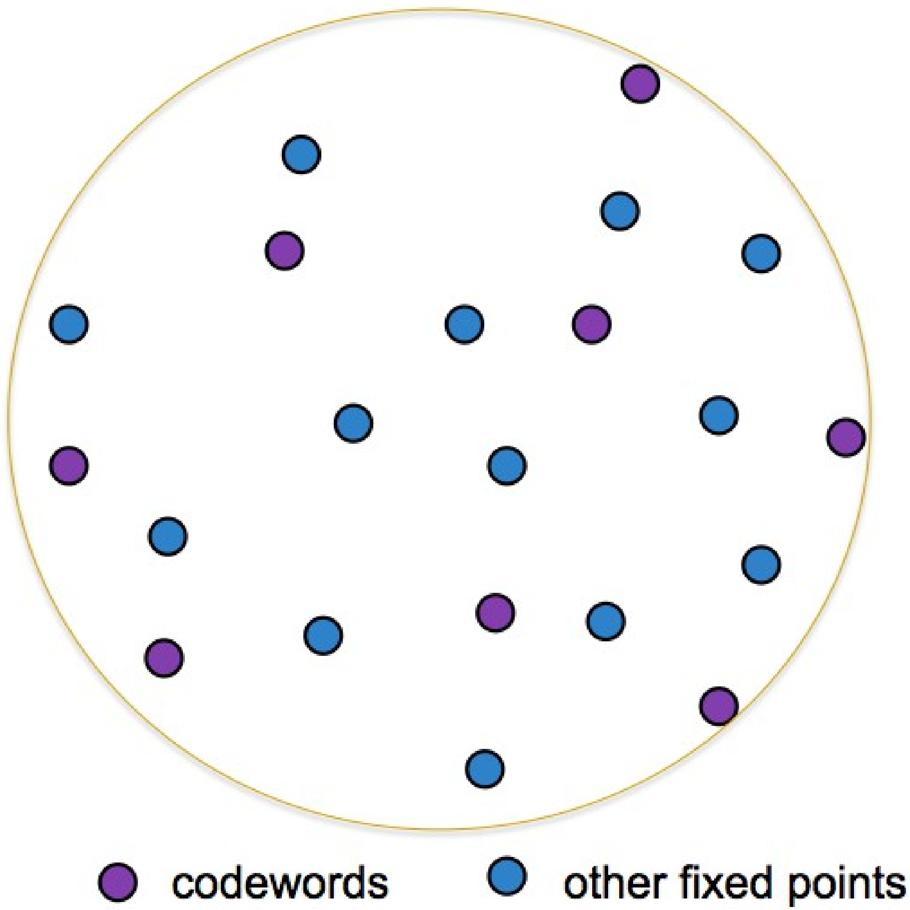

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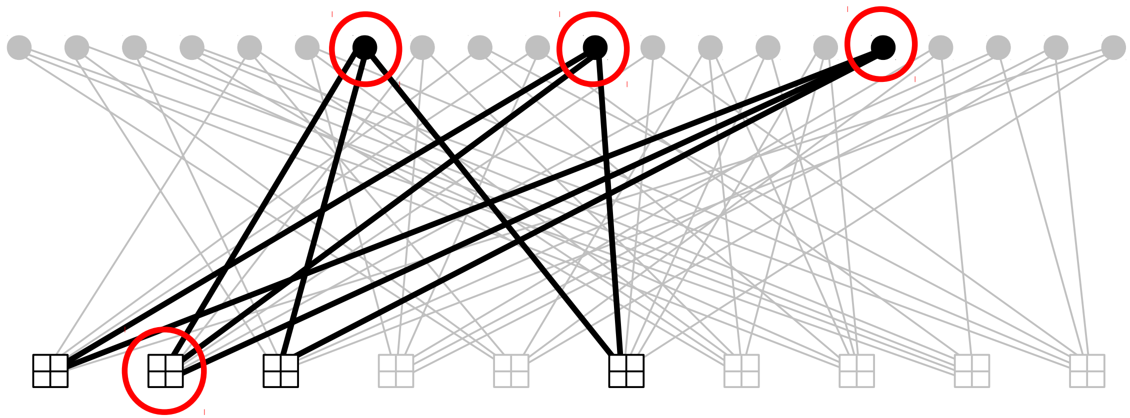
- What are these other fixed points? How many are there? How can we avoid them?

Error Floor Events - AWGNC

- On the AWGNC, failures are attributed to **trapping sets** [Richardson '03].

Definition: An (a,b) general trapping set $\tau_{a,b}$ is a set of a variable nodes which induce a subgraph with exactly b odd-degree check nodes.



A $(3,1)$ trapping set in a $(3,6)$-regular Tanner graph

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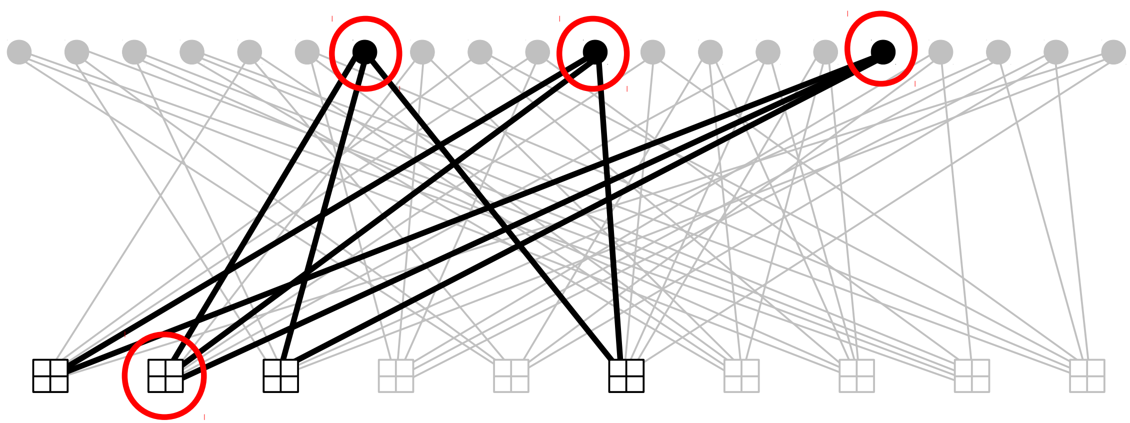

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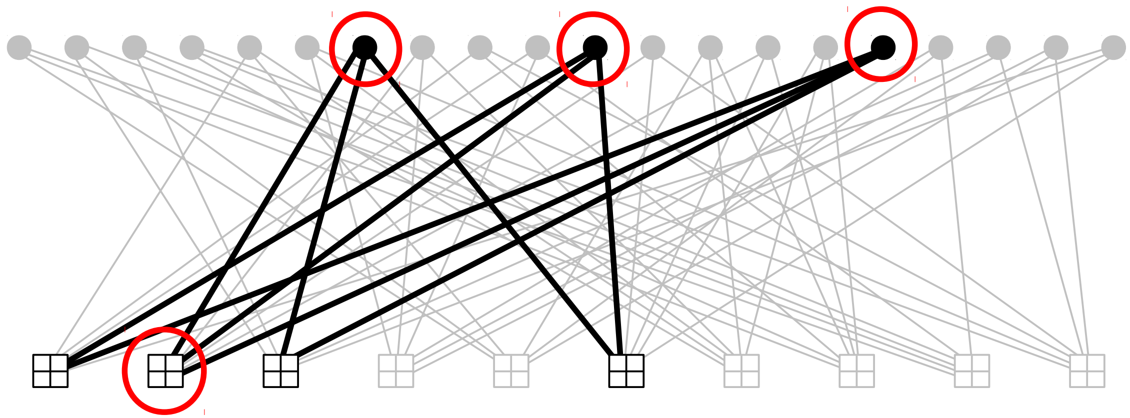

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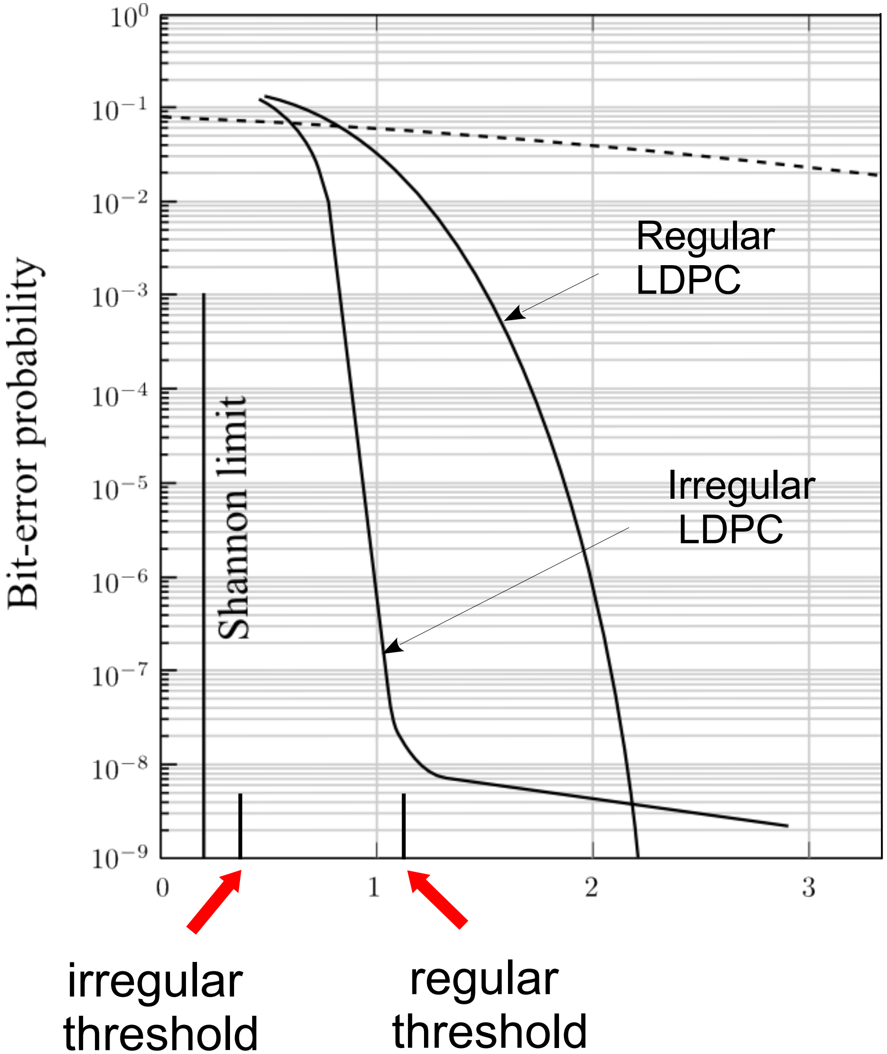
- Certain types of trapping sets with small a and b, such as **elementary trapping sets** and **absorbing sets**, are known to be particularly harmful.

Regular vs. Irregular LDPC codes



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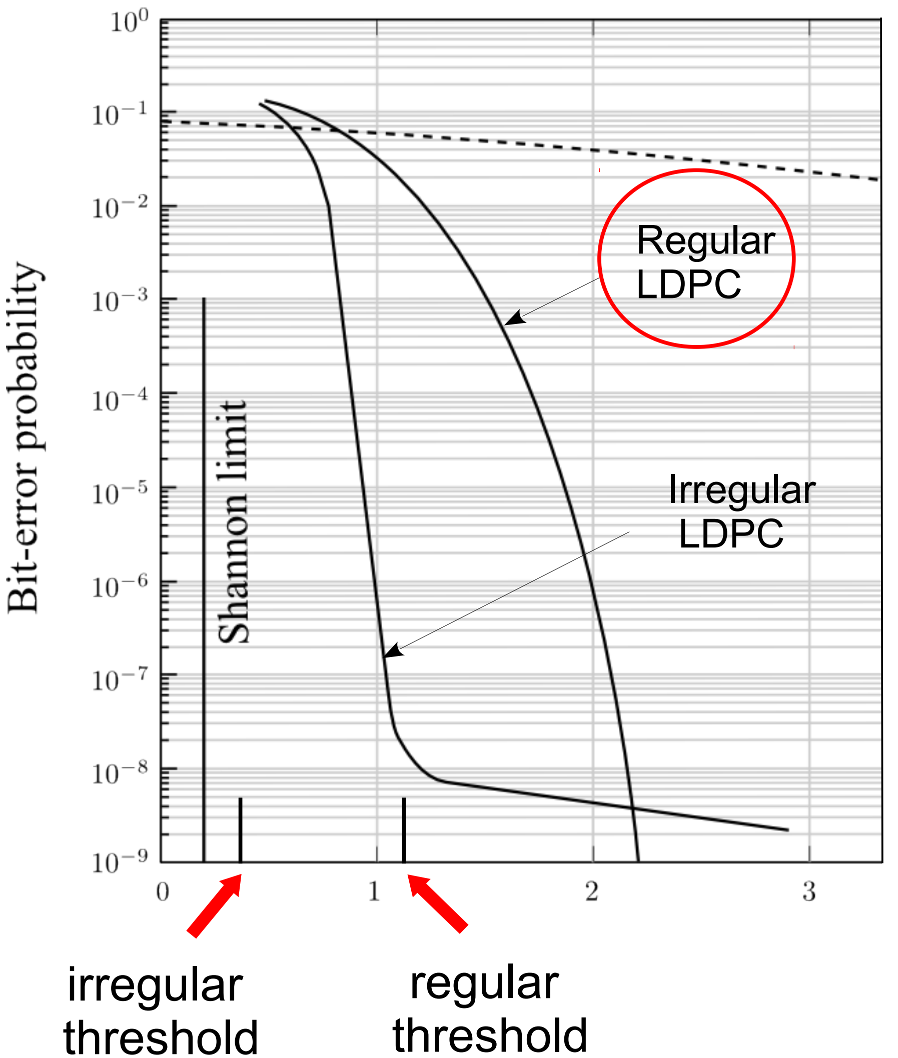

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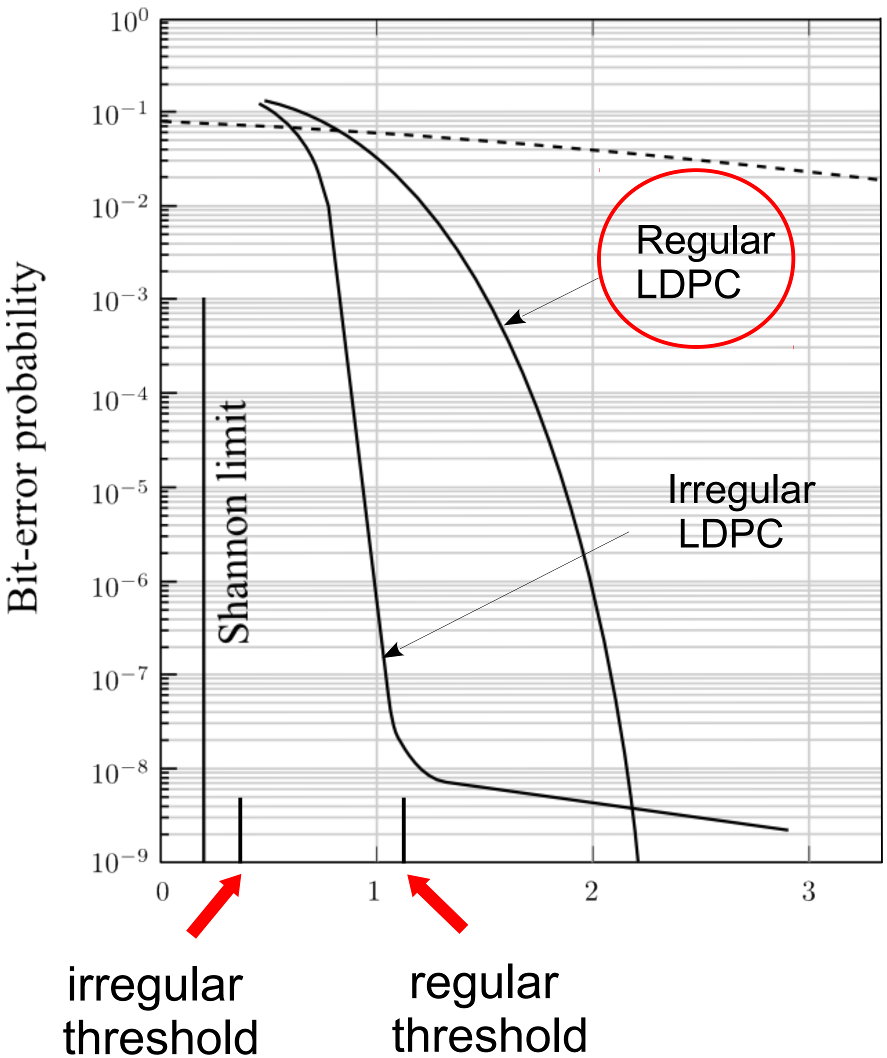

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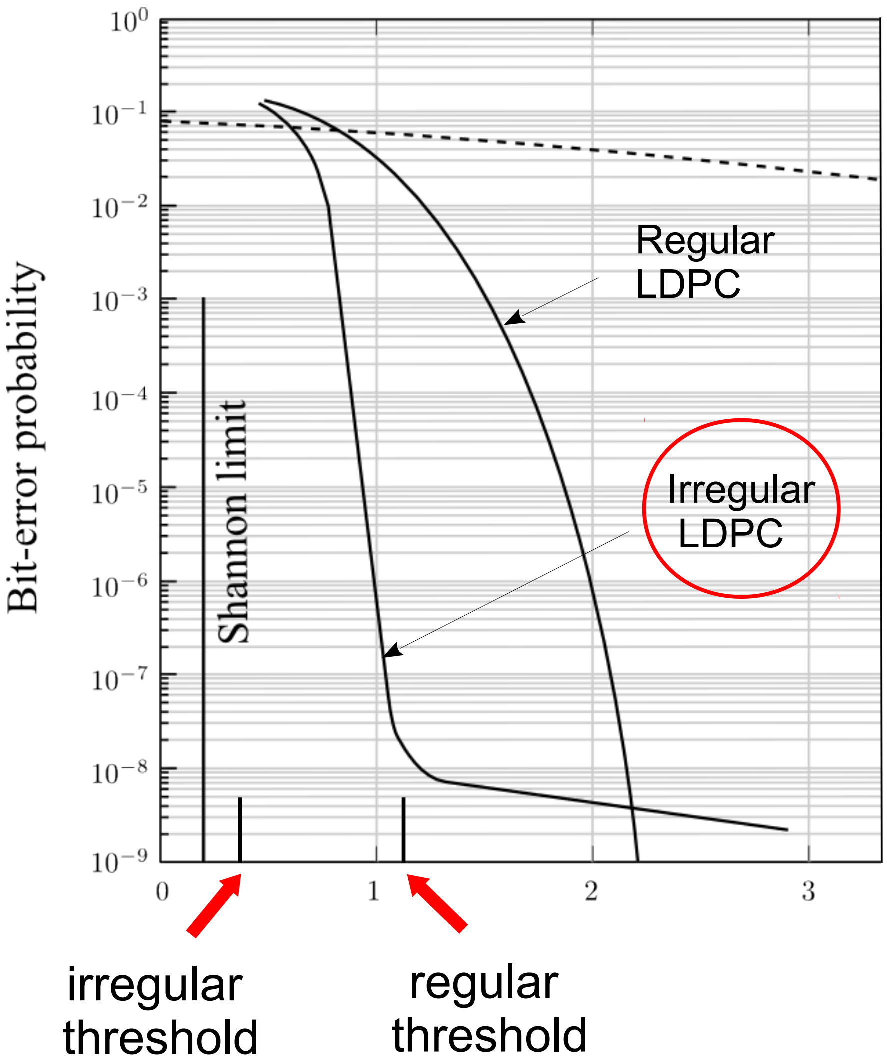

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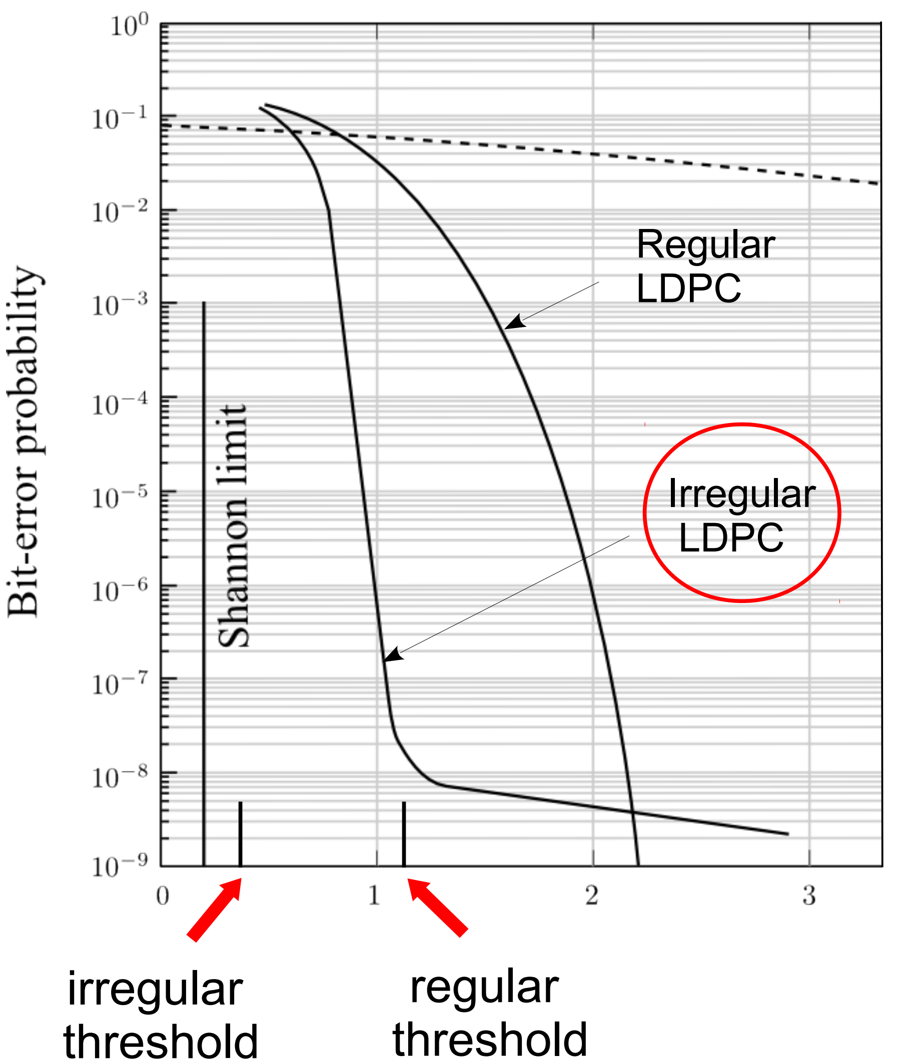

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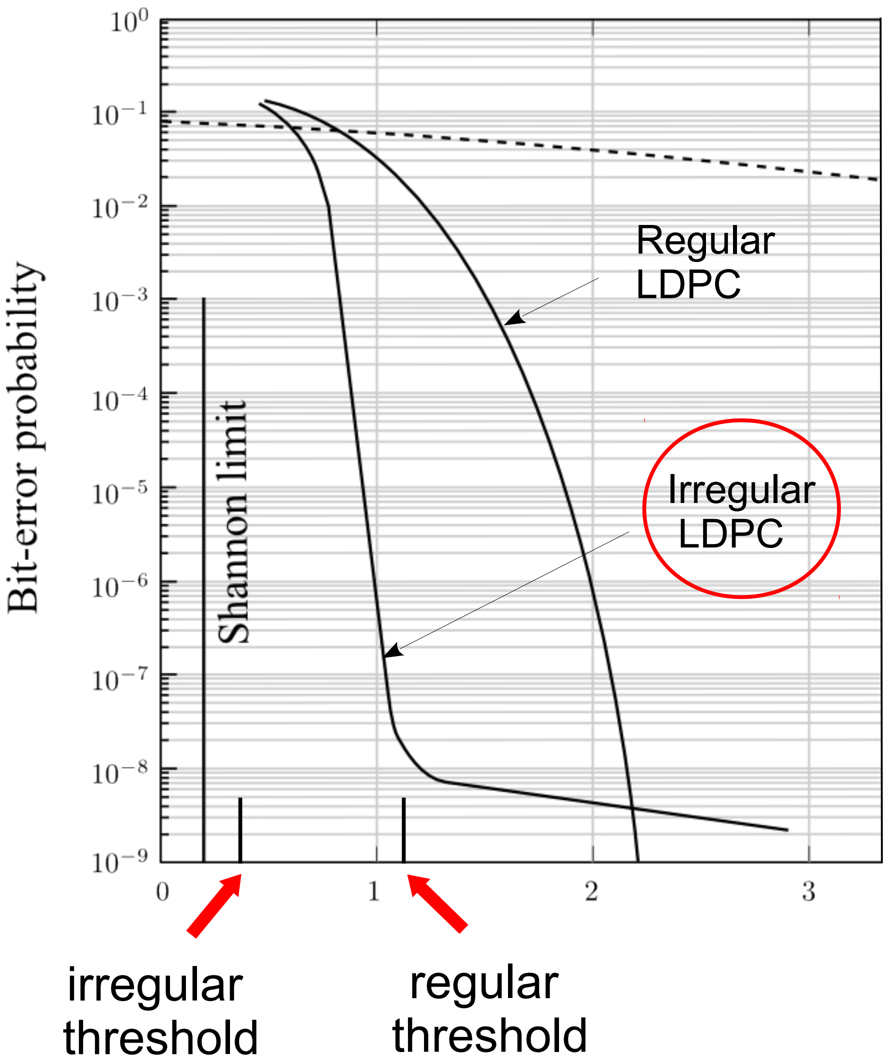

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- **Spatially coupled** LDPC codes combine all of the positive features!

Structured LDPC Codes

- Let $\mathbf{H}$ be composed of **permutation matrices** of size M:

$$\mathbf{H} = \begin{bmatrix} \mathbf{\Pi}_{1,1} & \mathbf{\Pi}_{1,2} & \cdots & \mathbf{\Pi}_{1,K} \\ \mathbf{\Pi}_{2,1} & \mathbf{\Pi}_{2,2} & \cdots & \mathbf{\Pi}_{2,K} \\ \vdots & \vdots & & \vdots \\ \mathbf{\Pi}_{J,1} & \mathbf{\Pi}_{J,2} & \cdots & \mathbf{\Pi}_{J,K} \end{bmatrix}_{J \times K}$$

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Regular example:

$J = 3, K = 6, n = 24,$

$M = 4$ ($\mathbf{\Pi}_{i,j}$ is 4×4)

$$\left[\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc}
0&1&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1\\
0&0&1&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&1&0&0&0\\
1&0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&1&0&0&0&0&1&0&0\\
0&0&0&1&1&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
\hline
0&0&0&1&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0\\
1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&1\\
0&1&0&0&0&1&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&1&0&1&0&0&0\\
\hline
0&1&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0\\
1&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
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Irregular example:

Replace some permutation matrices by **0** matrix of size M

$J_{\max} = 3, K_{\max} = 6$

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0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0\\
1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0\\
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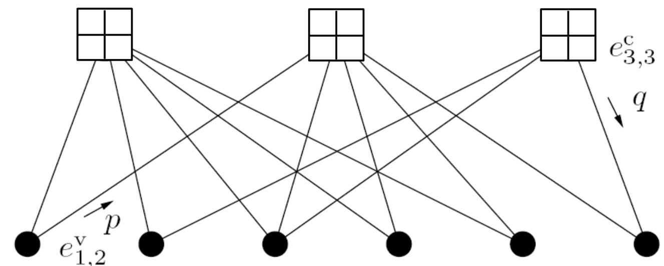

b_v variable nodes

protograph

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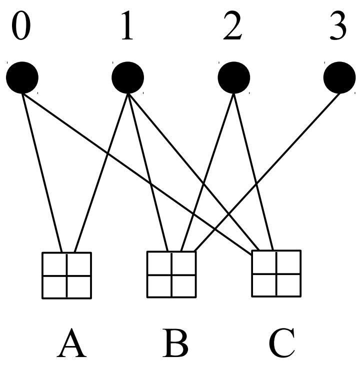

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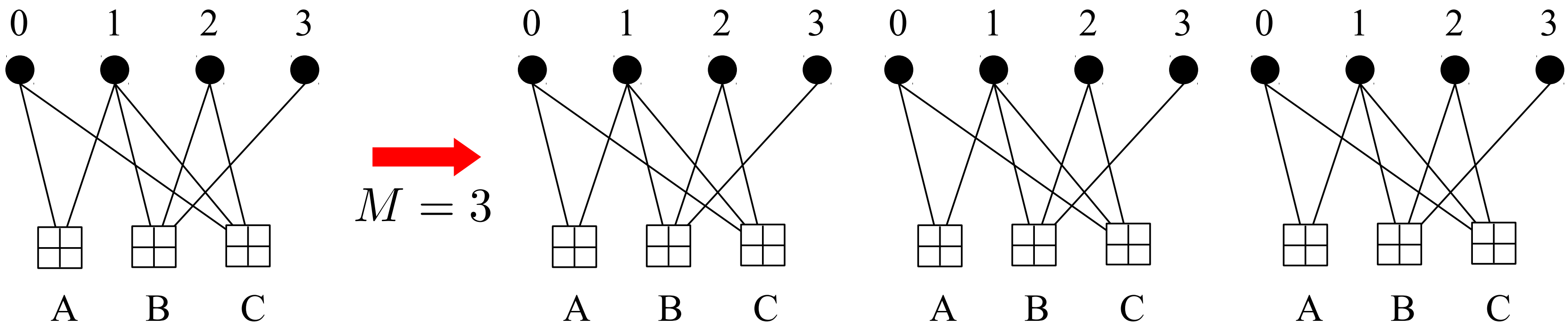

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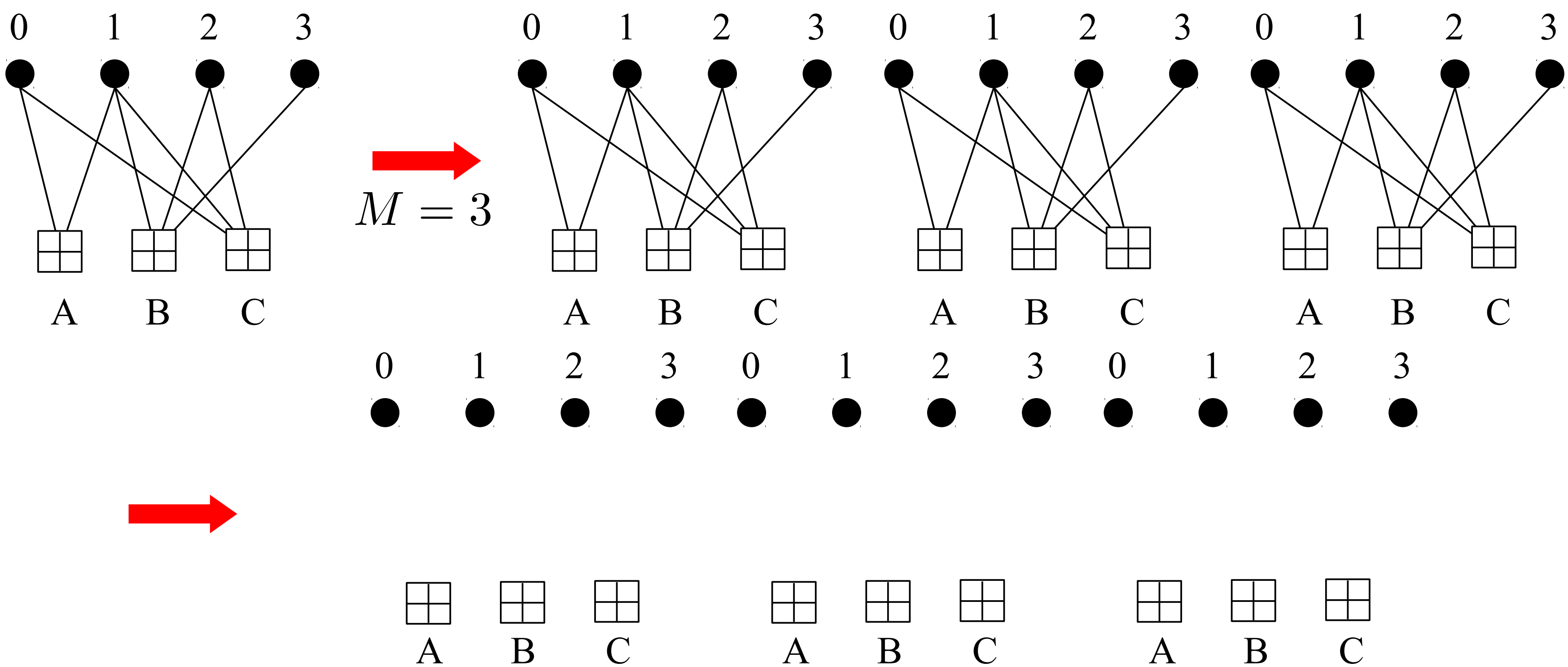

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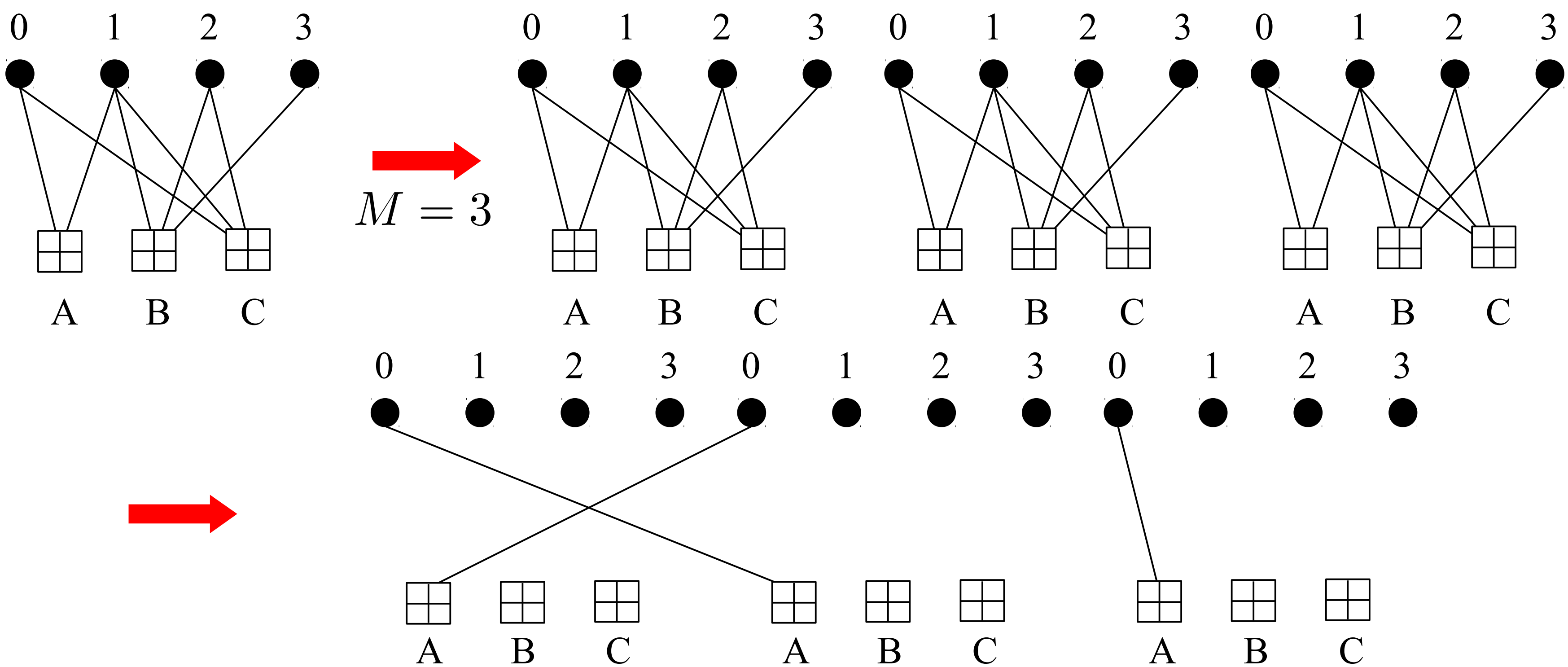

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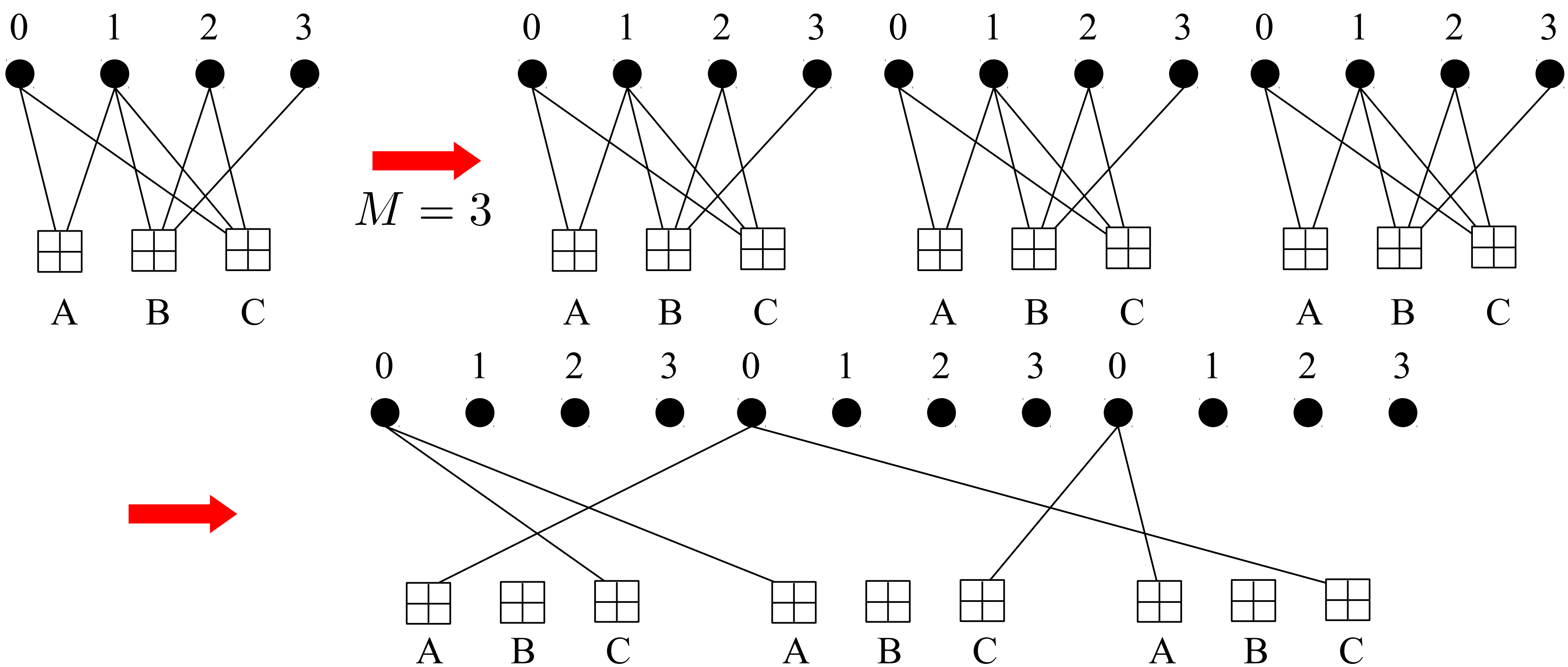

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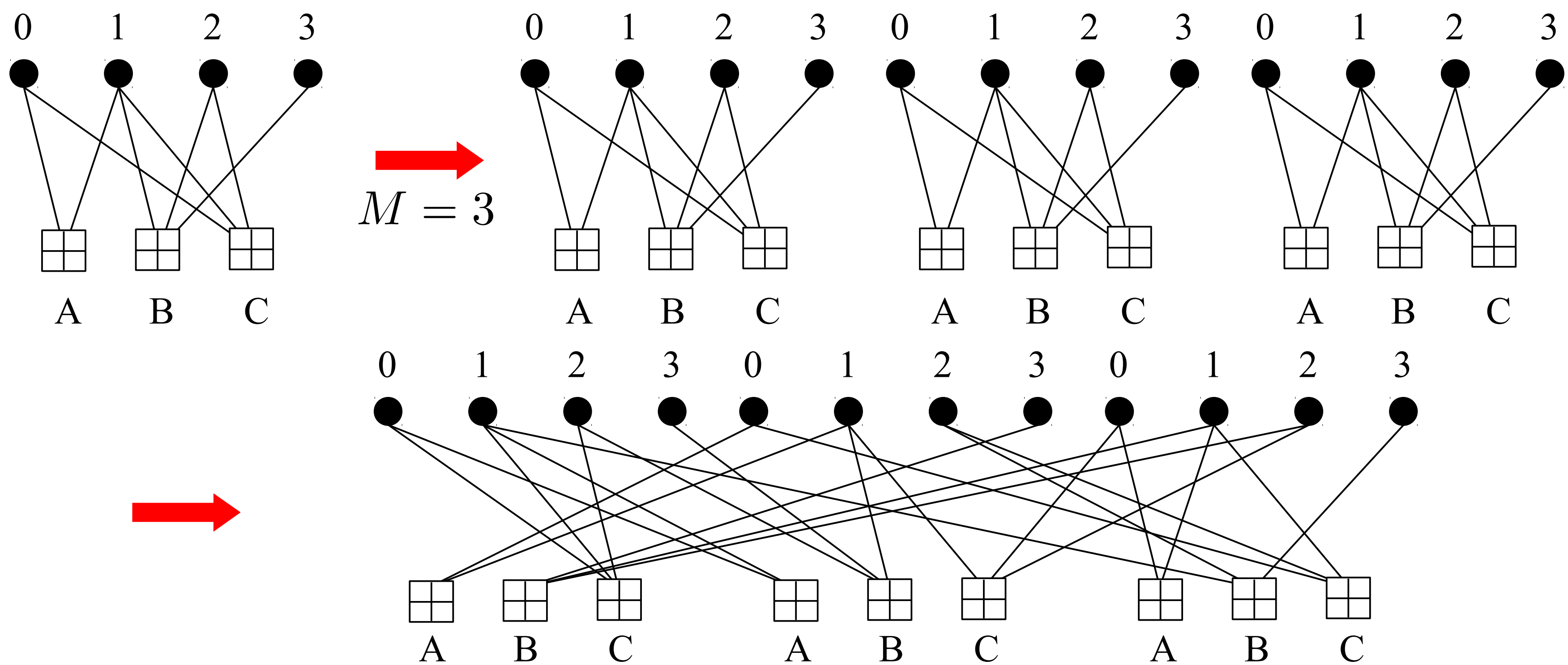

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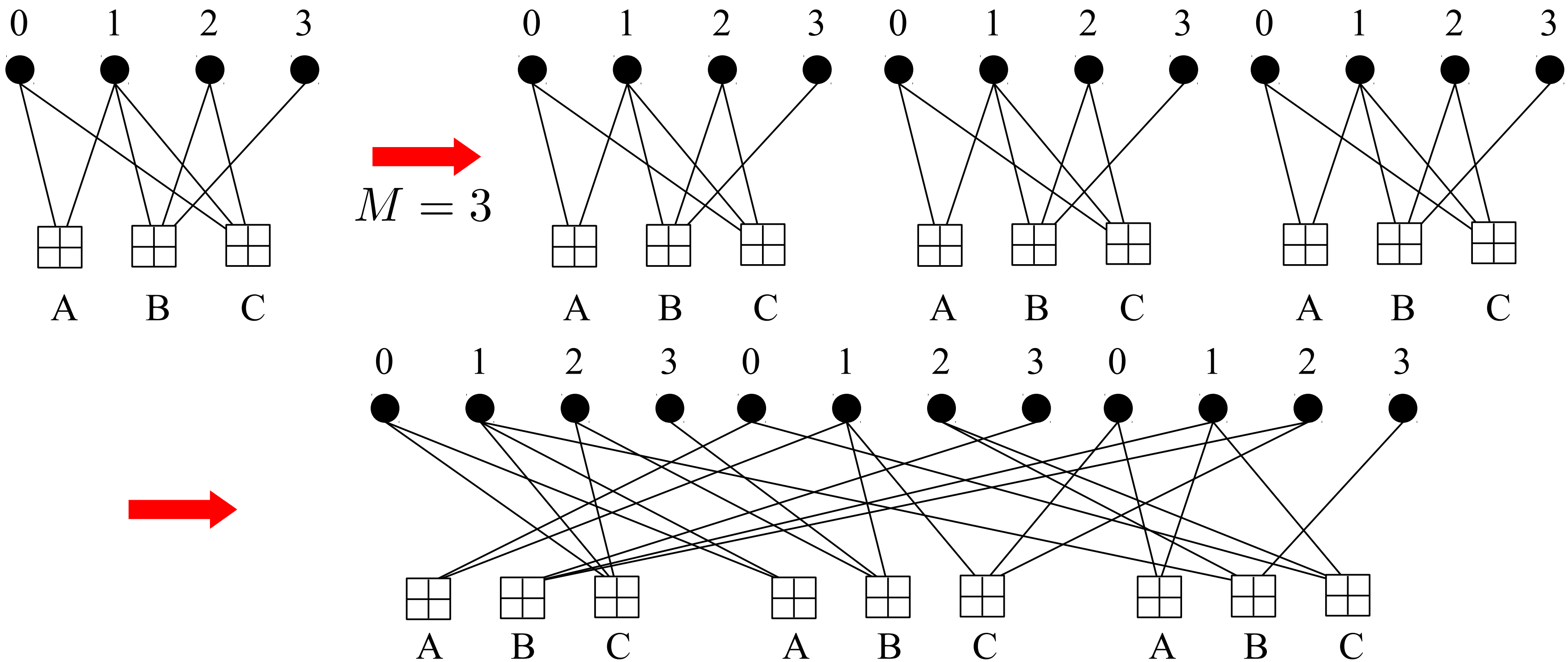
- Structured codes of varying lengths can be constructed from a **protograph** using a **copy-and-permute** operation, or ***M*-fold graph lifting**,



- Code rate $R \geq (b_v - b_c)/b_v$, code length $n = Mb_v$

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Quasi-Cyclic LDPC Codes

- **Quasi-cyclic** (QC) LDPC codes are of great interest to code designers, since they can be **encoded with low complexity** using simple feedback shift-registers and lend themselves to **efficient decoder implementation.**

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0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
\hline
0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
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$$M = 7$$
$$n = Mb_v = 21$$

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- The QC sub-ensemble contains **atypical** codes. Thus ensemble average results **do not apply** to these highly structured codes.

Outline

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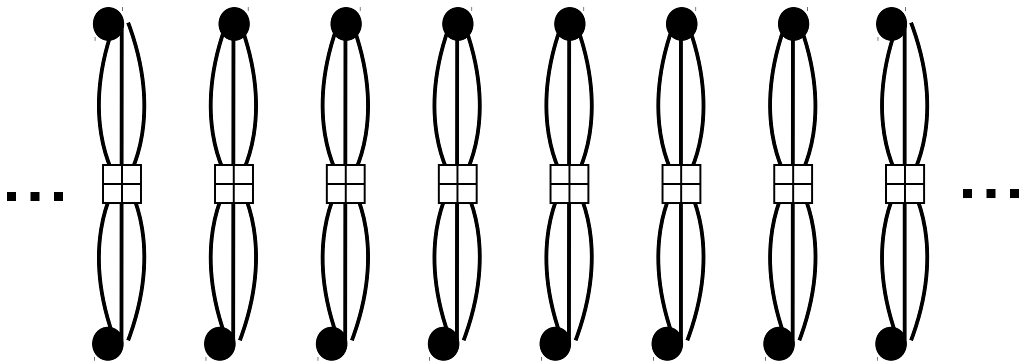
- Introduction: From Shannon to Modern Coding Theory
  - ➡ Channel capacity, structured codes, random codes, LDPC codes
- LDPC Block Codes
  - ➡ Parity-check matrix and Tanner graph representations, minimum distance bounds, iterative decoding thresholds, protograph-based constructions
- **Spatially Coupled LDPC Codes**
  - ➡ Protograph representation, edge-spreading construction, termination
  - ➡ Iterative decoding thresholds, threshold saturation, minimum distance
- Practical Considerations
  - ➡ Window decoding, performance, latency, and complexity comparisons to LDPC block codes, rate-compatibility, implementation aspects

Spatially Coupled Protographs

- Consider transmission of consecutive blocks (protograph representation):



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$(3,6)$-regular
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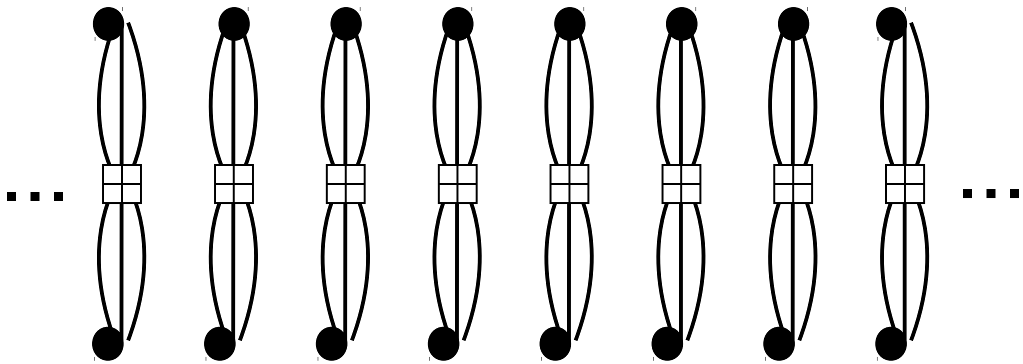

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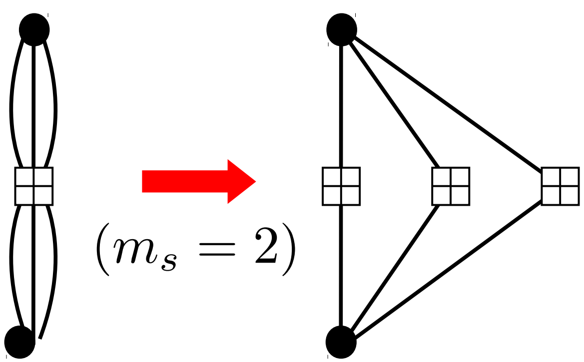

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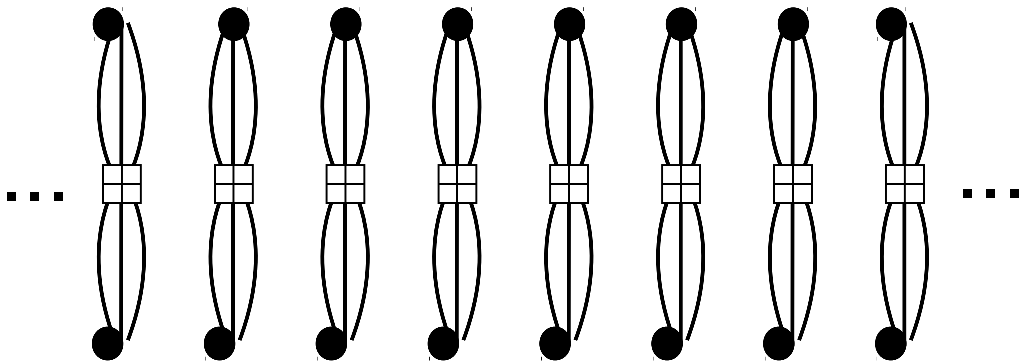

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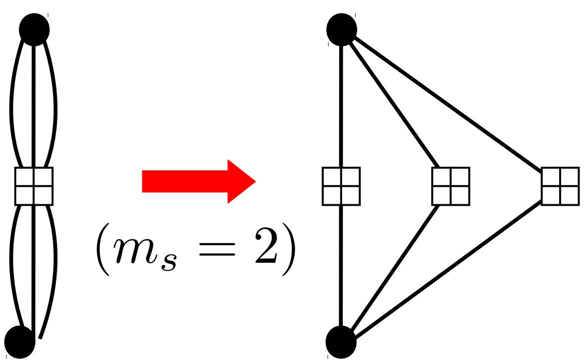

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- **Spreading constraint:** $\sum_{i=0}^{m_s} \mathbf{B}_i = \mathbf{B}$ ($\mathbf{B}_i$ has size $b_c \times b_v$)

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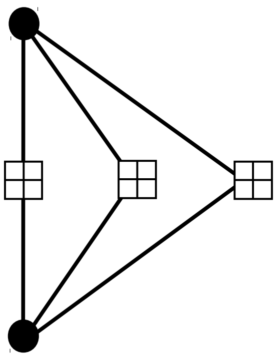
- Transmission of consecutive spatially coupled (SC) blocks results in a **convolutional protograph**:



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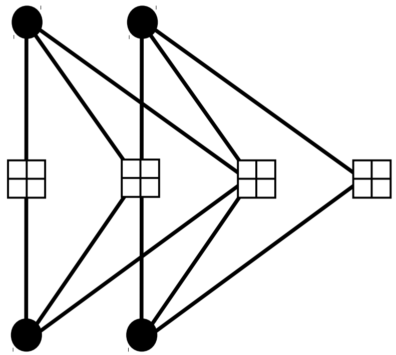

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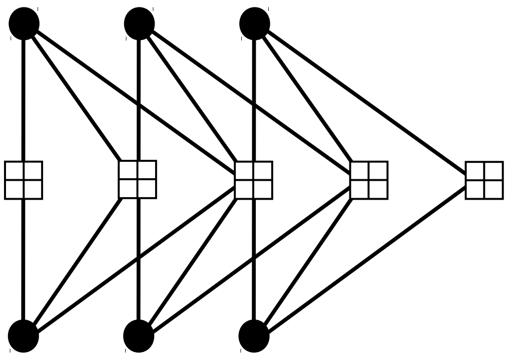

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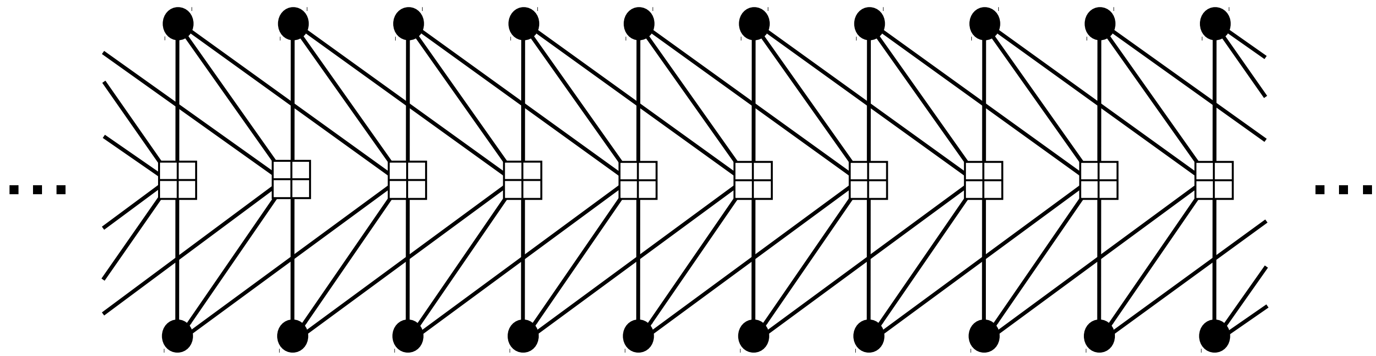

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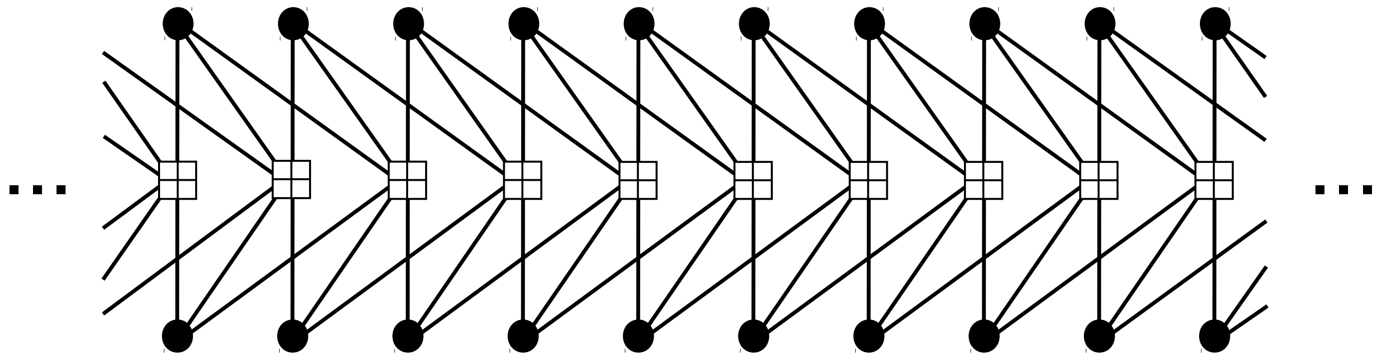

- The bi-infinite convolutional protograph corresponds to a bi-infinite **convolutional base matrix**:

$$\mathbf{B}_{[-\infty,\infty]} = \begin{bmatrix} \ddots & & \ddots & \ddots & & & \\ & \mathbf{B}_{m_s} & \cdots & \mathbf{B}_1 & \mathbf{B}_0 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & \mathbf{B}_{m_s} & \cdots & \mathbf{B}_1 & \mathbf{B}_0 \\ & & & & \ddots & & \ddots & \ddots \end{bmatrix}$$

Rate:

$$R = \frac{b_v - b_c}{b_v}$$

Constraint length:

$$\nu_s = b_v(m_s + 1)$$

SC-LDPC Code Ensembles

- An ensemble of (3,6)-regular SC-LDPC codes can be created from the **convolutional protograph** by the graph lifting operation

$$\mathbf{B}_{[-\infty,\infty]} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_0 & & & \\ & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_0 & & \\ & & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

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Graph lifting: $\Pi_{i,j}$ is an $M \times M$ permutation matrix

$$\nu_s = M b_v(m_s + 1) = 6M$$

$$\mathbf{H}_{\text{cc}} = \begin{bmatrix} \ddots & & \ddots & & \ddots & & & & & \\ \Pi_{5,t} & \Pi_{4,t} & \Pi_{3,t} & \Pi_{2,t} & \Pi_{1,t} & \Pi_{0,t} & & & & \\ & & \Pi_{5,t+1} & \Pi_{4,t+1} & \Pi_{3,t+1} & \Pi_{2,t+1} & \Pi_{1,t+1} & \Pi_{0,t+1} & & \\ & & & & \Pi_{5,t+2} & \Pi_{4,t+2} & \Pi_{3,t+2} & \Pi_{2,t+2} & \Pi_{1,t+2} & \Pi_{0,t+2} \\ & & & & & \ddots & & \ddots & & \ddots \end{bmatrix}$$

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- If each permutation matrix $\Pi_{i,j}$ is **circulant**, the codes are **quasi-cyclic**

Terminated Spatially Coupled Codes

- Consider **terminating** $\mathbf{B}_{[-\infty,\infty]}$ to a (block code) **base matrix** of length Lb_v:

$$\mathbf{B}_{[0,L-1]} = \begin{bmatrix} \mathbf{B}_0 & & \\ \vdots & \ddots & \\ \mathbf{B}_{m_s} & & \mathbf{B}_0 \\ & \ddots & \vdots \\ & & \mathbf{B}_{m_s} \end{bmatrix}_{(L+m_s)b_c \times Lb_v}$$

Code rate:

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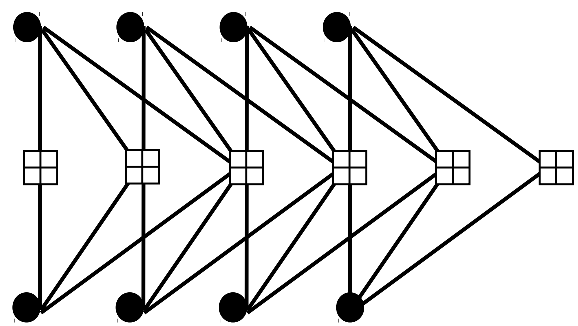
- **Example:** $(3,6)$-regular base matrix $\mathbf{B} = [3 \;\; 3]$, $m_s = 2$, $L = 4$, $R_4 = 1/4$

$$\mathbf{B}_{[0,3]} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$



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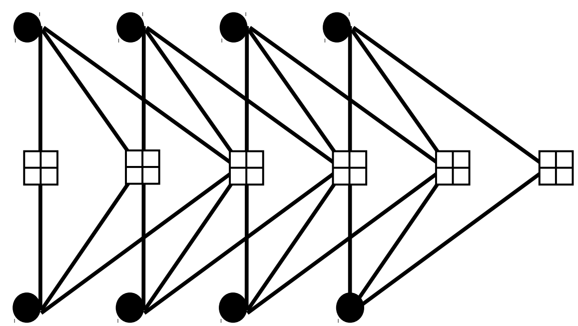

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- Codes can be **lifted** to **different lengths and rates** by varying M and L.

Thresholds of SC-LDPC Codes

- Variable nodes all have the **same degree** as the block code.
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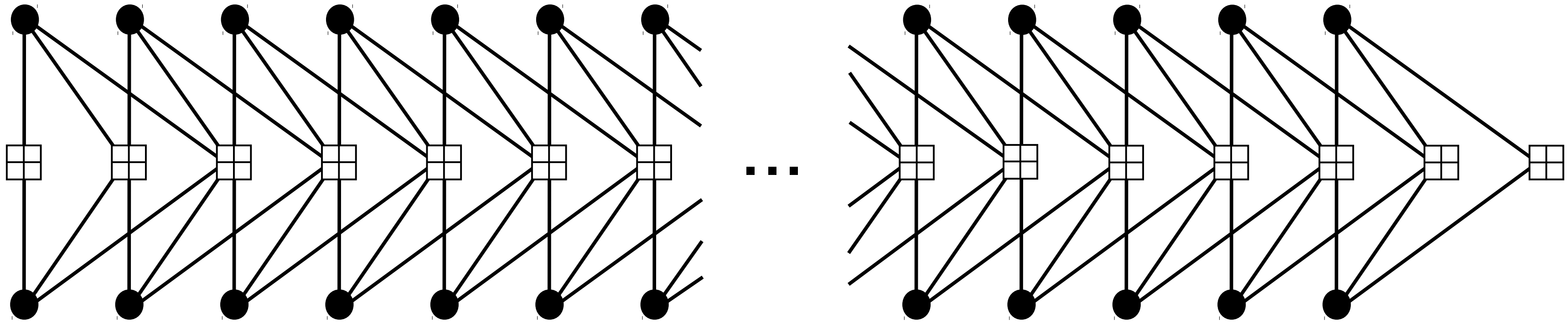

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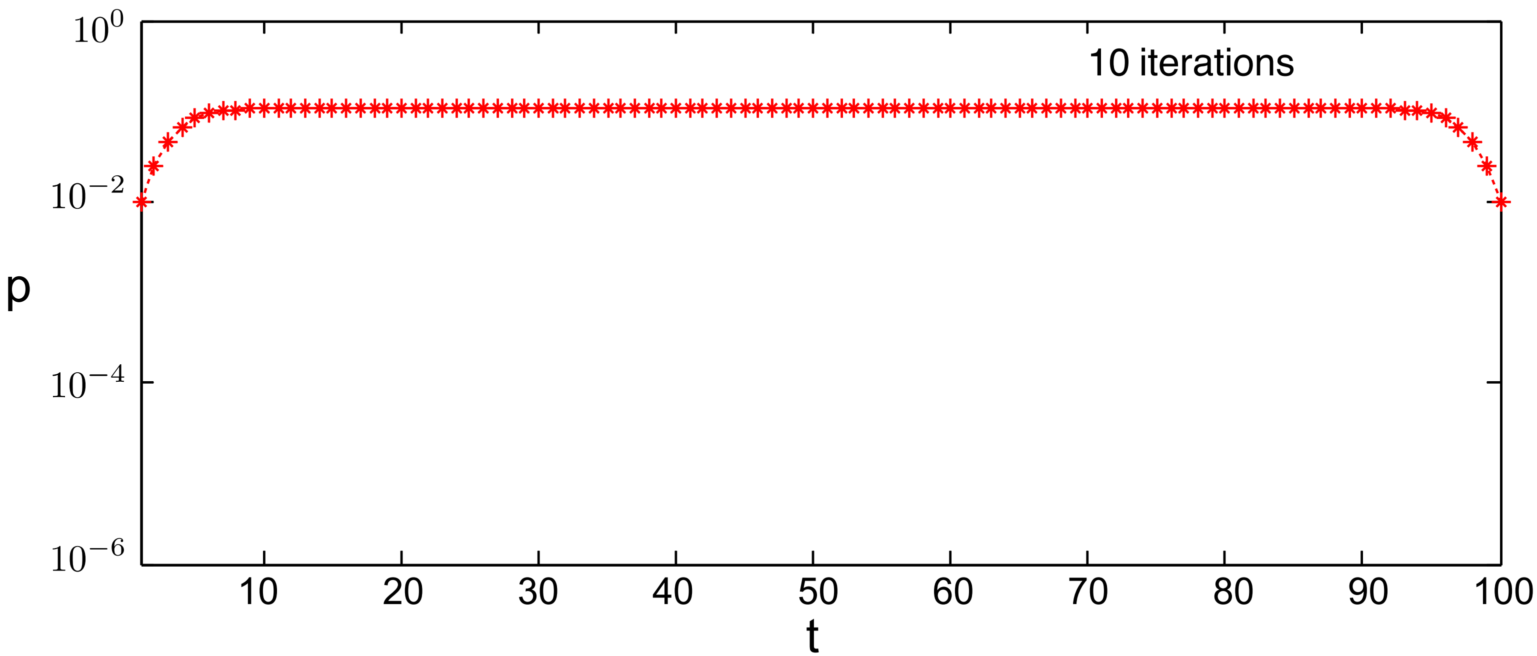

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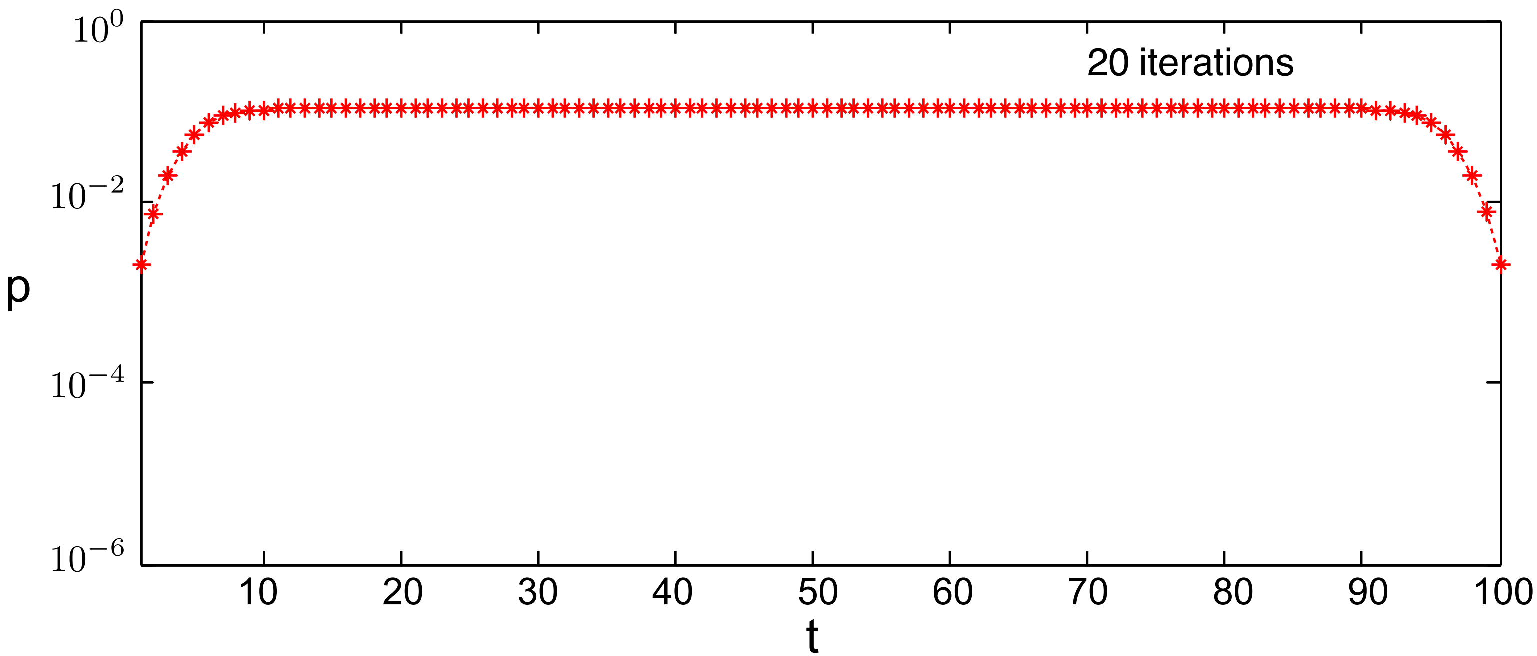

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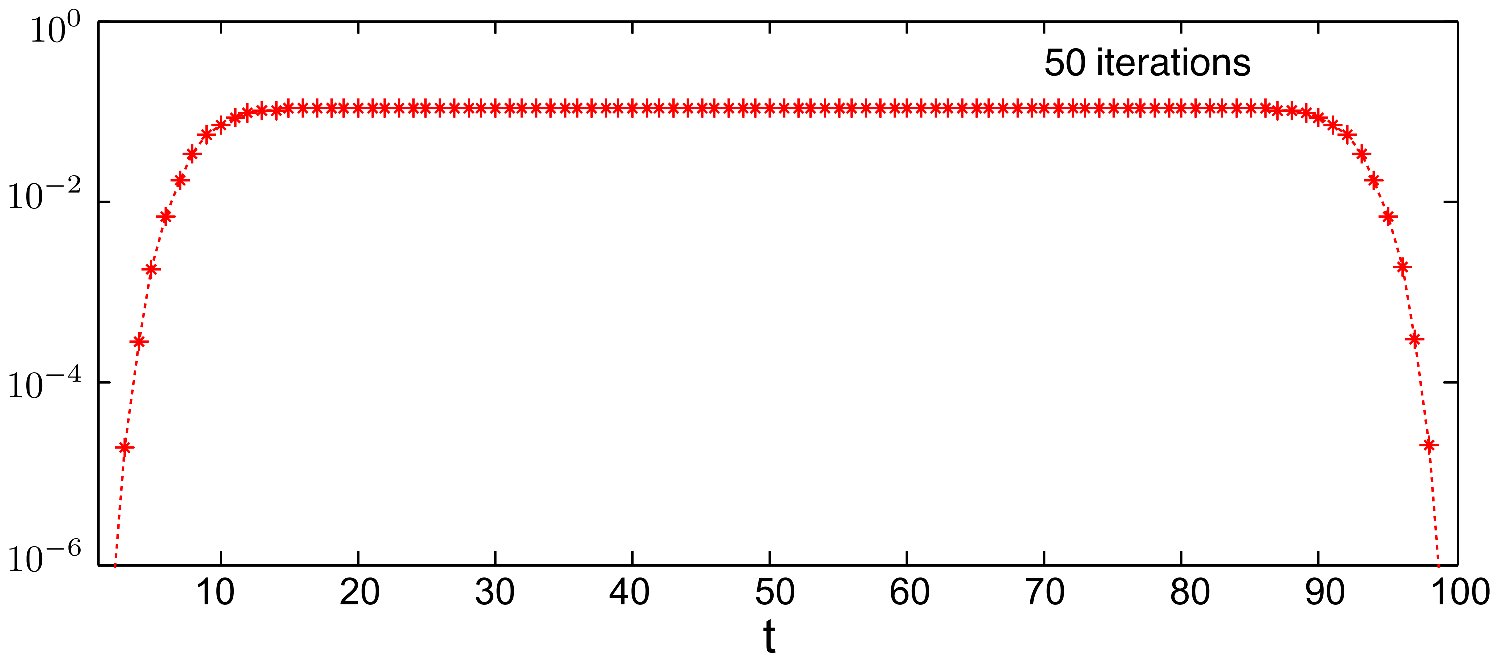

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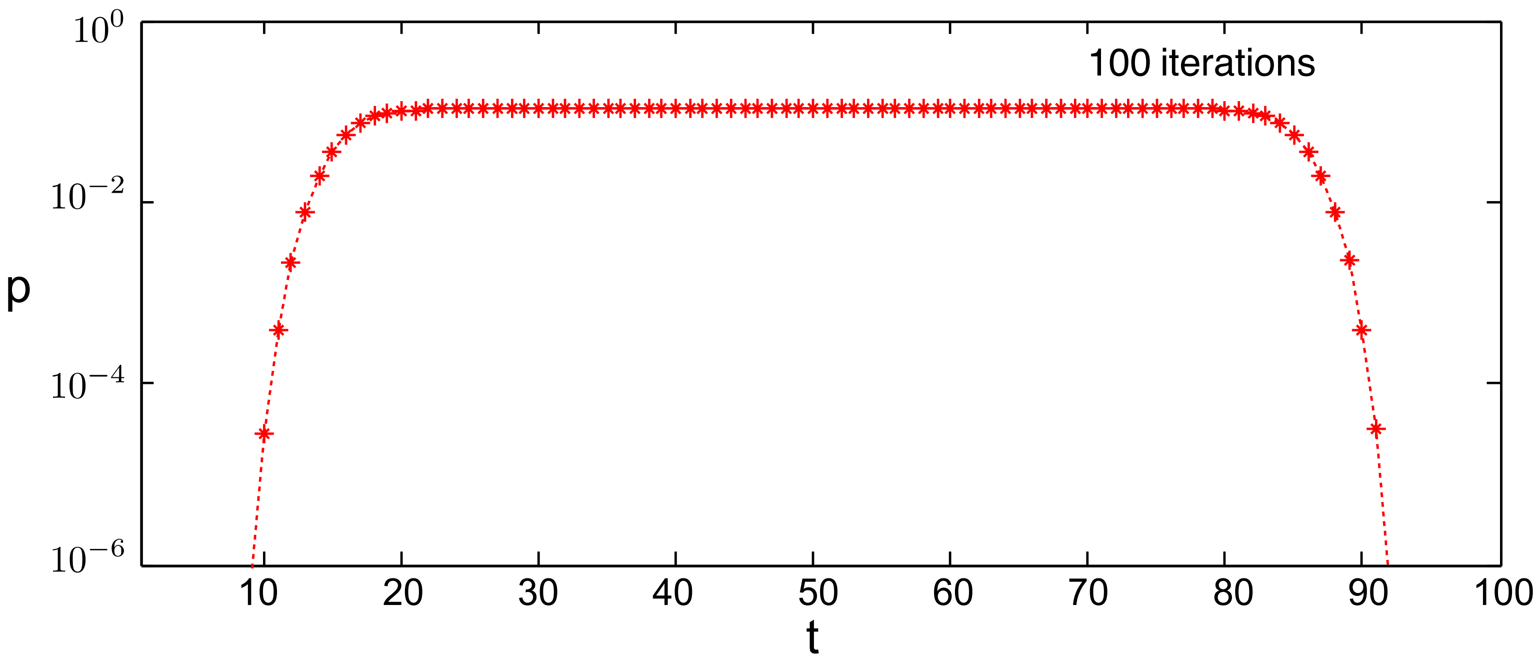

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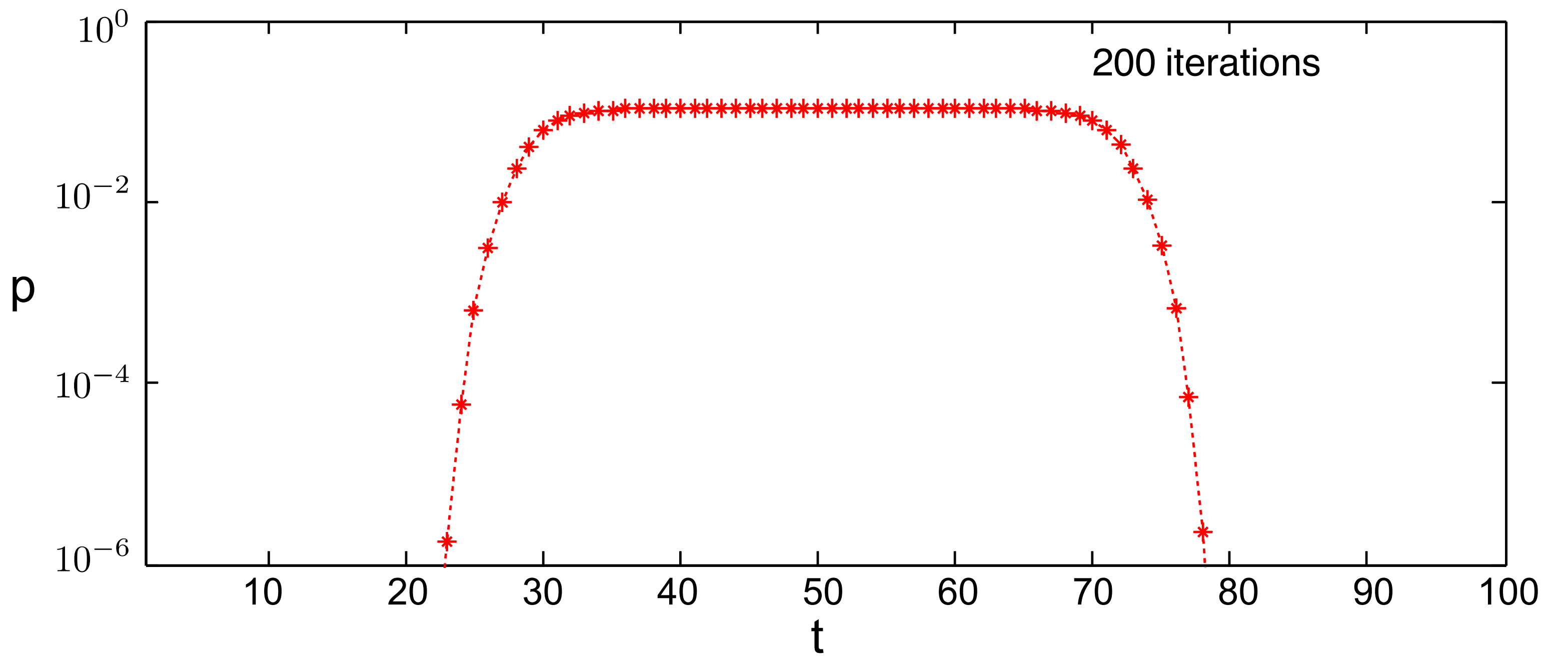

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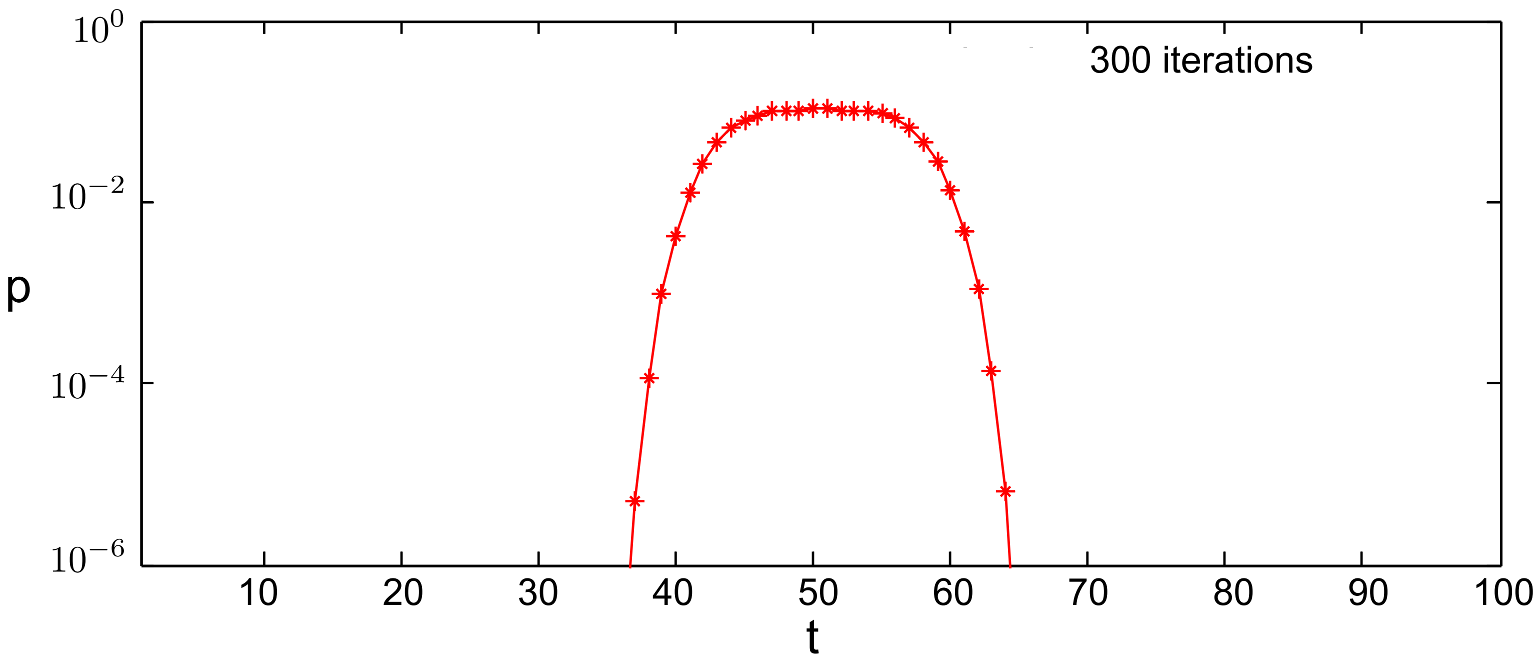

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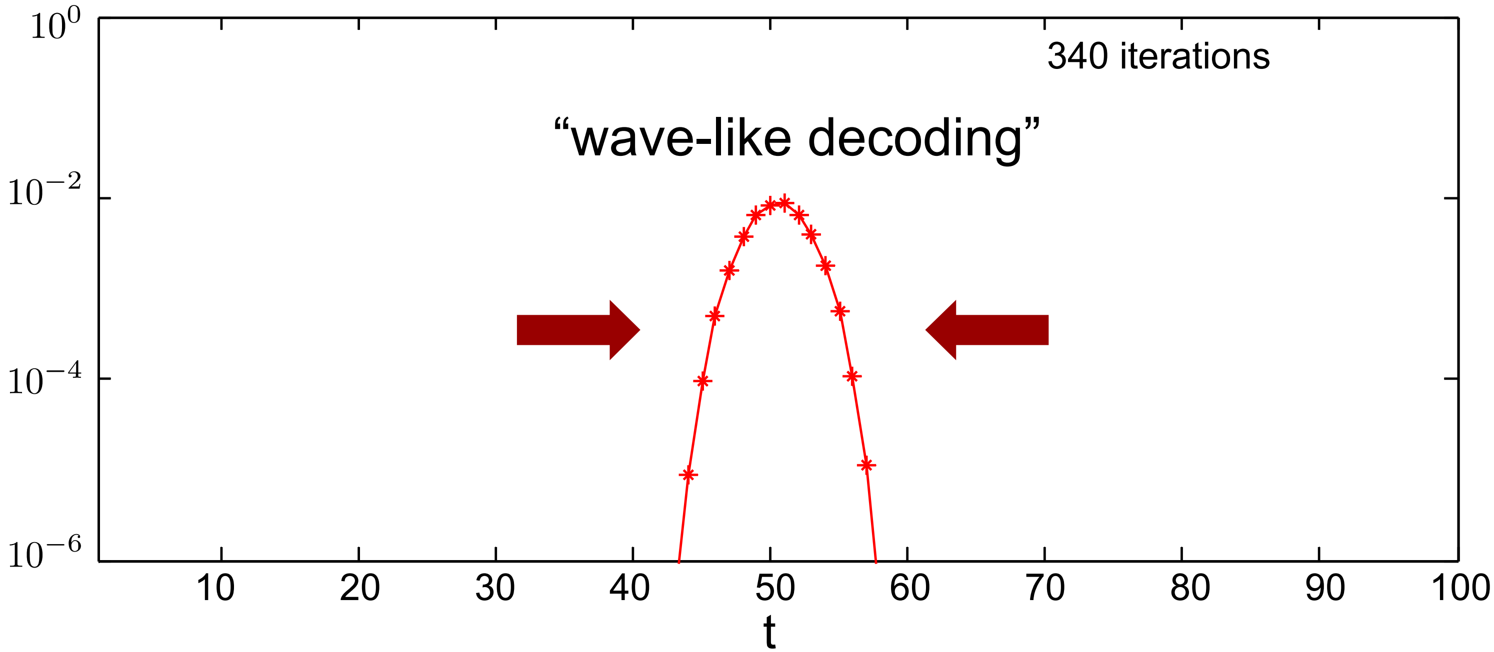

Thresholds of SC-LDPC Codes

- Variable nodes all have the **same degree** as the block code.
- Check nodes with **lower degrees** (at the ends) improve the BP decoder.

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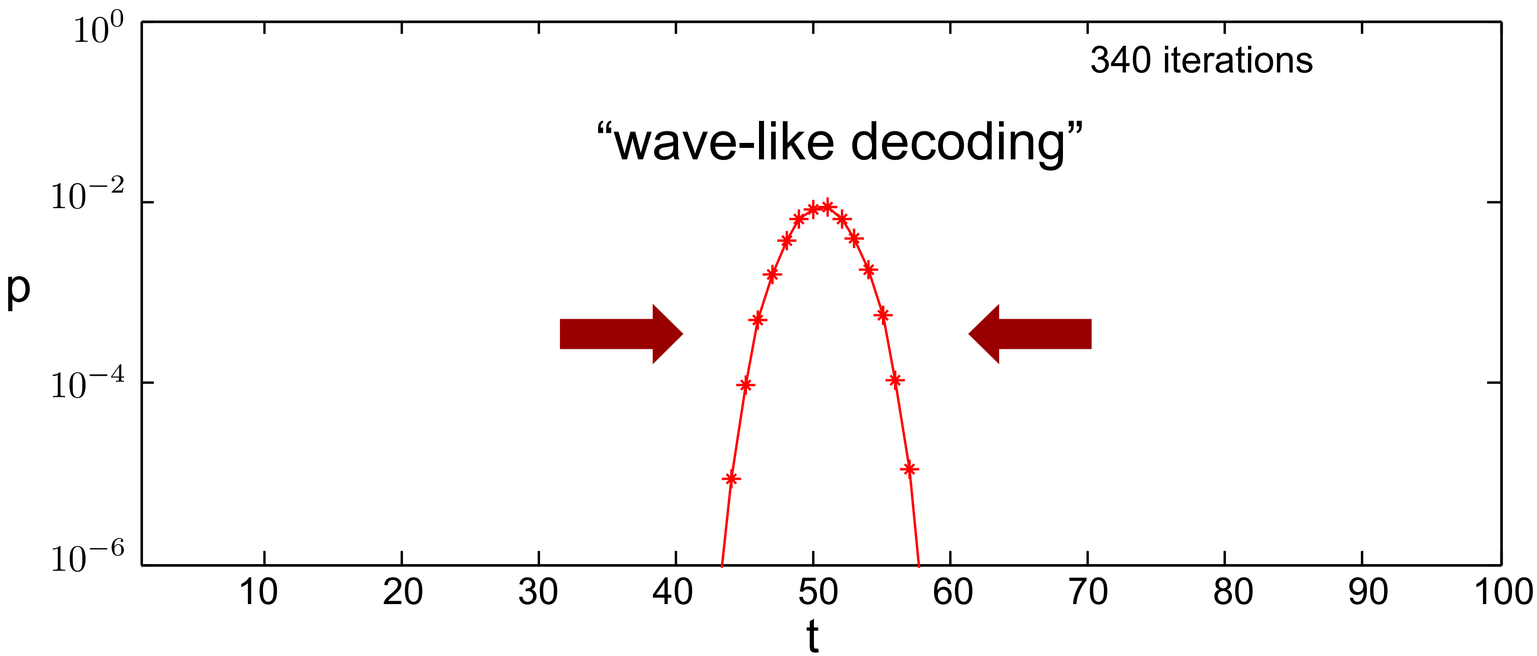
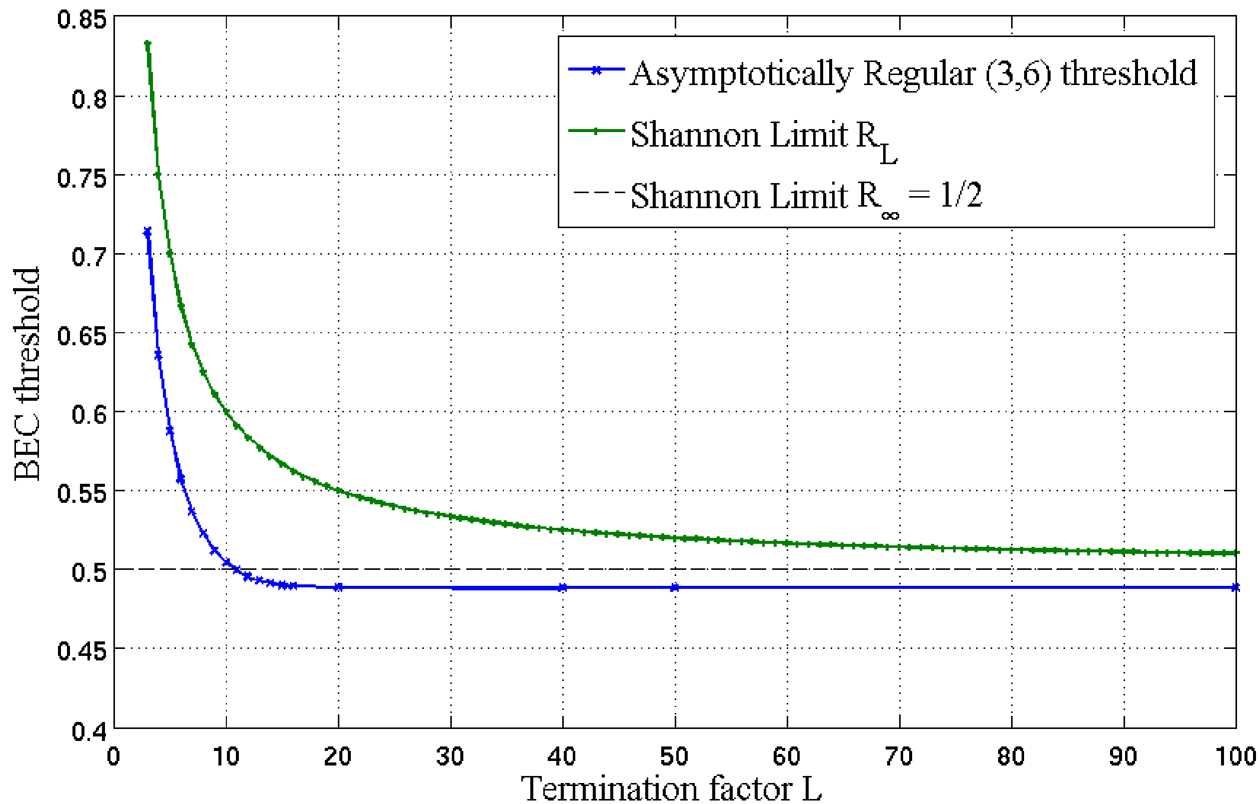


- Note: the **fraction** of **lower degree** nodes tends to zero as $L \to \infty$, i.e., the codes are **asymptotically regular**.

Thresholds of SC-LDPC Codes

- **Density evolution** can be applied to the protograph-based ensembles with $M \to \infty$ [Sridharan et al. '04]:

Example: BEC



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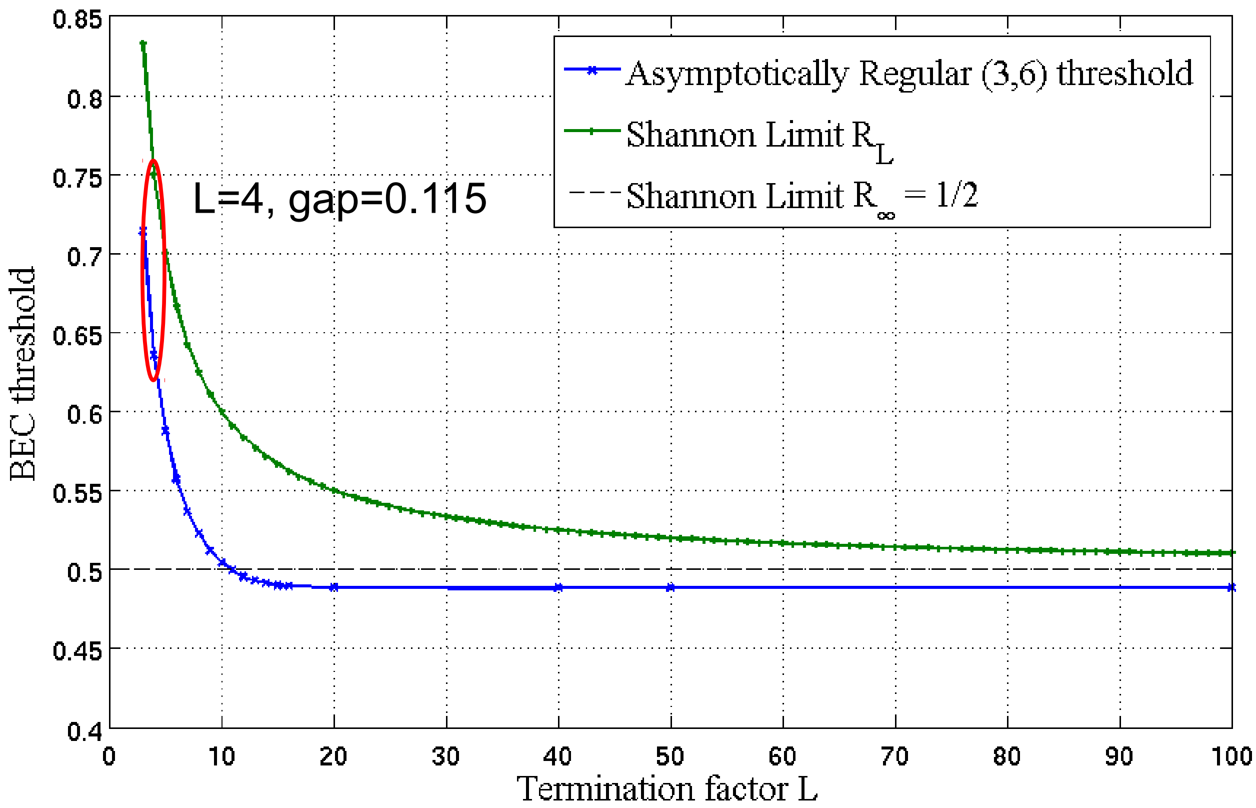
$L = 4$, $R = 1/4$

$\varepsilon^* = 0.635$, $\varepsilon_{\text{Sh}} = 0.75$



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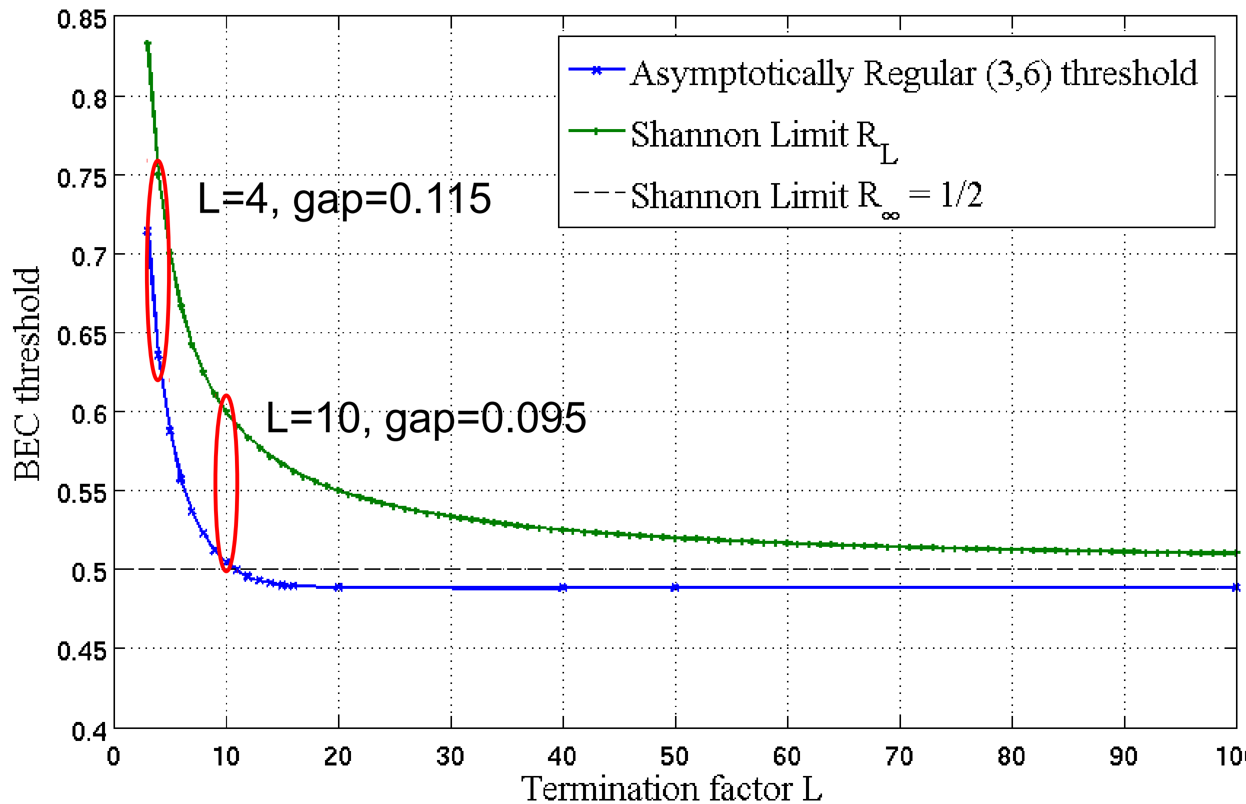

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⋮

$L \to \infty$, $R \to 1/2$
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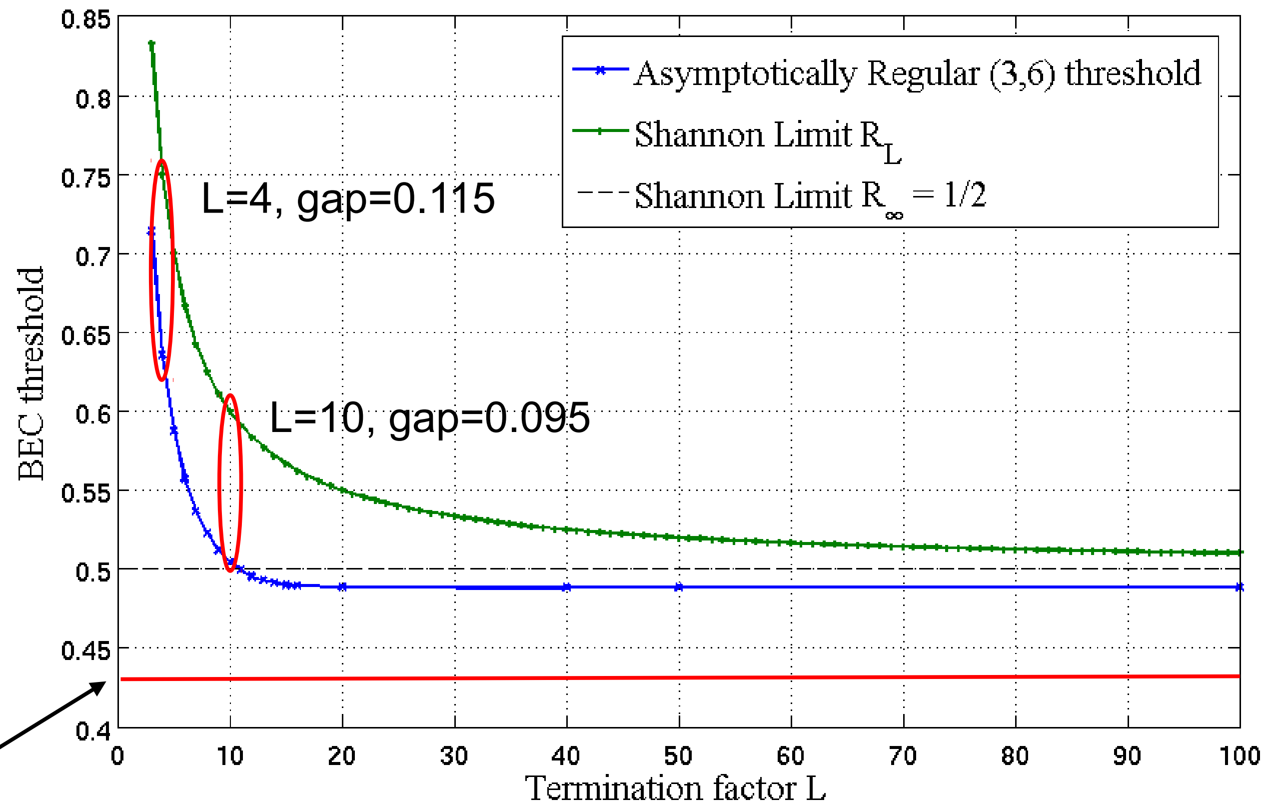

(3,6)-regular block code:

$\varepsilon^* = 0.429$

Thresholds of SC-LDPC Codes

Iterative decoding thresholds (protograph-based ensembles)

BEC

(J, K)	ϵ^*_{SC}	$\epsilon^*_{\mathrm{blk}}$
(3,6)	0.488	0.429
(4,8)	0.497	0.383
(5,10)	0.499	0.341

AWGN

(J, K)	$E_b/N_{o\,\mathrm{SC}}$	$E_b/N_{o\,\mathrm{blk}}$
(3,6)	0.46 dB	1.11 dB
(4,8)	0.26 dB	1.61 dB
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- We observe a **significant improvement** in the thresholds of SC-LDPC codes compared to the associated LDPC block codes (LDPC-BCs) due to the lower degree check nodes at the ends of the graph and the wave-like decoding.

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Thresholds of SC-LDPC Codes

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Why are SC-LDPC Codes Better?

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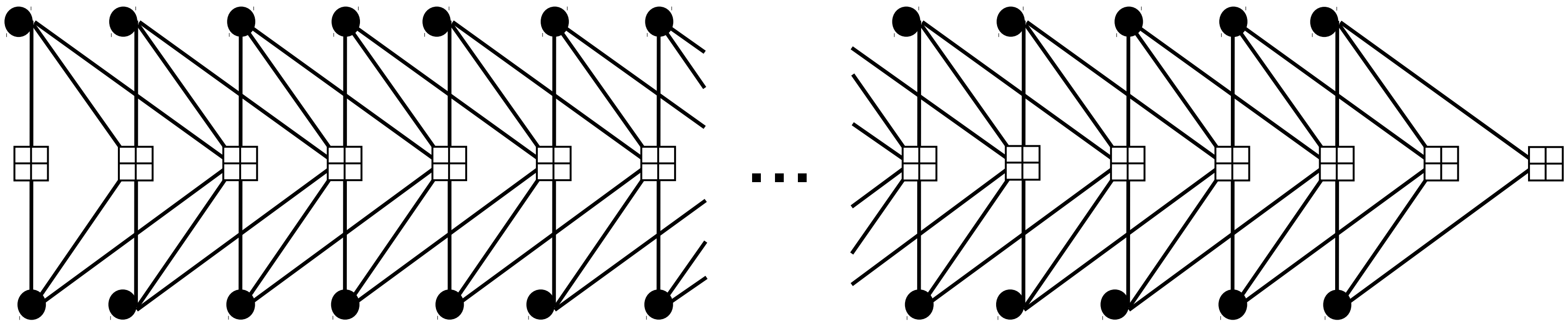
- When symbols are perfectly known (BEC), all adjacent edges can be removed from the Tanner graph.



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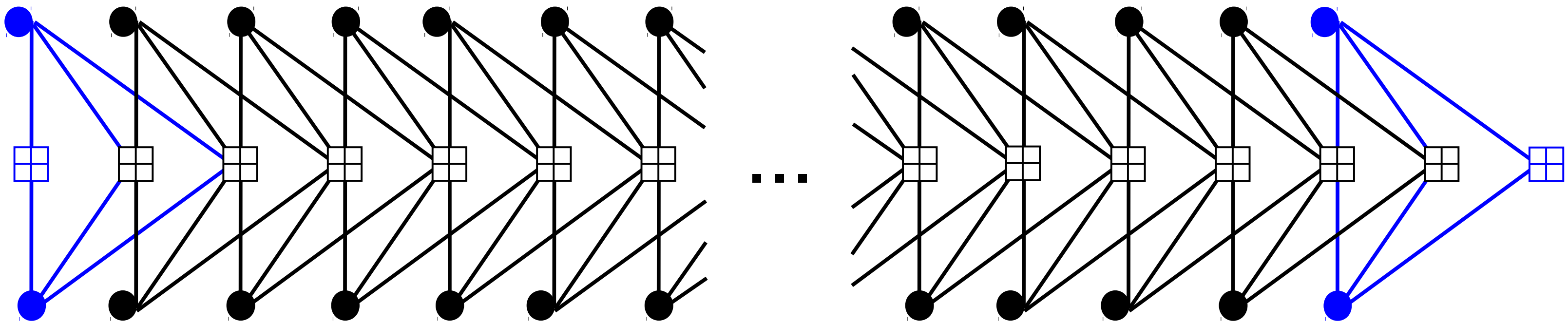

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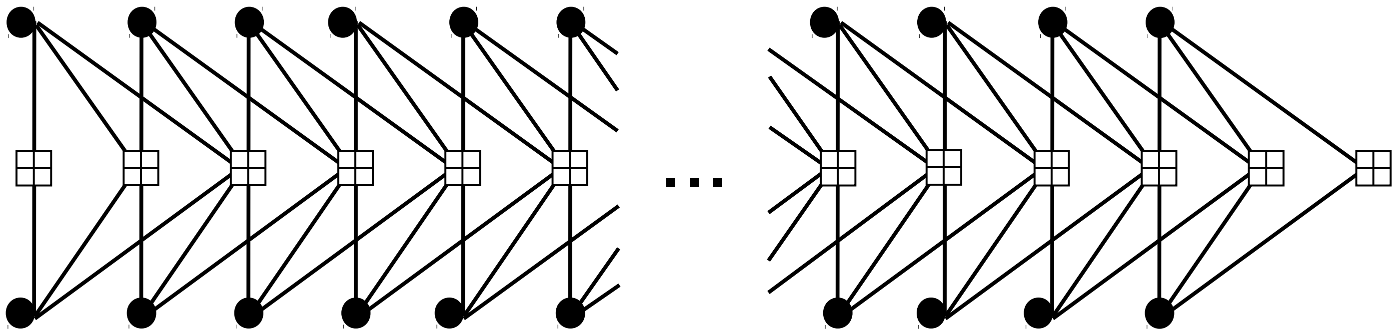

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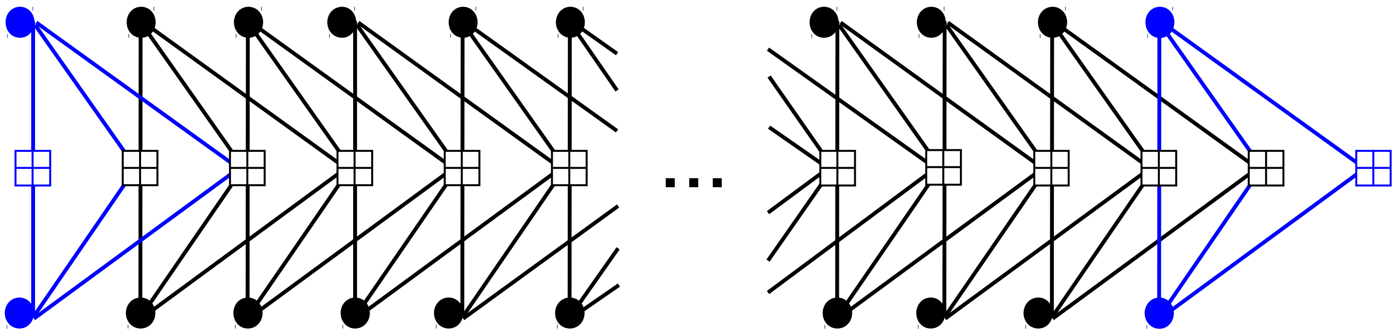

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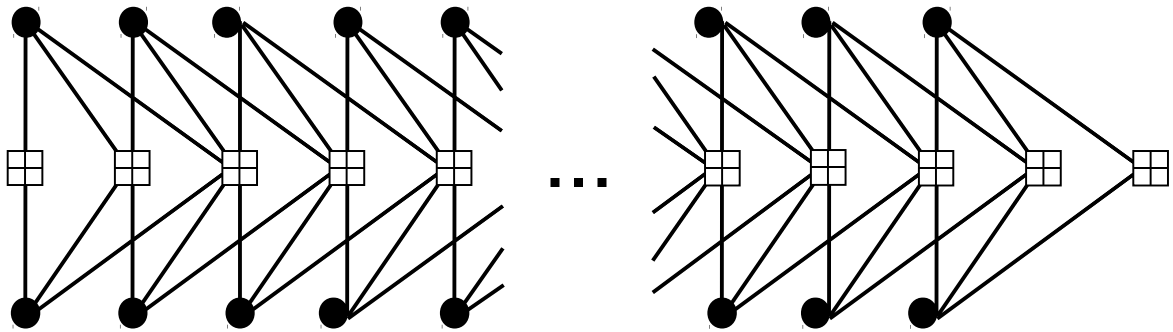

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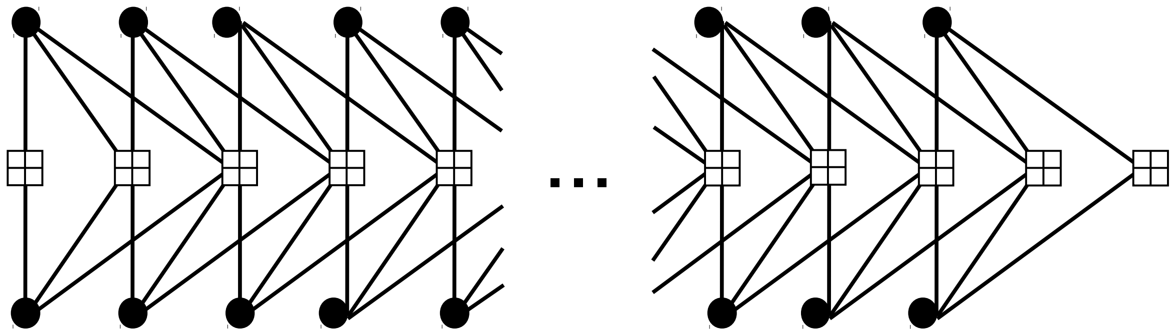

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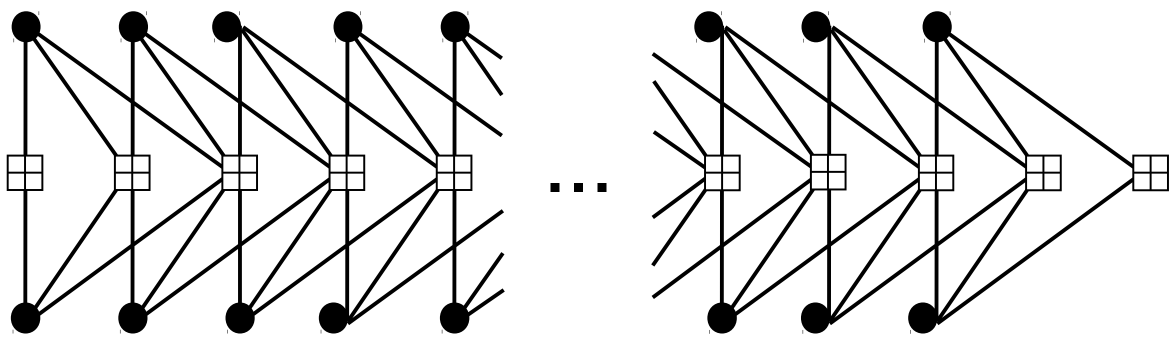

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- For a more random-like ensemble, this has been proven analytically, first for the BEC [KRU11], then for all BMS channels [KRU13].

[LSCZ10] M. Lentmaier, A. Sridharan, D. J. Costello, Jr., and K.Sh. Zigangirov, "Iterative decoding threshold analysis for LDPC convolutional codes," *IEEE Trans. Inf. Theory*, 56:10, Oct. 2010.
[KRU11] S. Kudekar, T. J. Richardson and R. Urbanke, "Threshold saturation via spatial coupling: why convolutional LDPC ensembles perform so well over the BEC", *IEEE Trans. on Inf. Theory*, 57:2, 2011
[KRU13] S. Kudekar, T. J. Richardson and R. Urbanke, "Spatially coupled ensembles universally achieve capacity under belief propagation", *IEEE Trans. on Inf. Theory*, 59:12, 2013.

Threshold Saturation (BEC)

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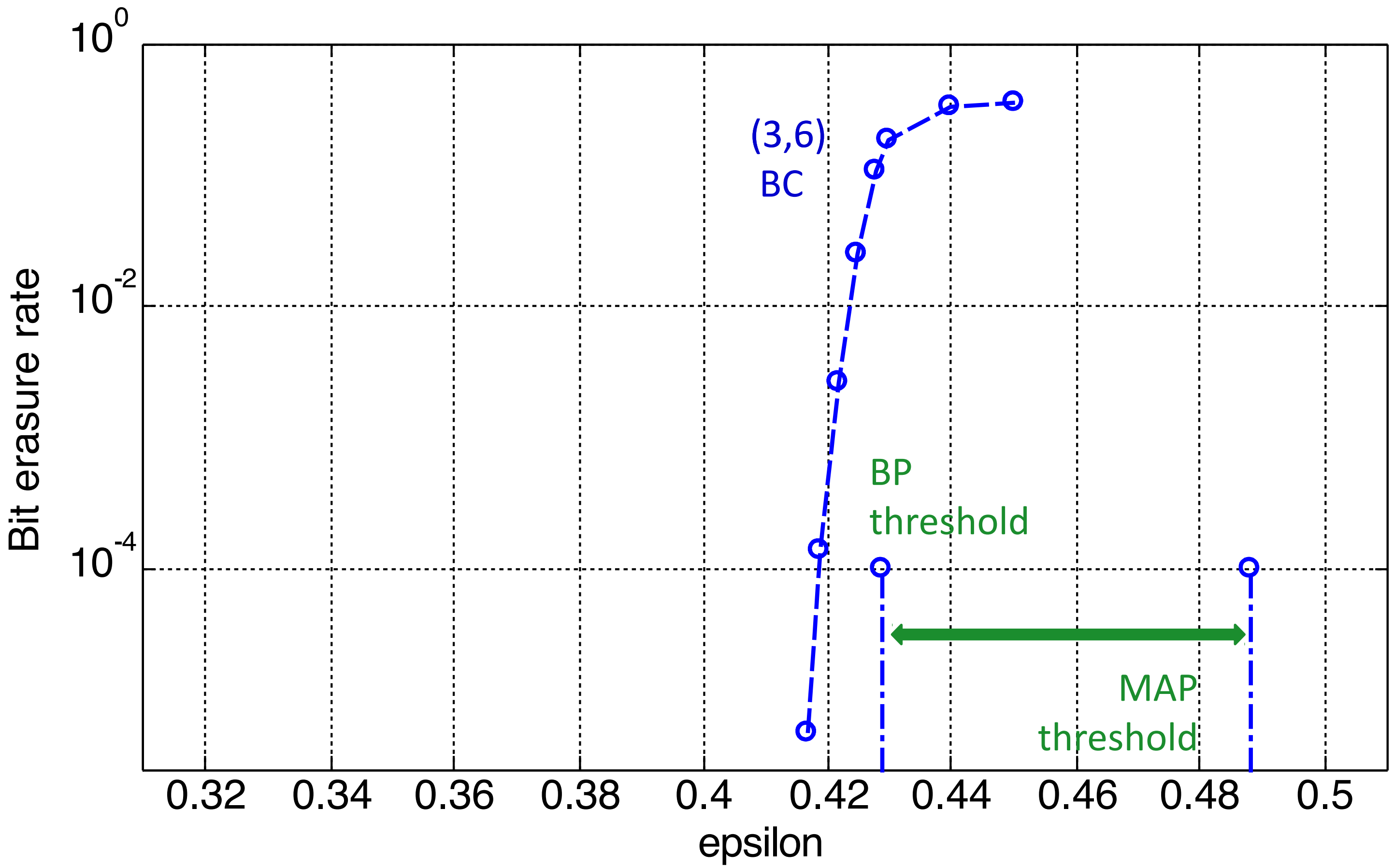
BP = iterative (suboptimal) decoding threshold
MAP = (optimal) maximum a posteriori threshold



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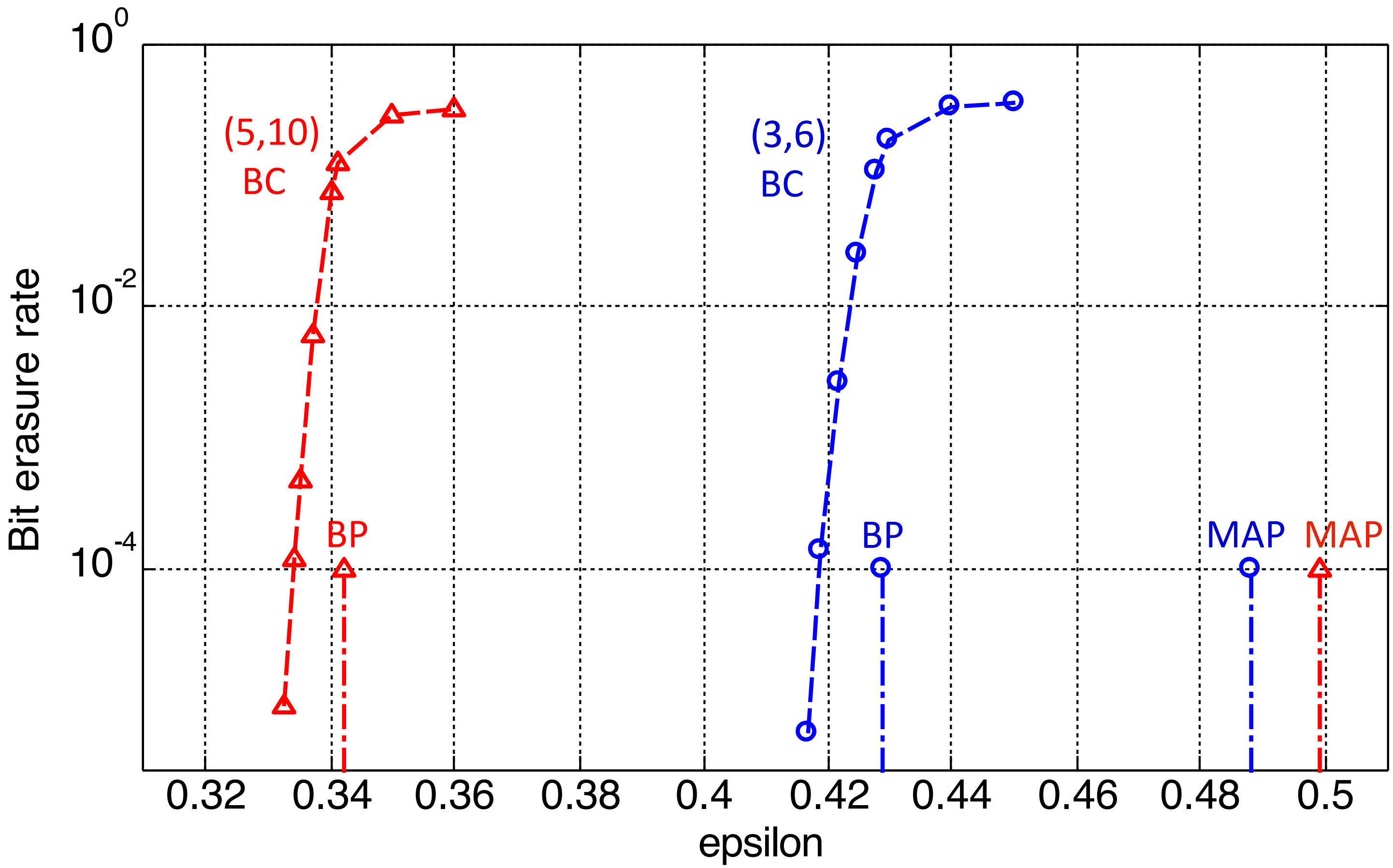

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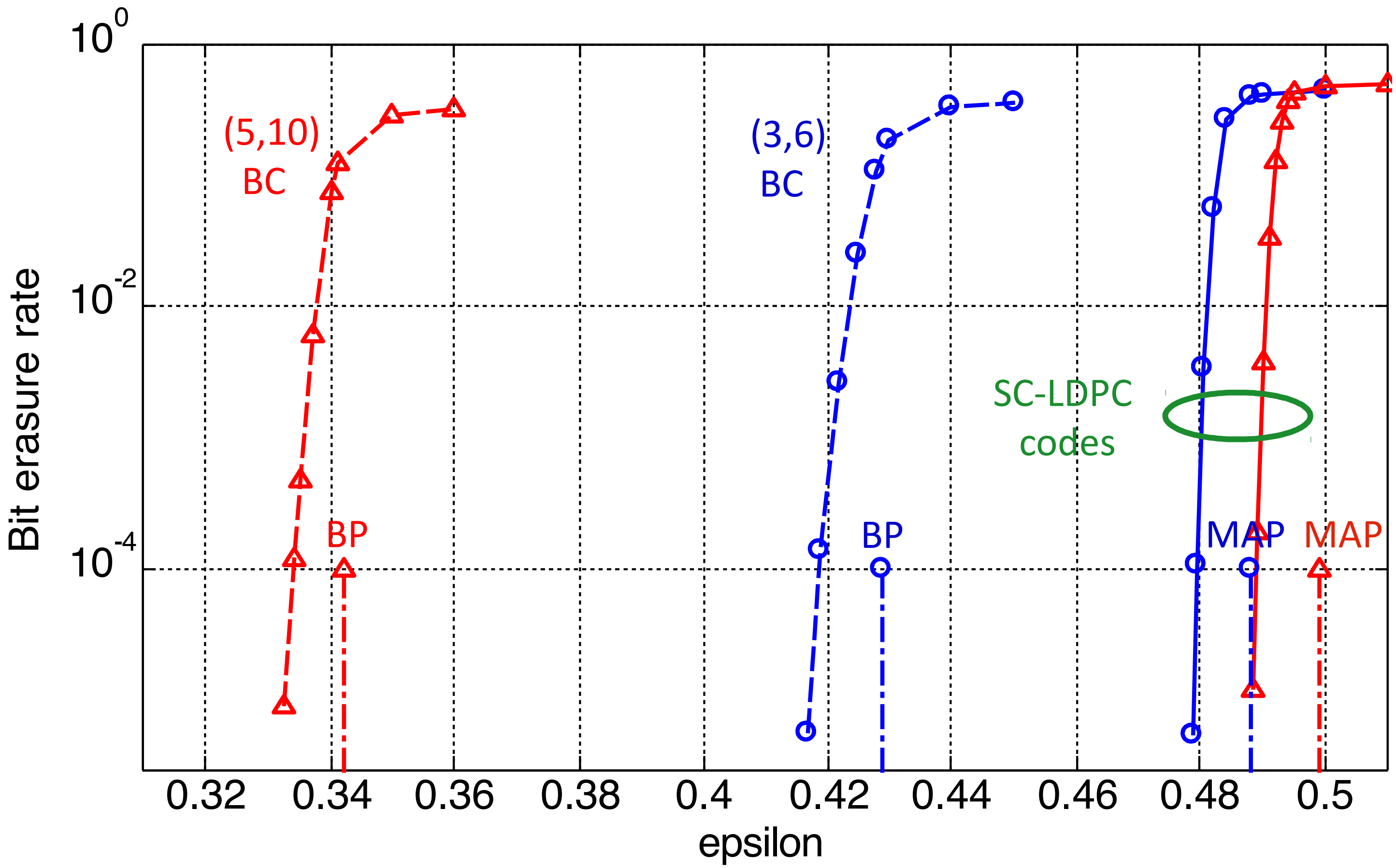

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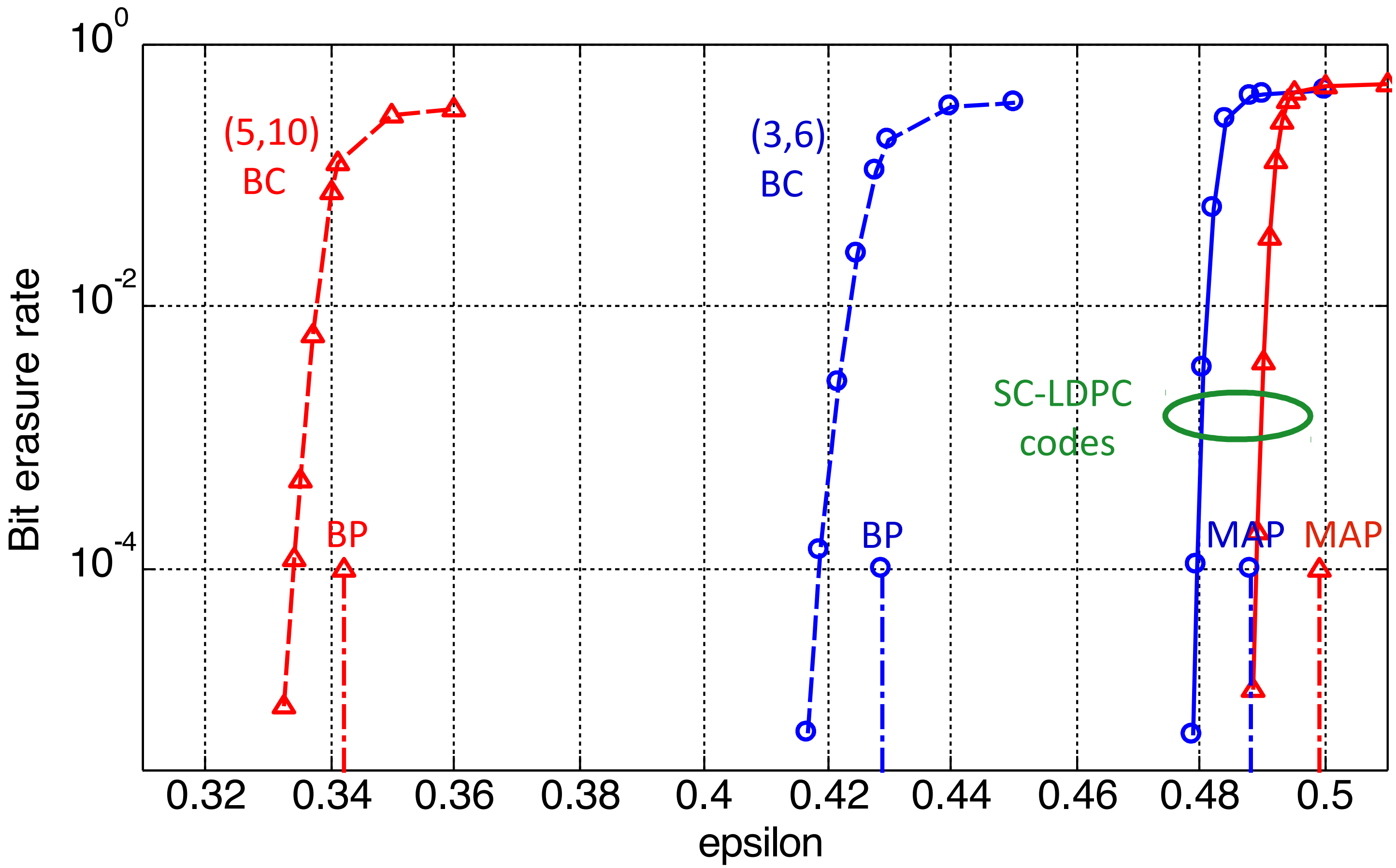

➡ **optimal** decoding performance with a **suboptimal** iterative algorithm!

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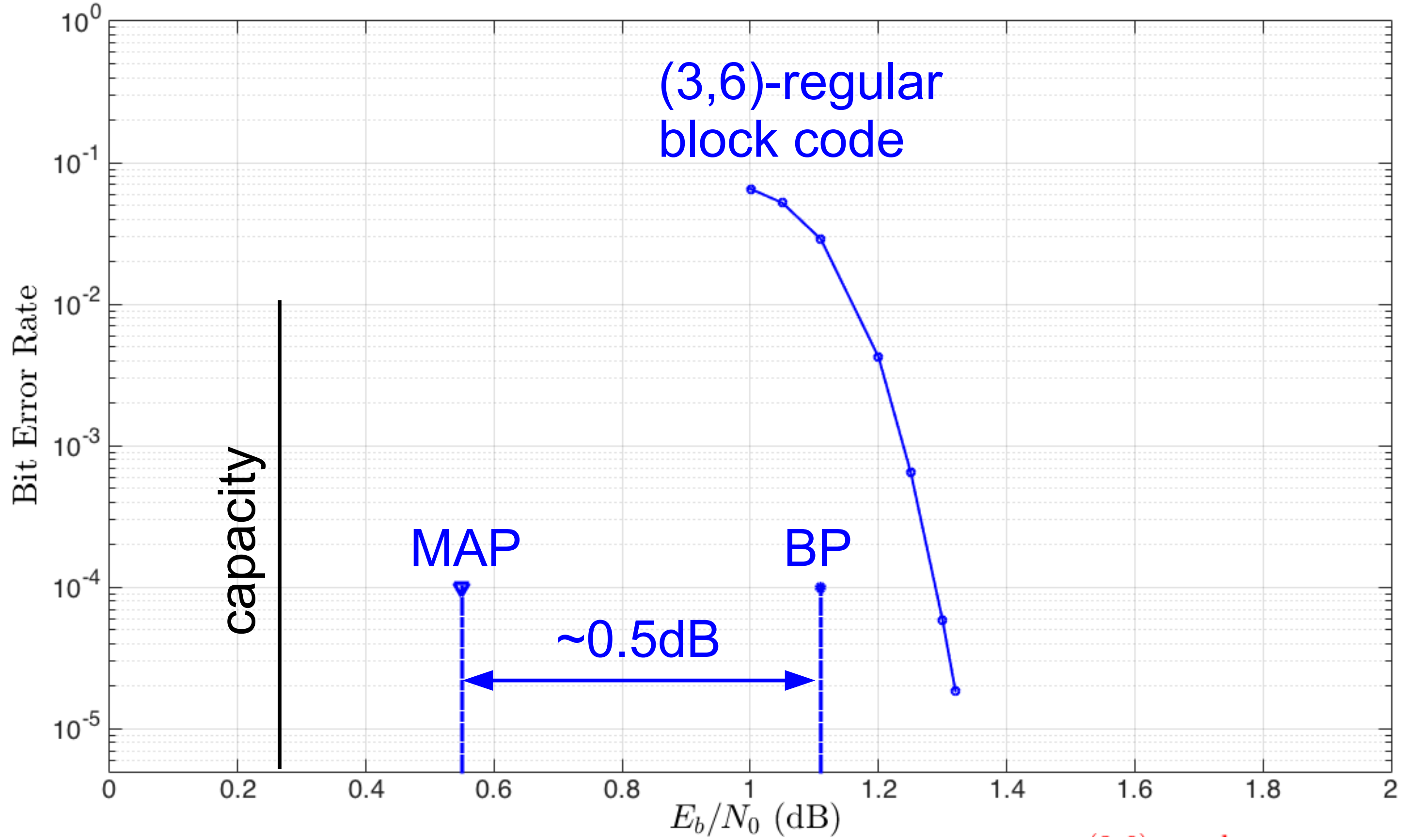

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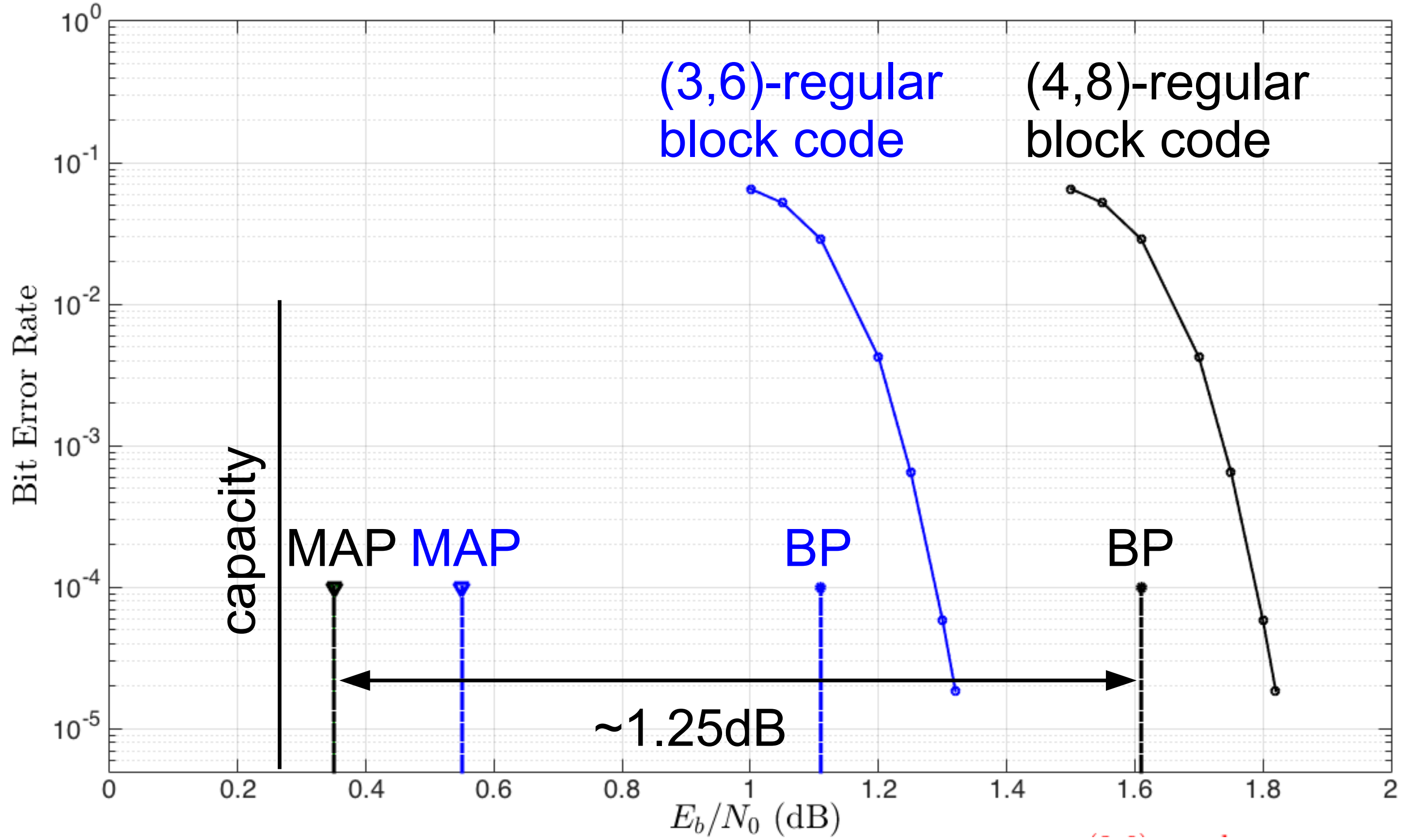

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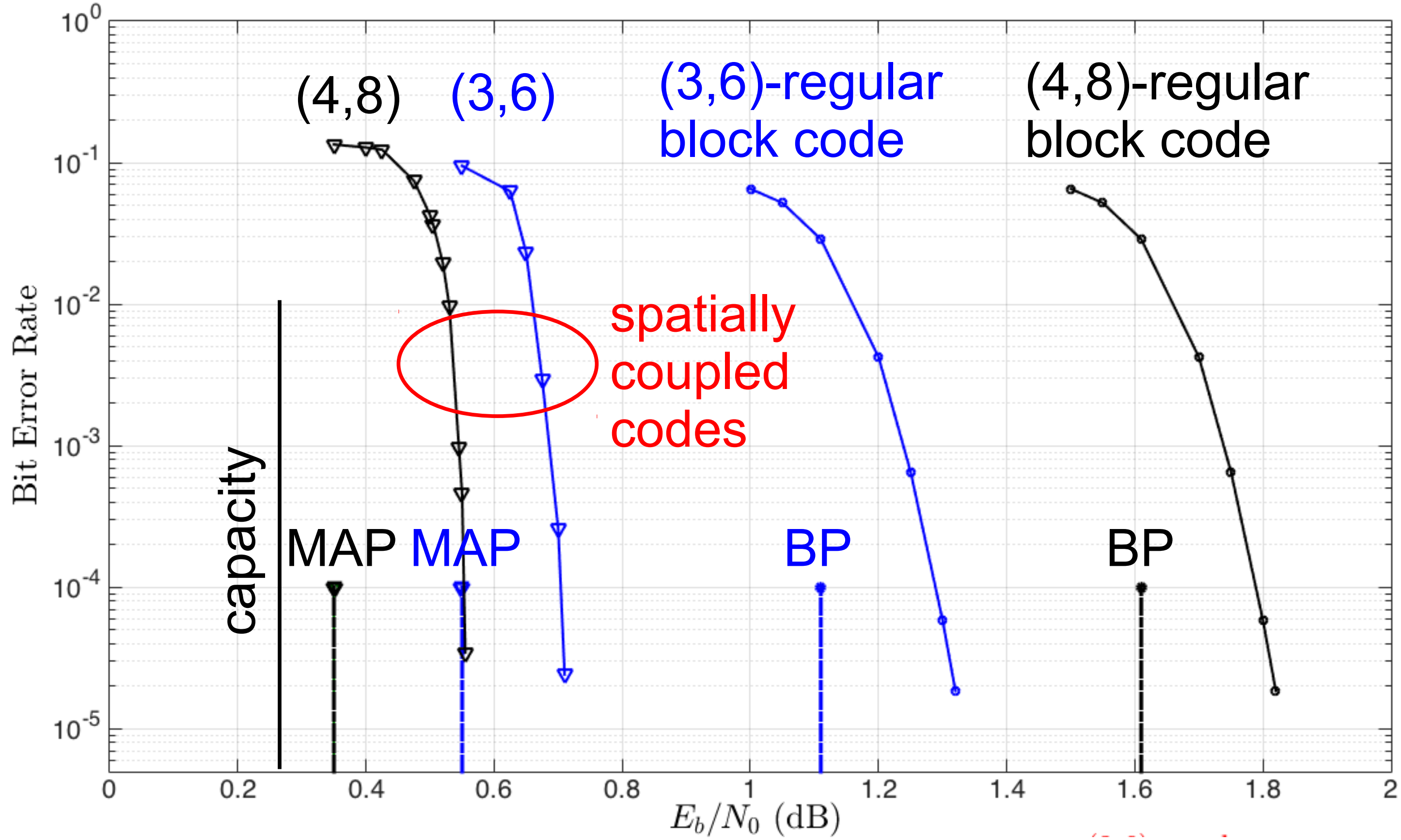

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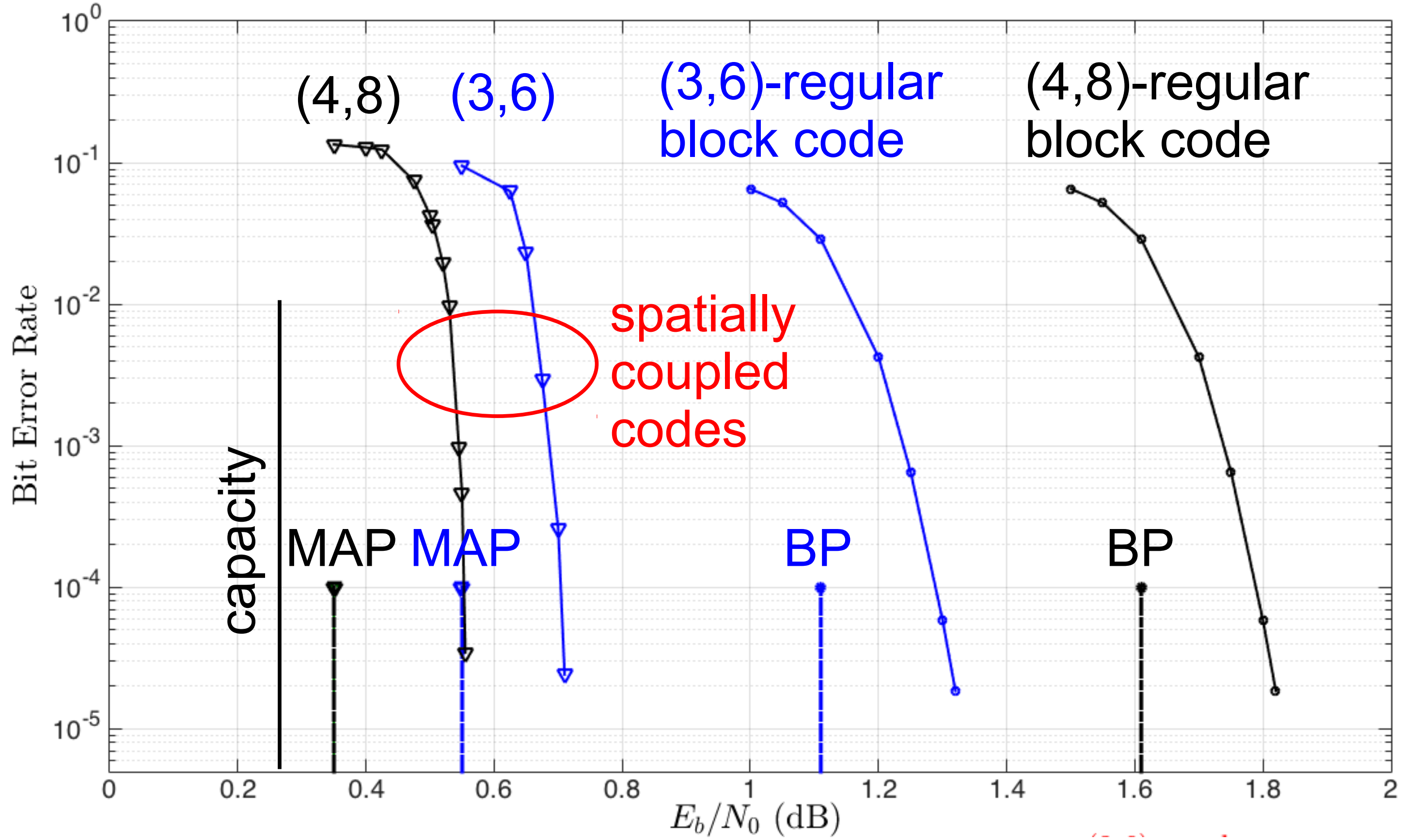

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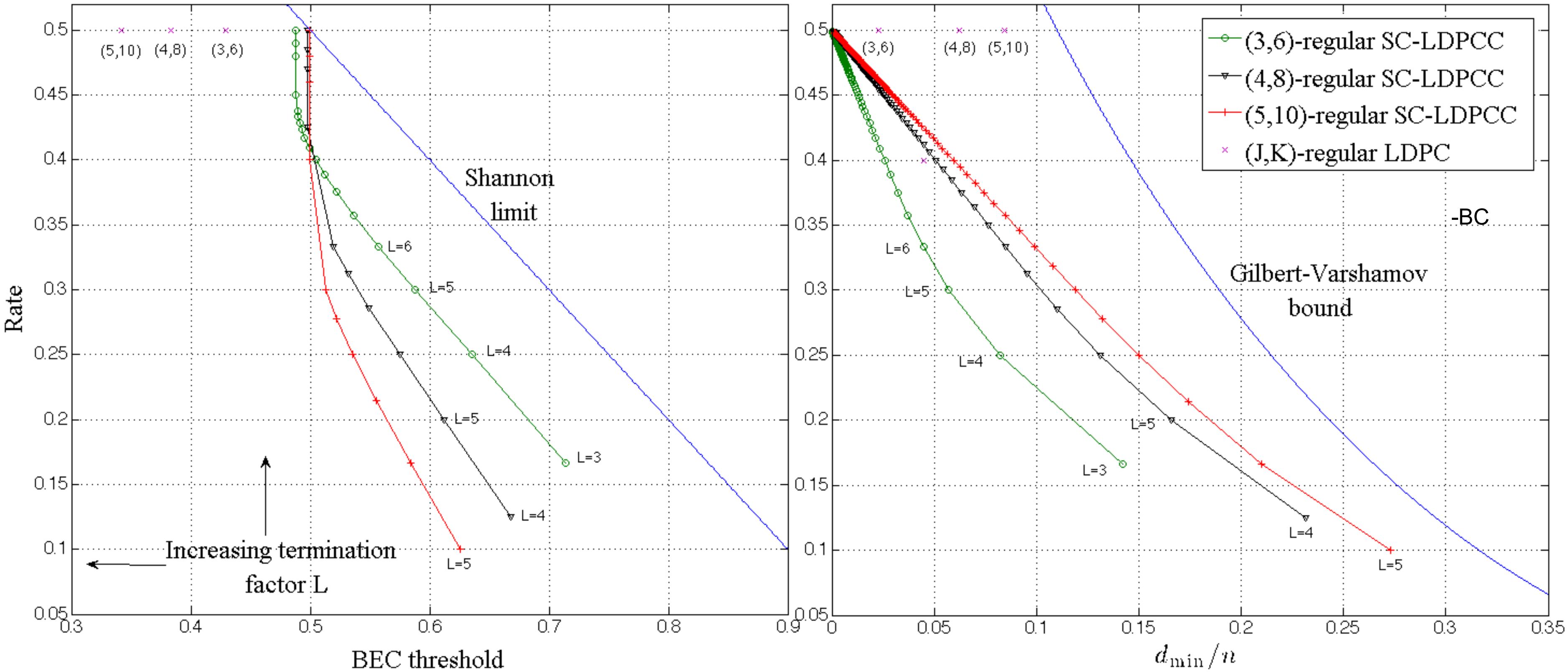
BEC Thresholds vs Distance Growth

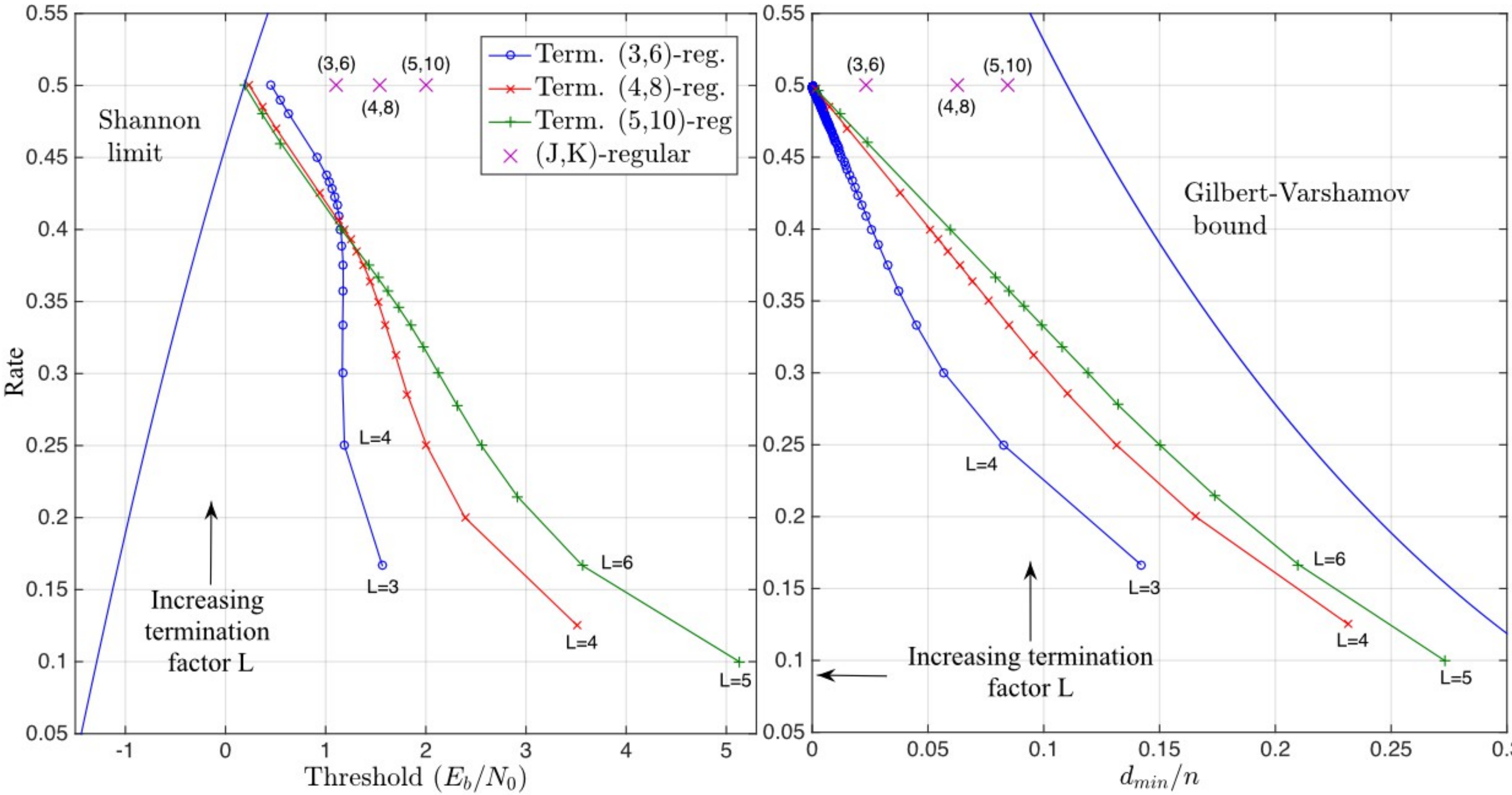
- By increasing J and K, we obtain **capacity achieving** (J,K)-regular SC-LDPC code ensembles with linear minimum distance growth.



- (J,K)-regular SC-LDPC codes combine the best features of irregular and regular LDPC-BCs, i.e., capacity approaching thresholds and linear distance growth.

AWGNC Thresholds vs. Distance Growth

- Similar results are obtained for the AWGNC



[MLC10] D. G. M. Mitchell, M. Lentmaier and D. J. Costello, Jr., "AWGN Channel Analysis of Terminated LDPC Convolutional Codes", *Proc. Information Theory and Applications Workshop*, San Diego, Feb. 2011.

Outline

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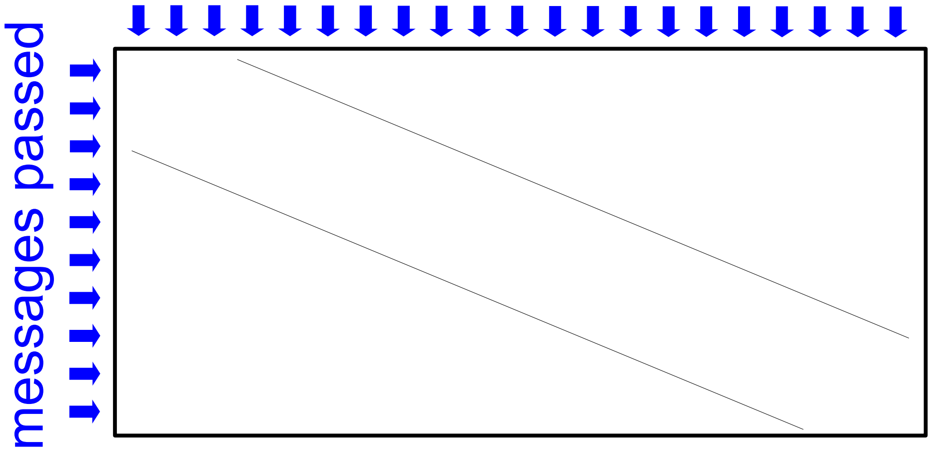
- Introduction: From Shannon to Modern Coding Theory
  - ➡ Channel capacity, structured codes, random codes, LDPC codes
- LDPC Block Codes
  - ➡ Parity-check matrix and Tanner graph representations, minimum distance bounds, iterative decoding thresholds, protograph-based constructions
- Spatially Coupled LDPC Codes
  - ➡ Protograph representation, edge-spreading construction, termination
  - ➡ Iterative decoding thresholds, threshold saturation, minimum distance
- **Practical Considerations**
  - ➡ Window decoding, performance, latency, and complexity comparisons to LDPC block codes, rate-compatibility, implementation aspects

Block Decoding of SC-LDPC Codes

- SC-LDPC codes can be decoded with standard iterative decoding schedules.



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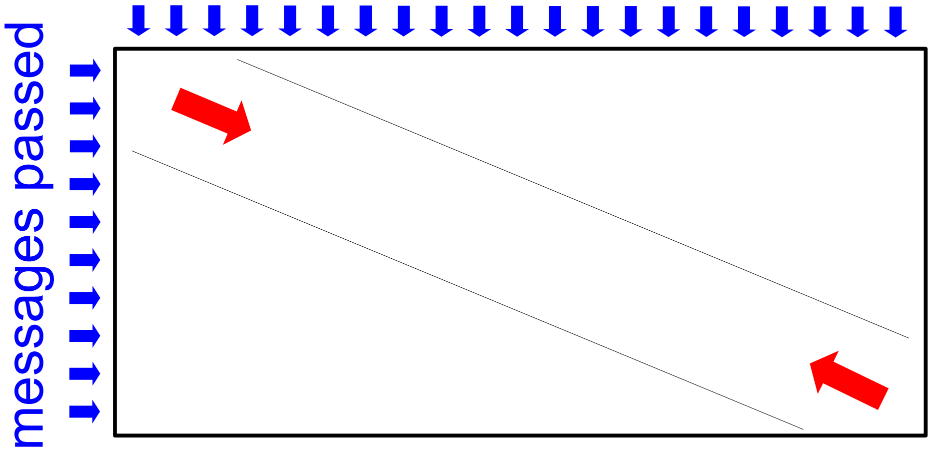

Reliable messages from the ends propagate through the graph toward the center as iterations proceed.

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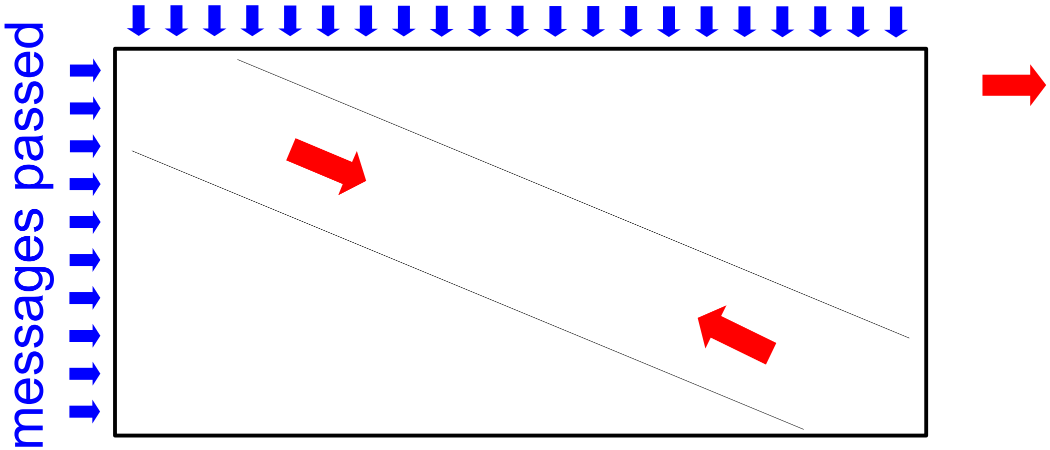

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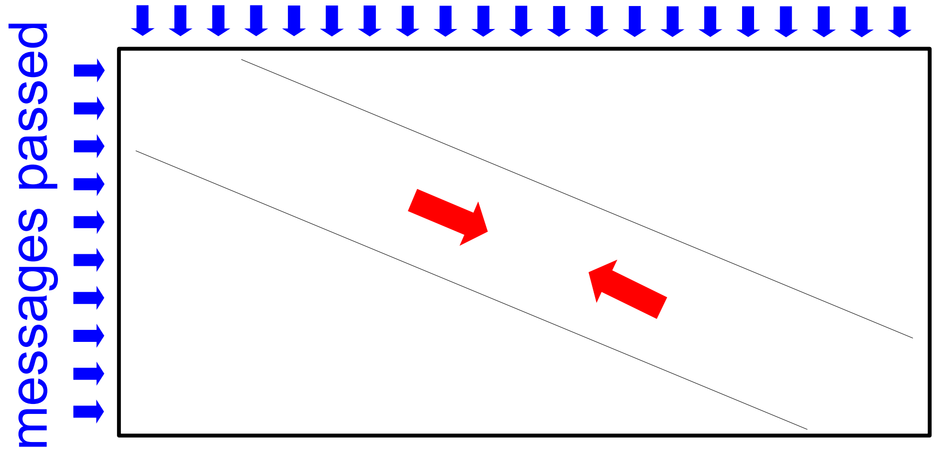

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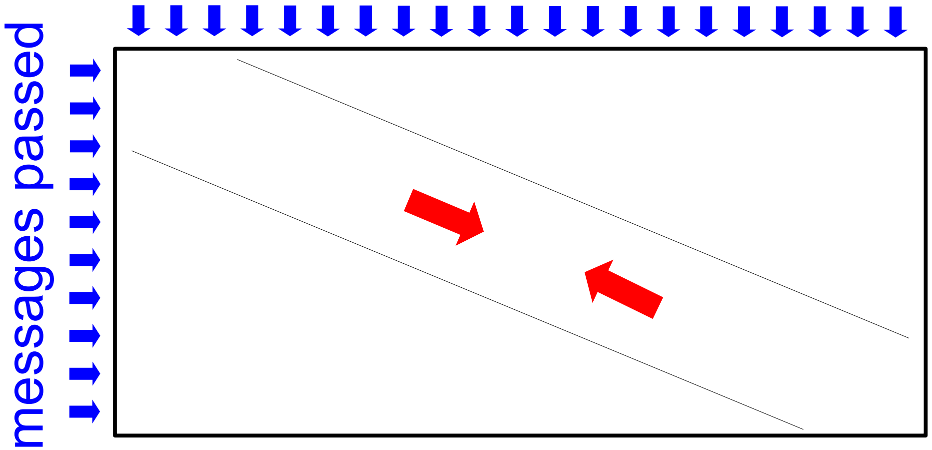
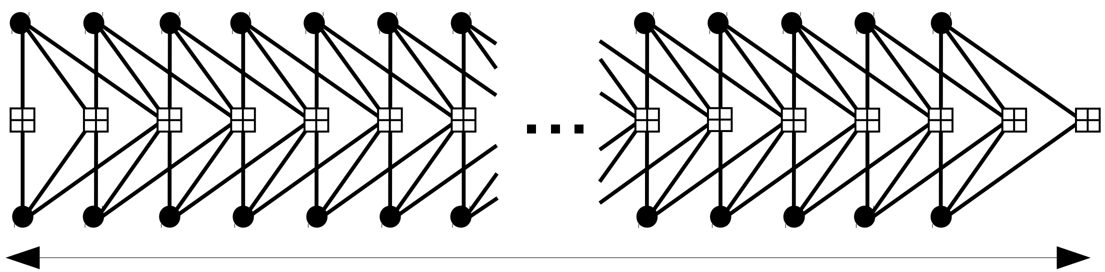


➡ Reliable messages from the ends propagate through the graph toward the center as iterations proceed.

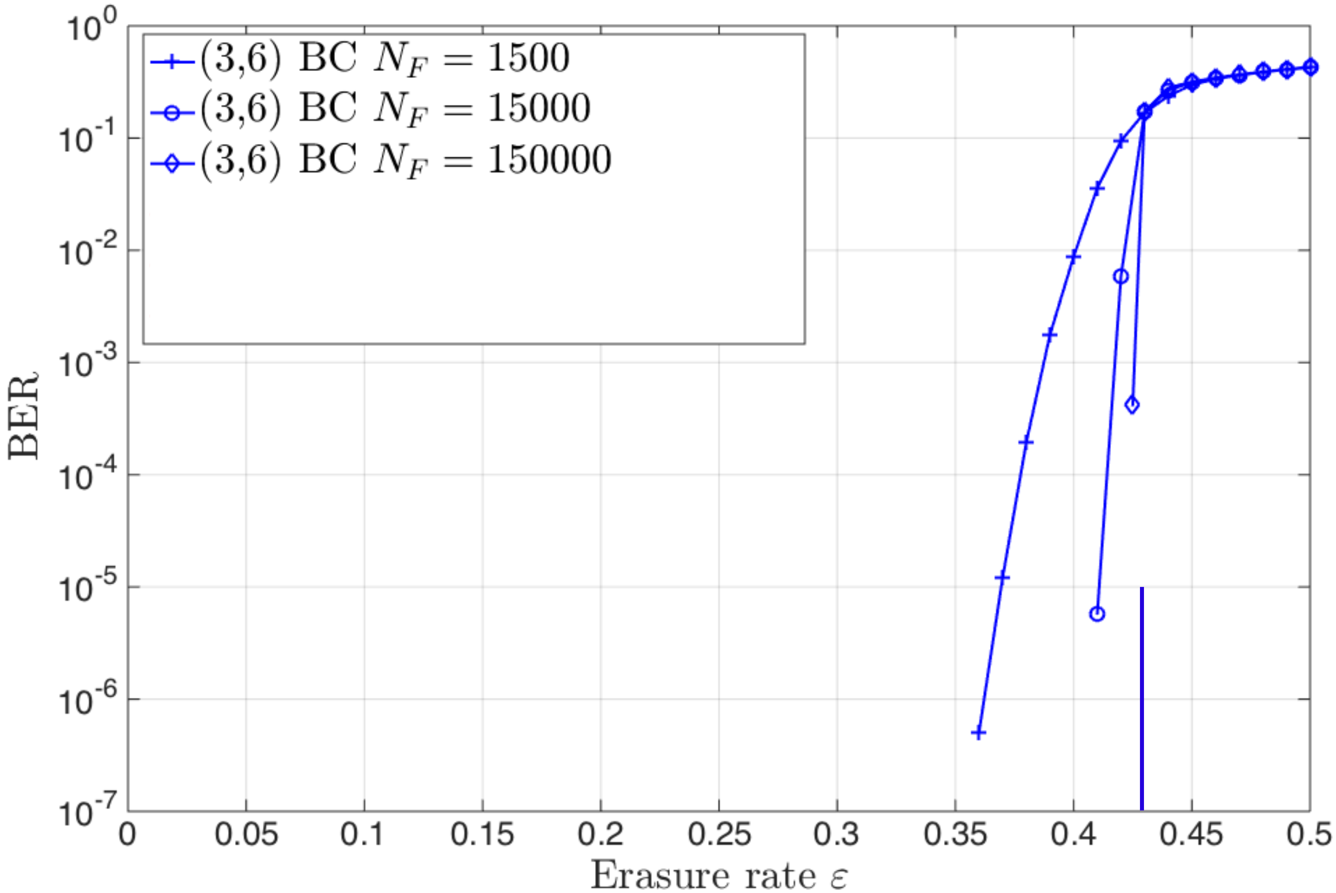
- The **frame error rate (FER)** of a terminated graph can be analyzed



➡ The **FER depends on L** (FER $\xrightarrow[L\to\infty]{} 1$)

Flooding Decoding Scaling (1) – BEC

- Consider LDPC-BCs and SC-LDPC codes with increasing frame length N_F



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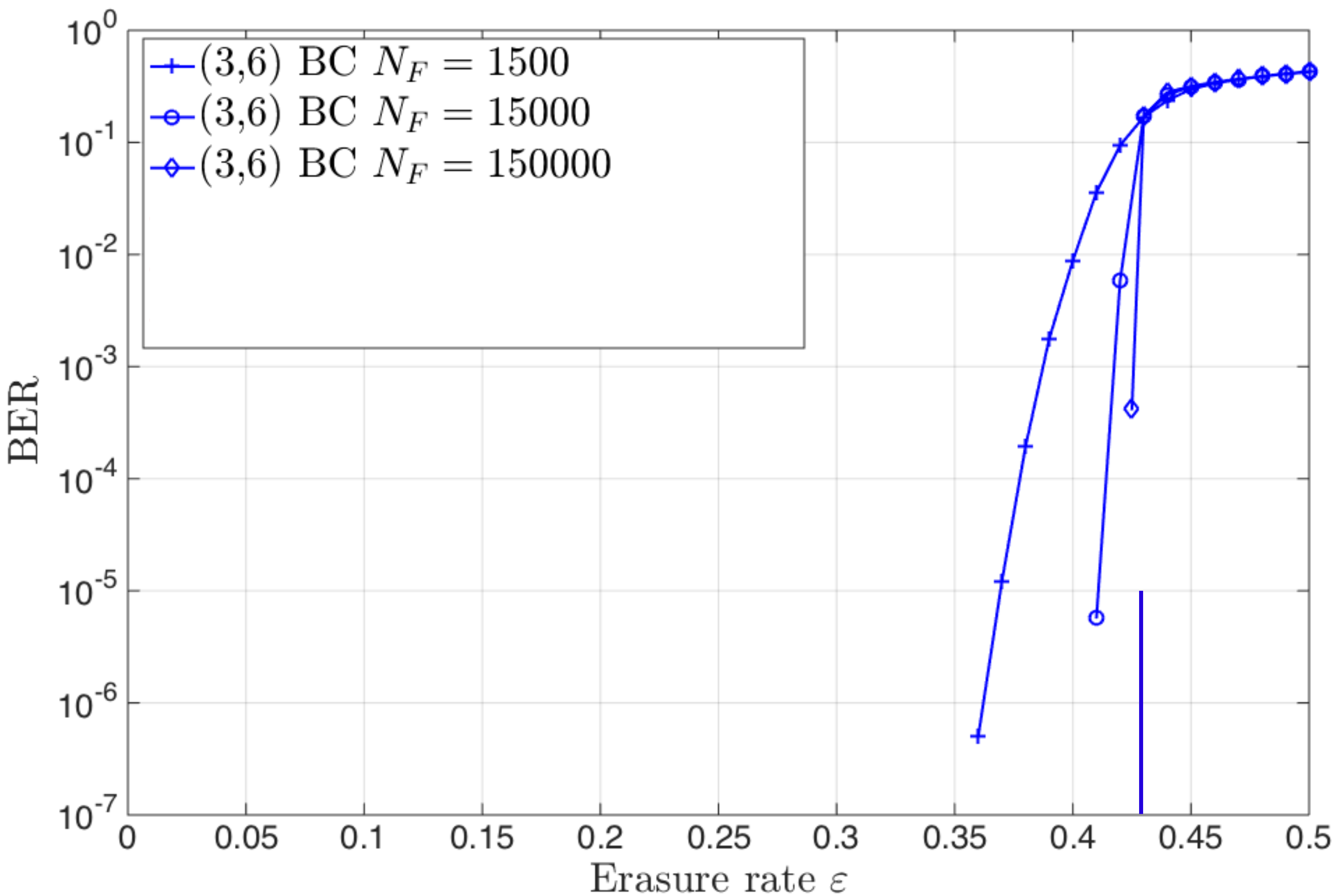

- As N_F increases the LDPC-BC performance approaches $\varepsilon^* = 0.429$

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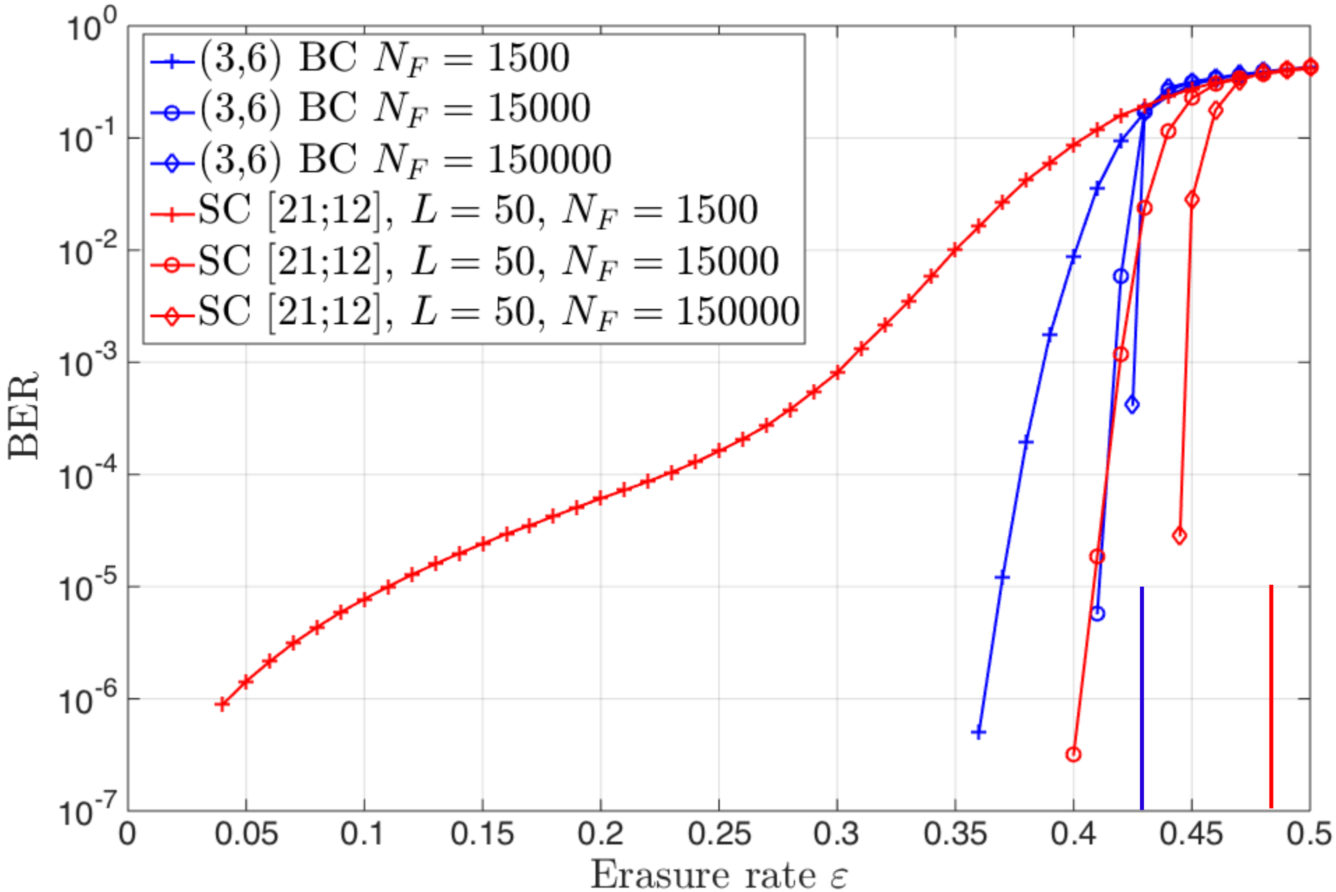

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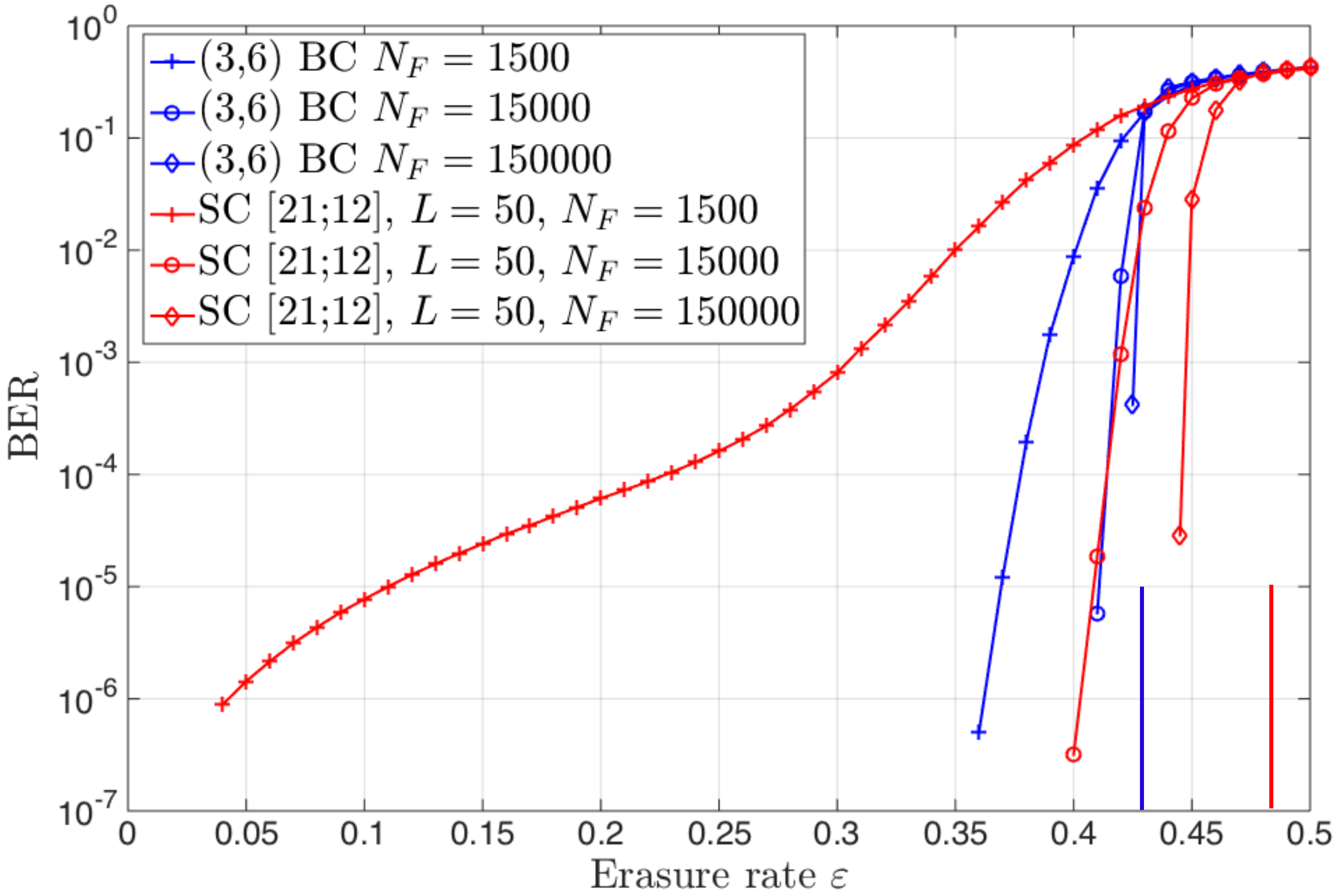

- As N_F increases the LDPC-BC performance approaches

  $\varepsilon^* = 0.429$

- As $N_F = 2LM$ increases (fixed L and increasing M = 15,150,1500), the SC-LDPC code performance approaches

  $\varepsilon^* = 0.488$, outperforming the LDPC-BC

9

Flooding Decoding Scaling (2) – BEC

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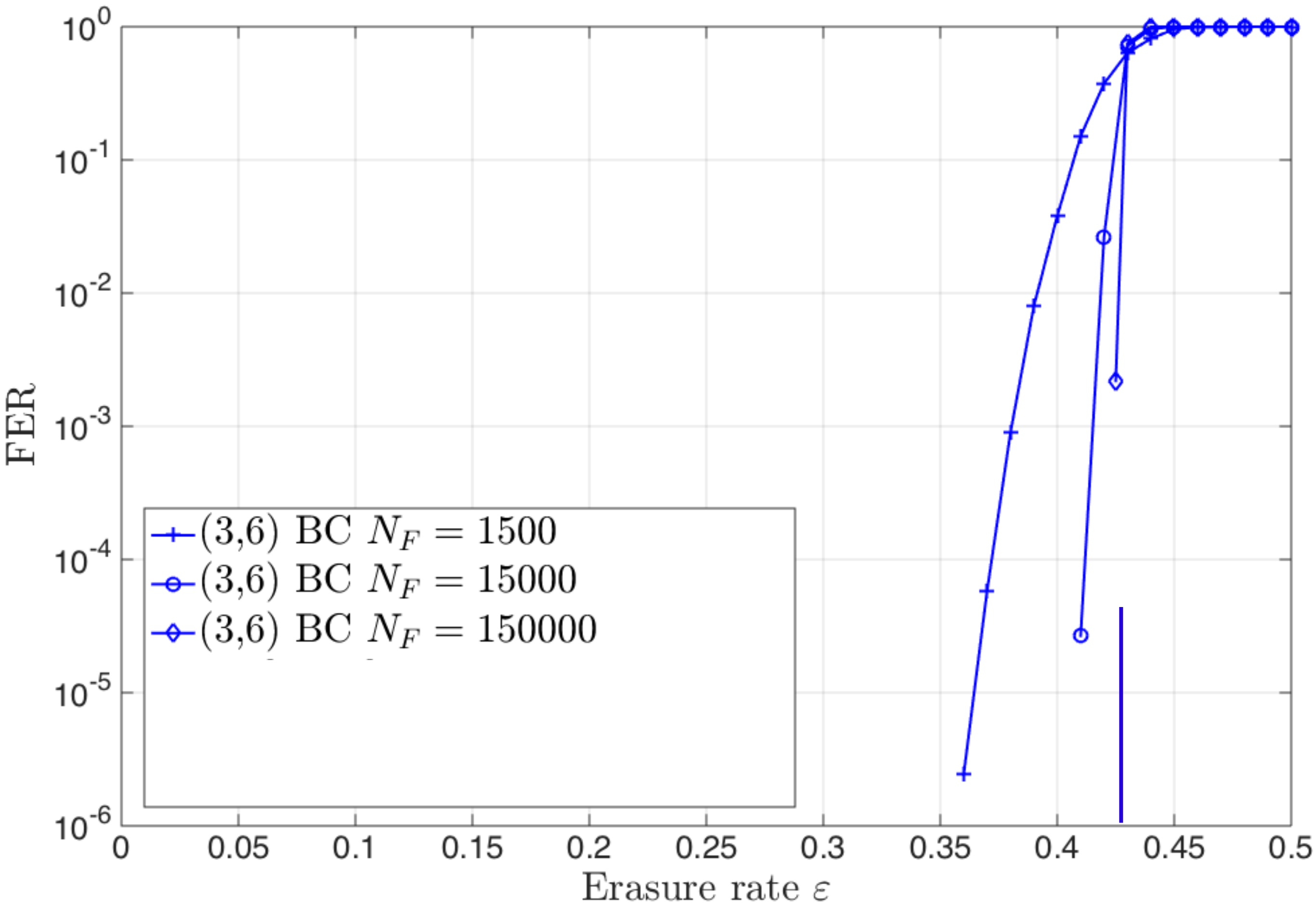

- We now consider the FER performance

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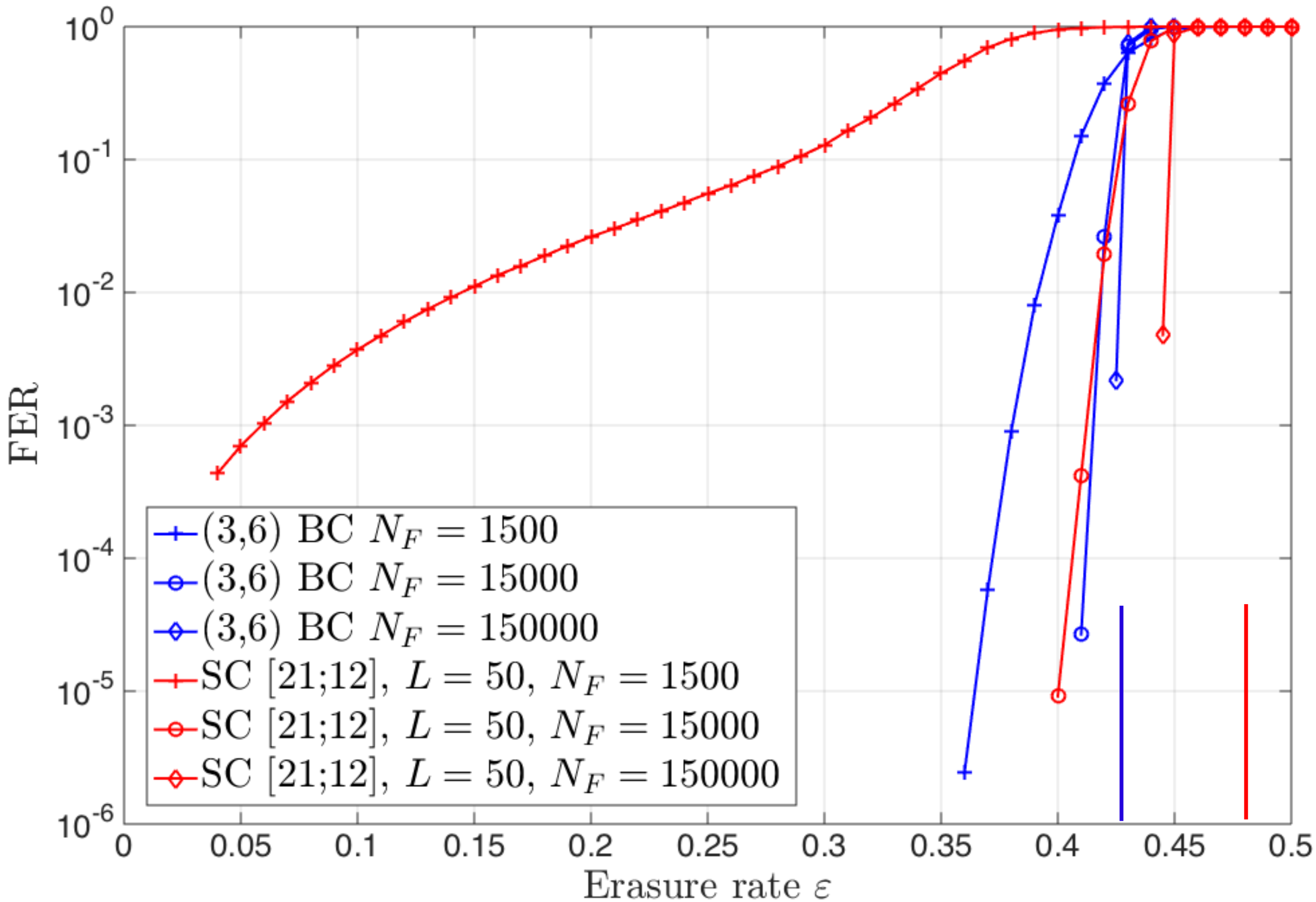

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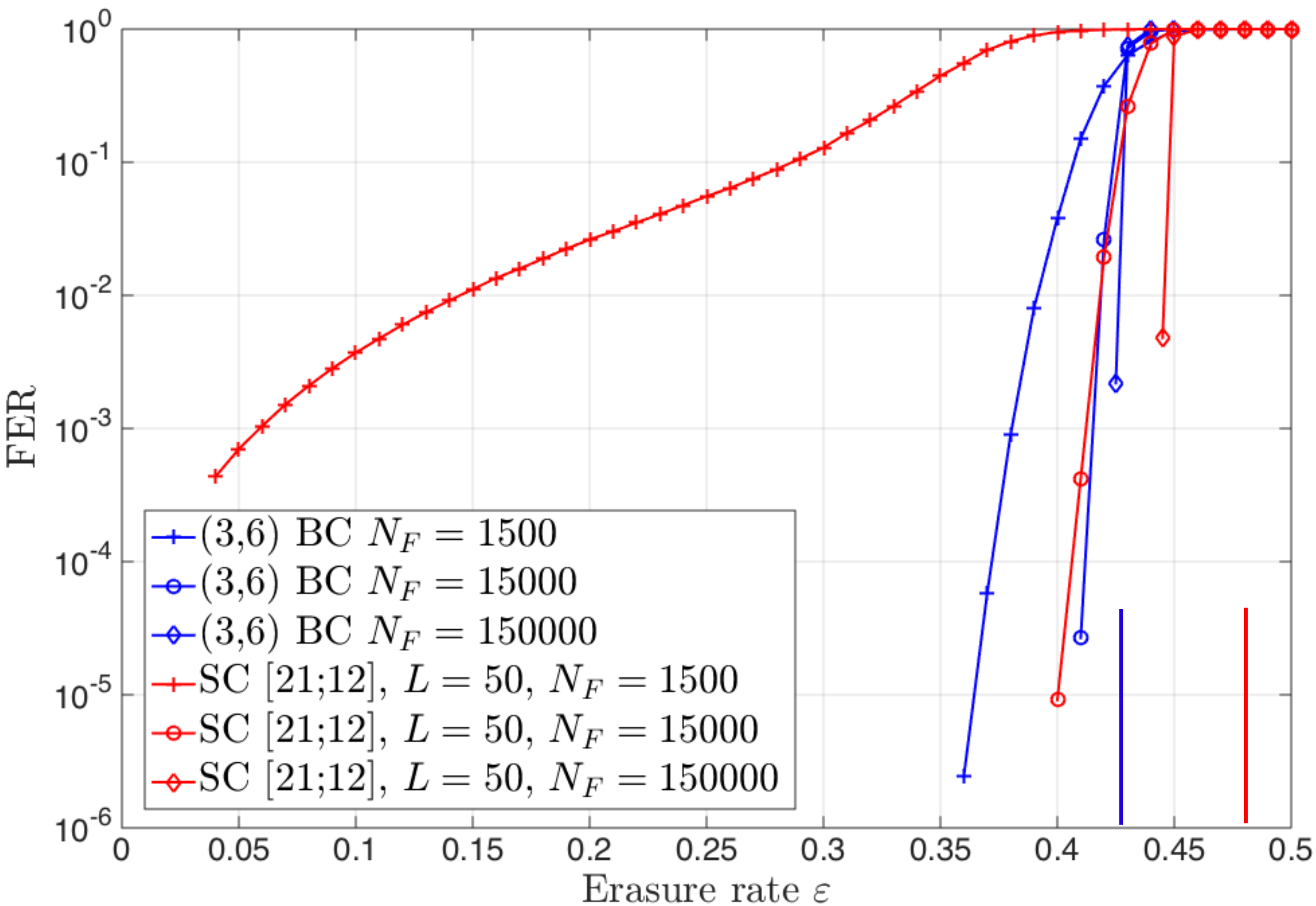
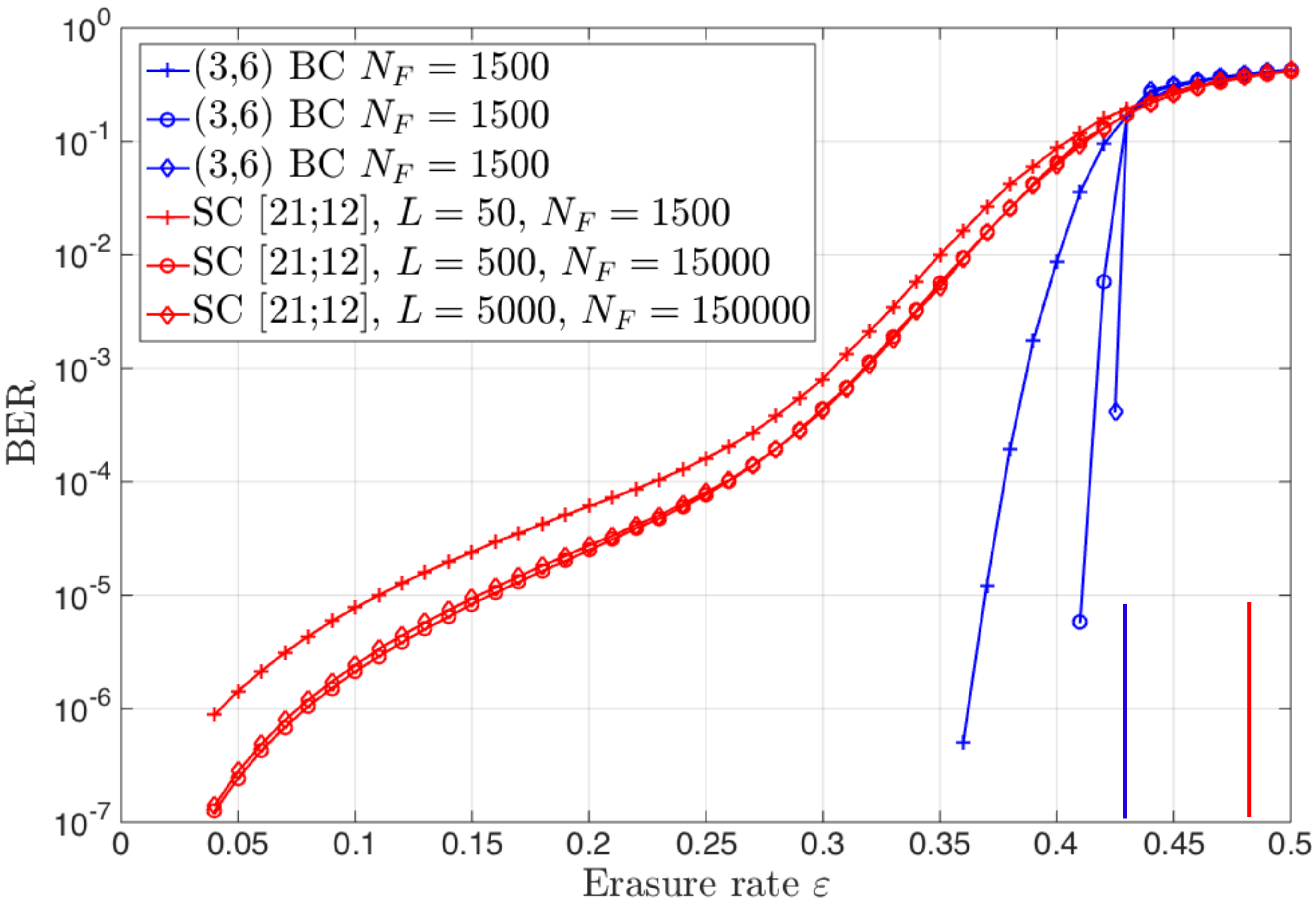


- We now consider the FER performance
- The FER curves display similar behavior for fixed L and increasing M

➡ The SC-LDPC codes outperform LDPC-BCs for large N_F

Flooding Decoding Scaling (3) – BEC

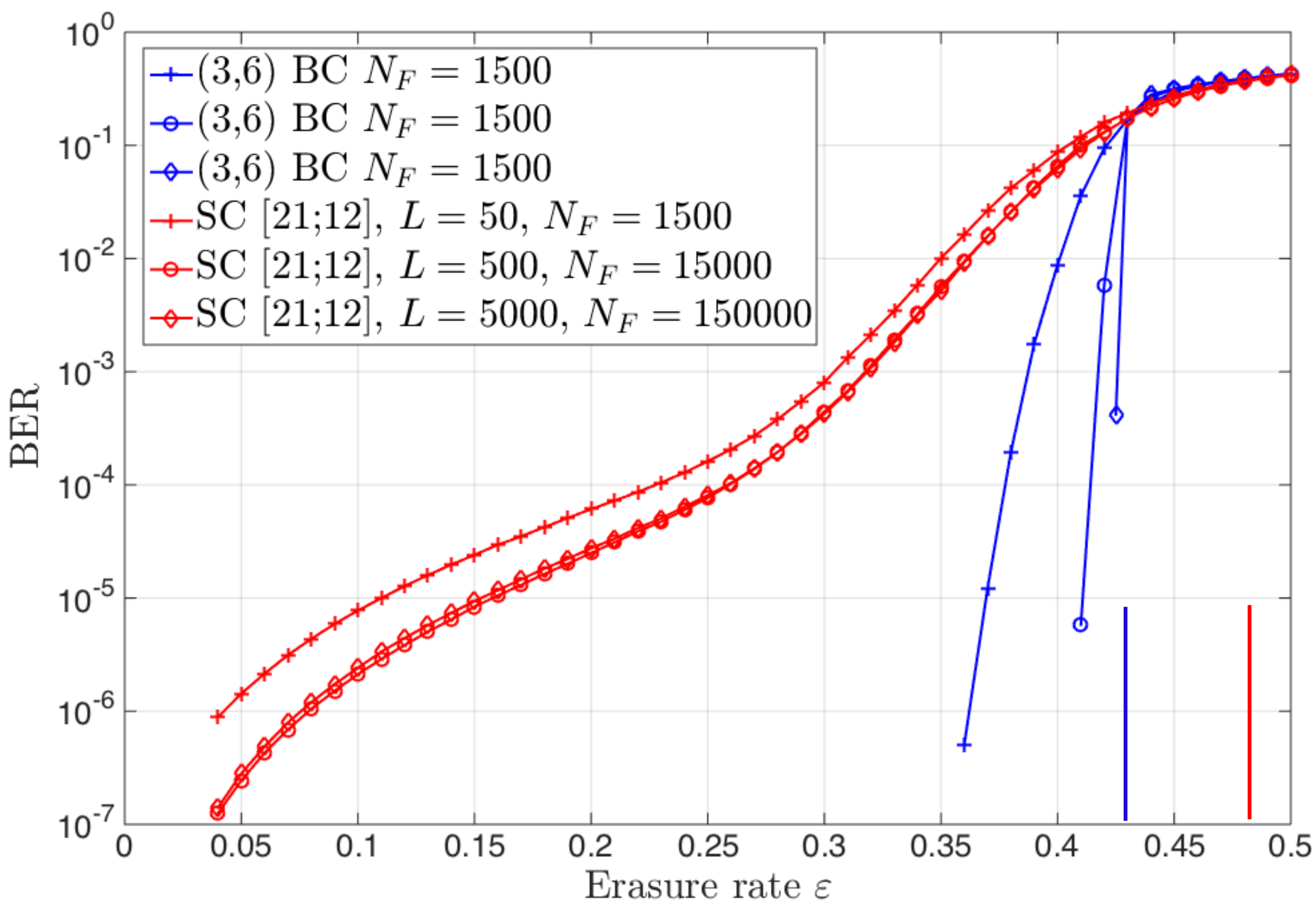
- Consider LDPC-BCs and SC-LDPC codes with increasing frame length N_F



- We now consider increasing the frame length $N_F = 2LM$ with fixed M = 15 and increasing L

Flooding Decoding Scaling (3) – BEC

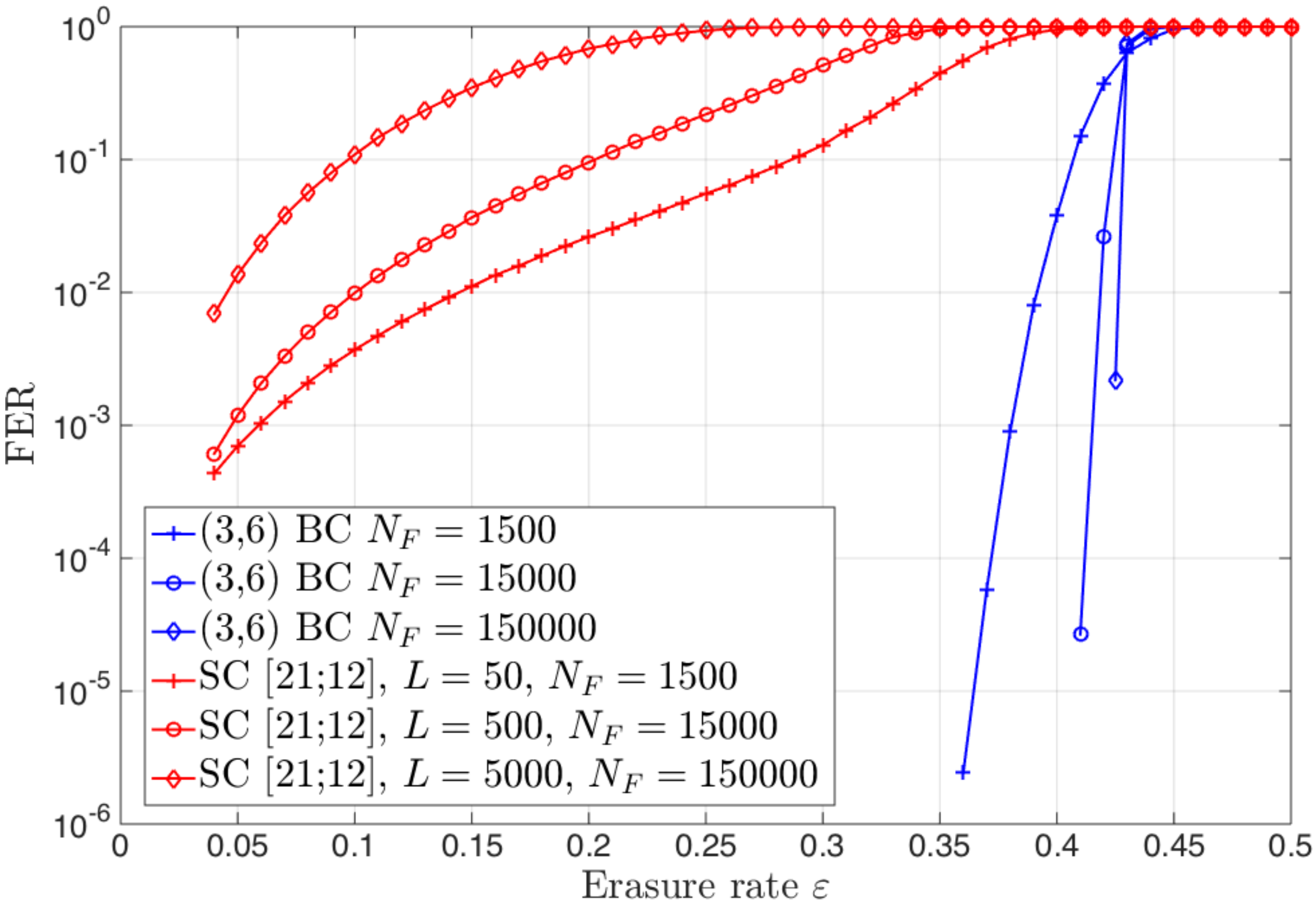
- Consider LDPC-BCs and SC-LDPC codes with increasing frame length N_F



- We now consider increasing the frame length $N_F = 2LM$ with fixed M = 15 and increasing L
- Note that the BER performance is poor for the SC-LDPC codes (small M) and does not improve substantially with increasing L

Flooding Decoding Scaling (4) – BEC

- Consider LDPC-BCs and SC-LDPC codes with increasing frame length N_F



Flooding Decoding Scaling (4) – BEC

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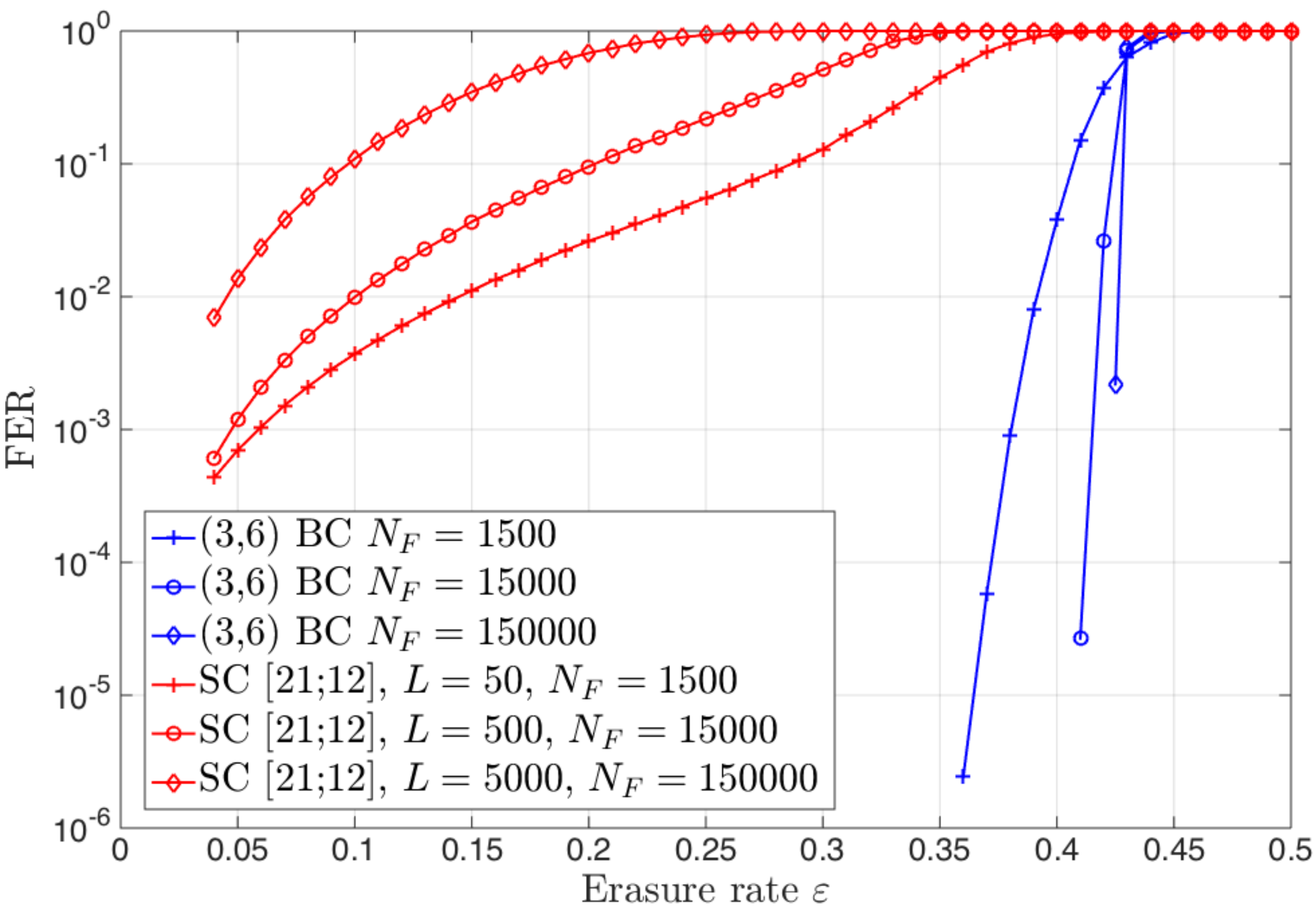

- The FER performance is even more extreme since the **FER depends on L** (FER $\xrightarrow[L\to\infty]{} 1$)

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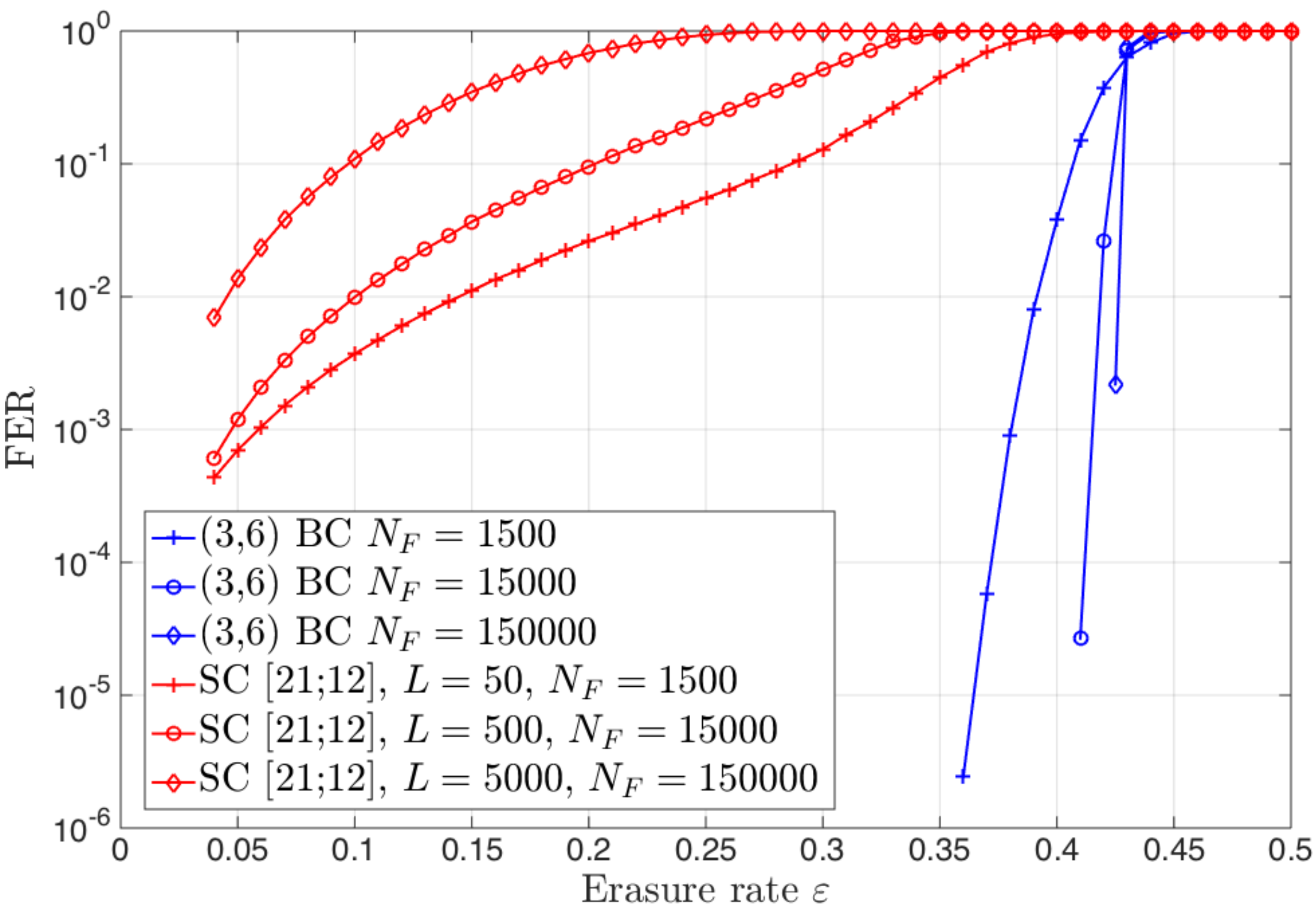


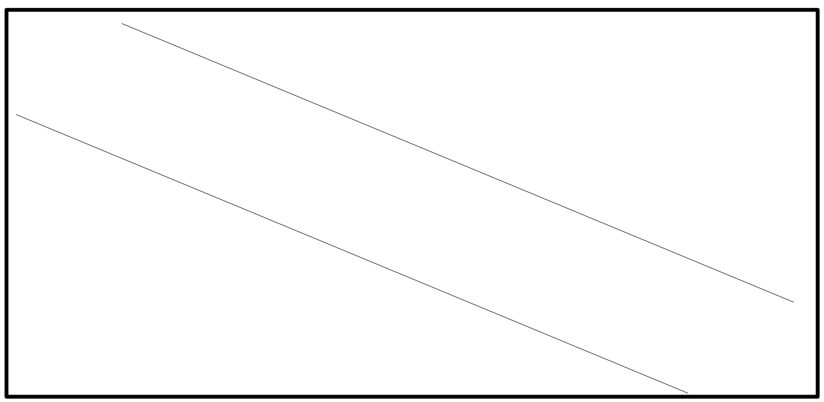
- The FER performance is even more extreme since the **FER depends on L**

  (FER $\xrightarrow[L\to\infty]{} 1$)

- Correspondingly, we note that the larger L codes perform worse (the order is reversed)

Window Decoding of SC-LDPC Codes

- The **highly localized (convolutional) structure** is well-suited for efficient decoding schedules that reduce memory and latency requirements.



Window Decoding of SC-LDPC Codes

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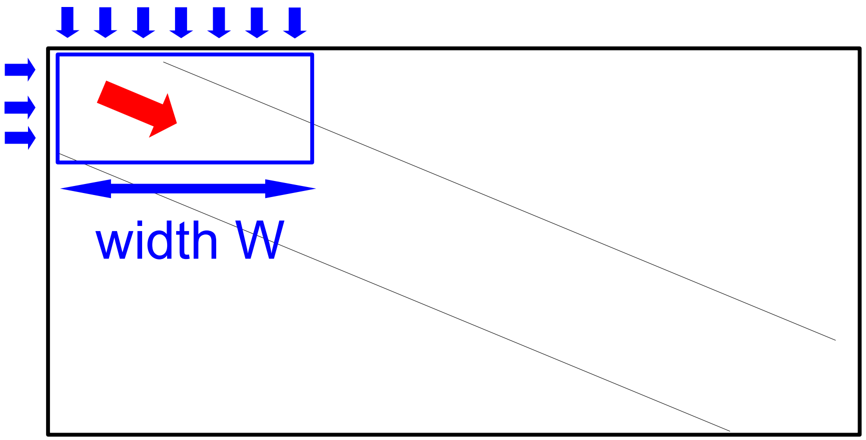

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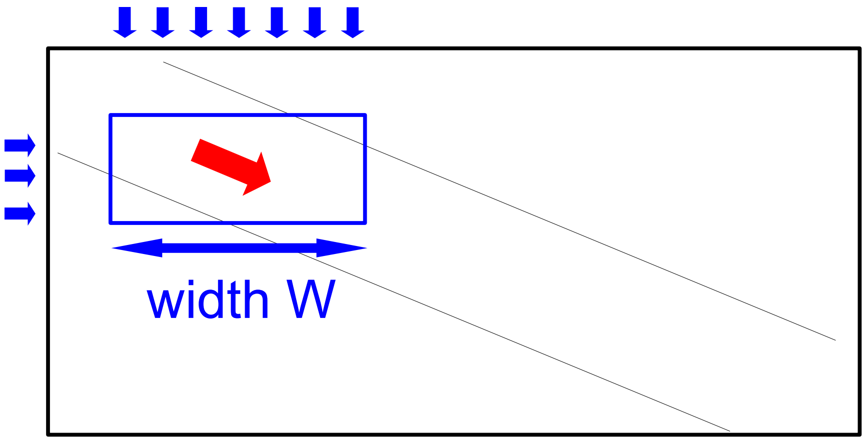

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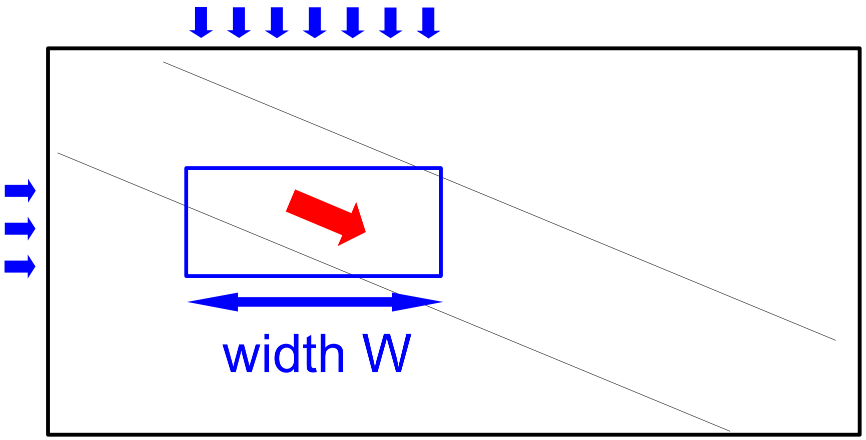

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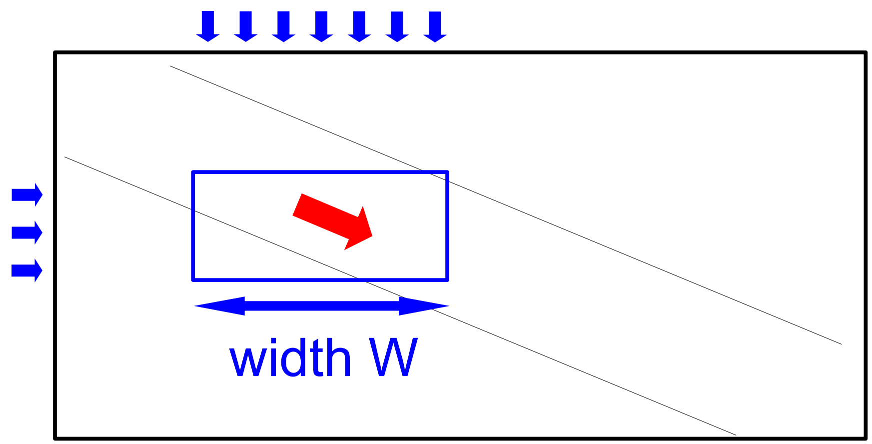

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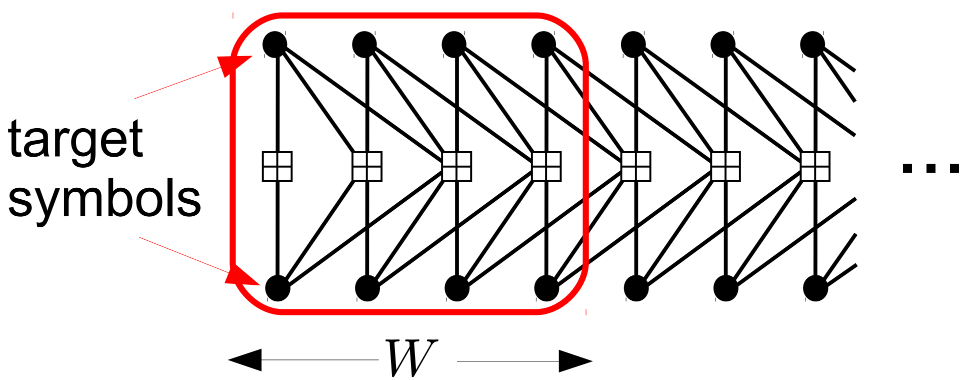

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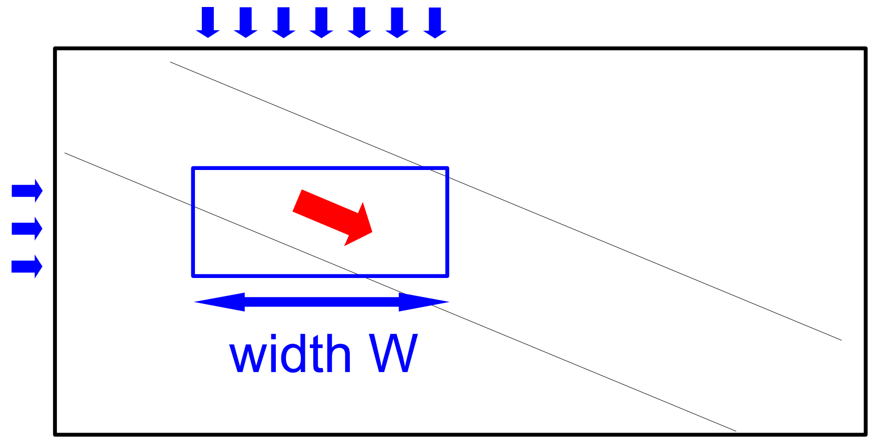

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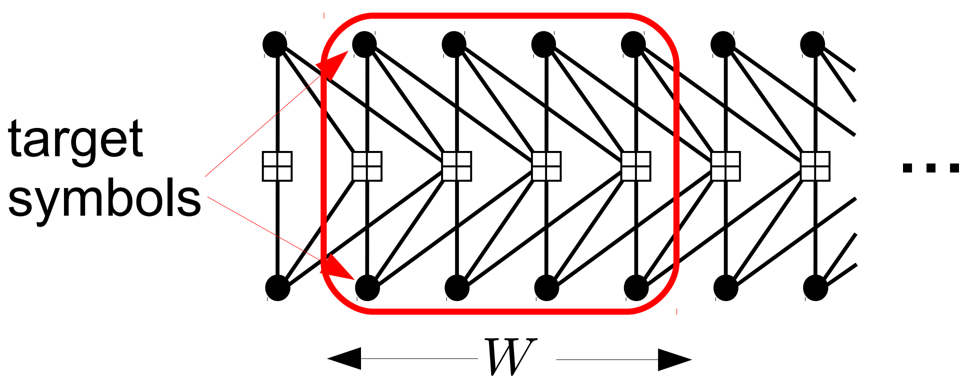

➡ One block of cM **target symbols** is decoded in each window position

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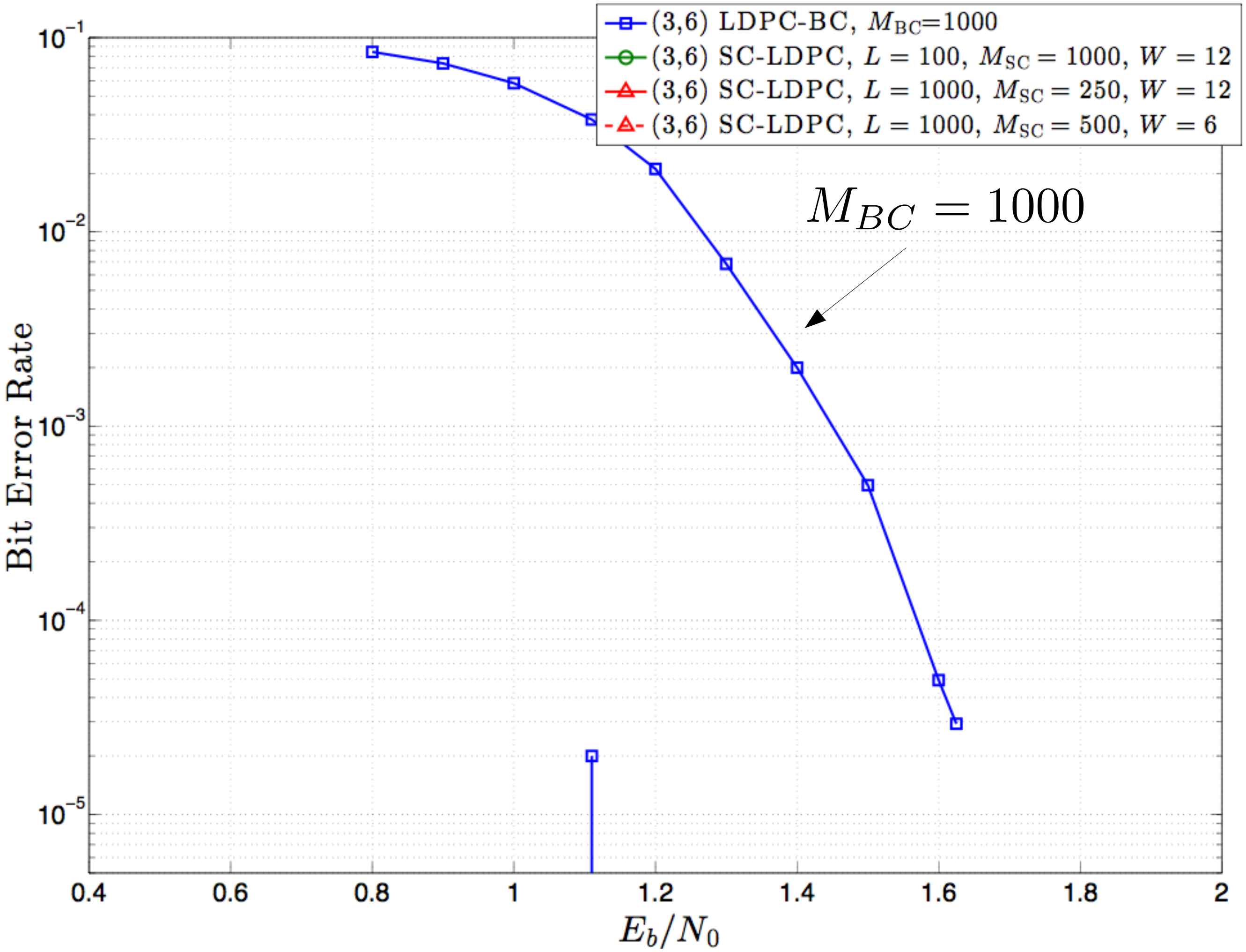
Window Decoding Performance



Latencies:
LDPC: $6M_{BC}$
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[LPF11] M. Lentmaier, M. M. Prenda, and G. Fettweis, "Efficient Message Passing Scheduling for Terminated LDPC Convolutional Codes", *Proc. IEEE ISIT*, St. Petersburg, Russia, July 2011.

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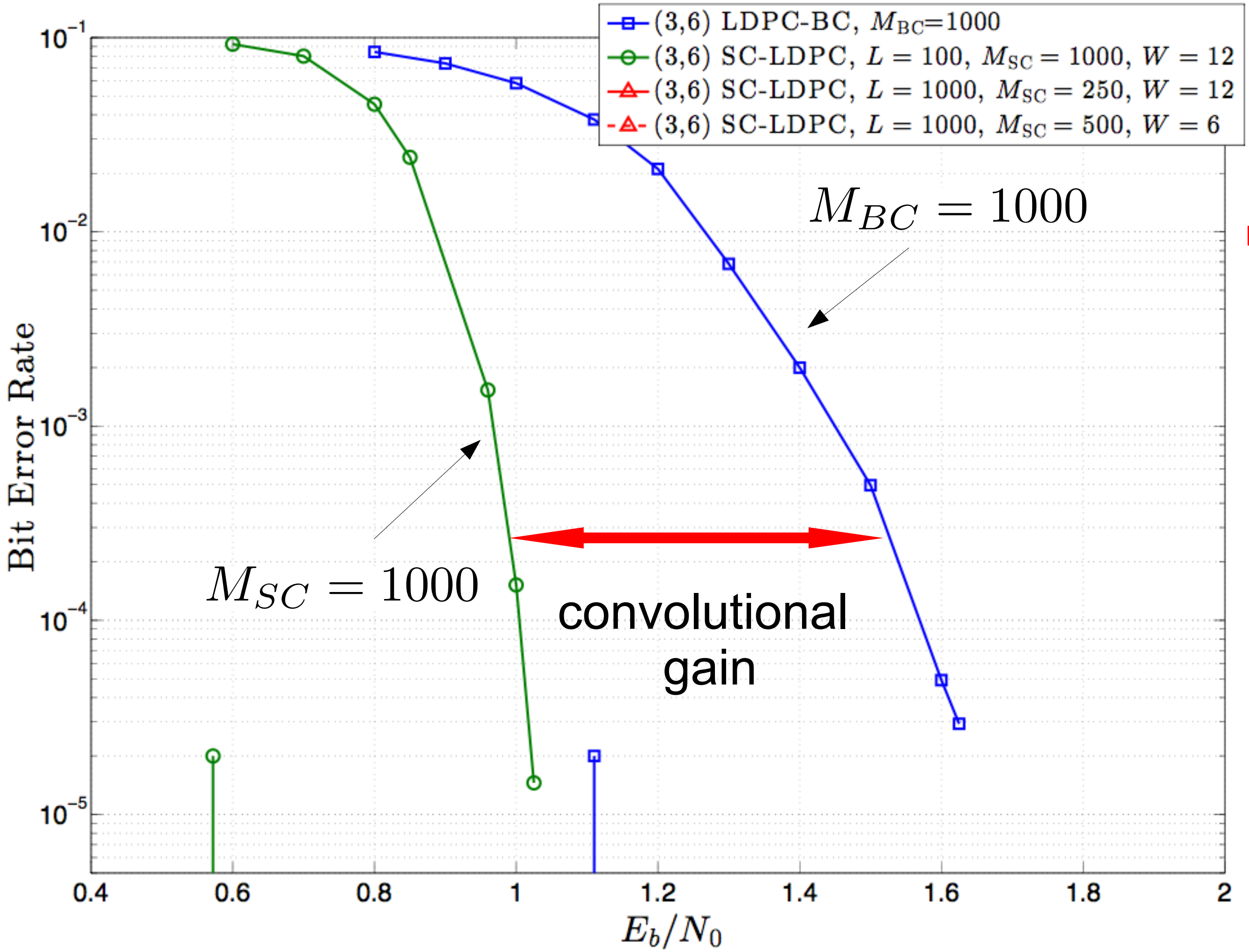

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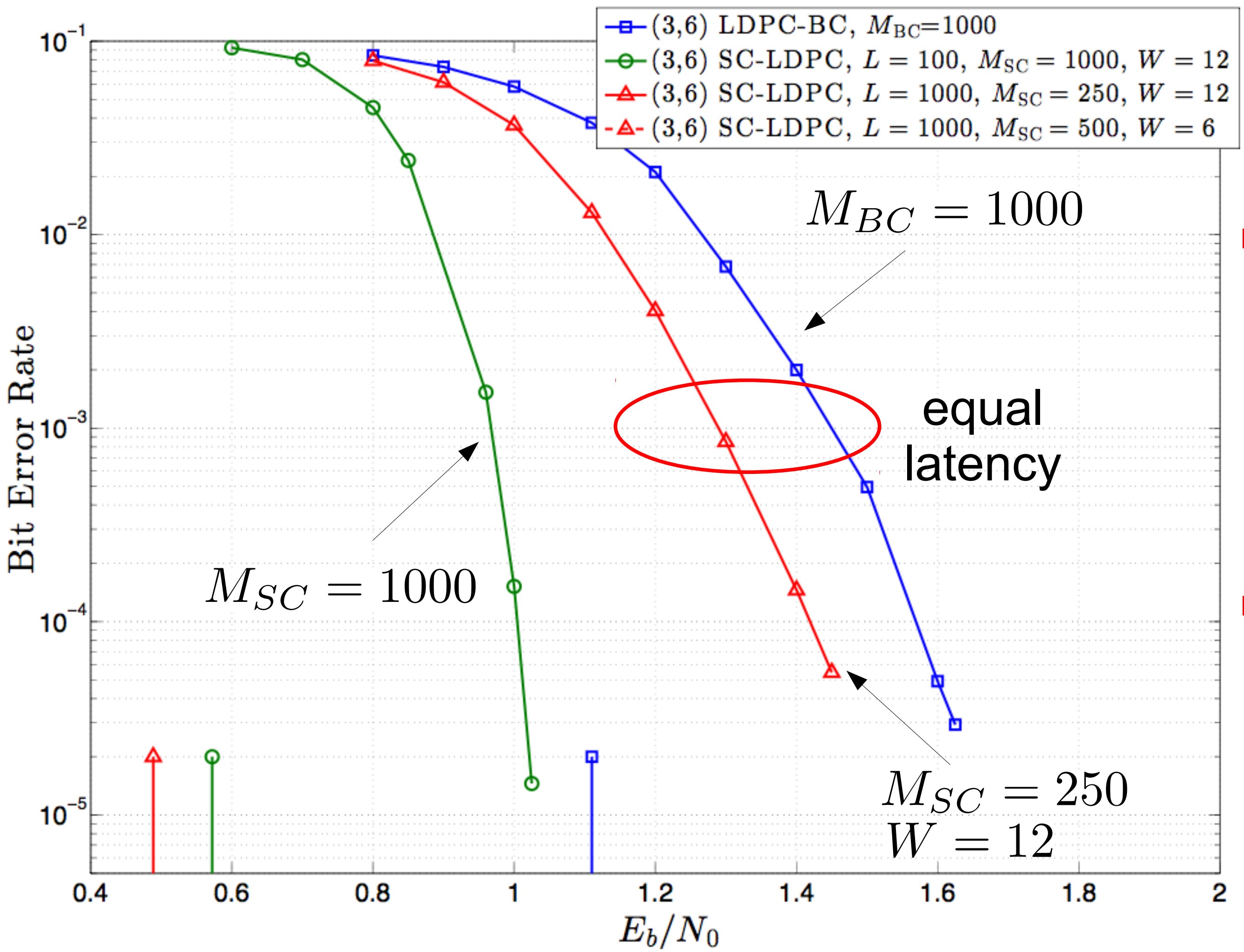

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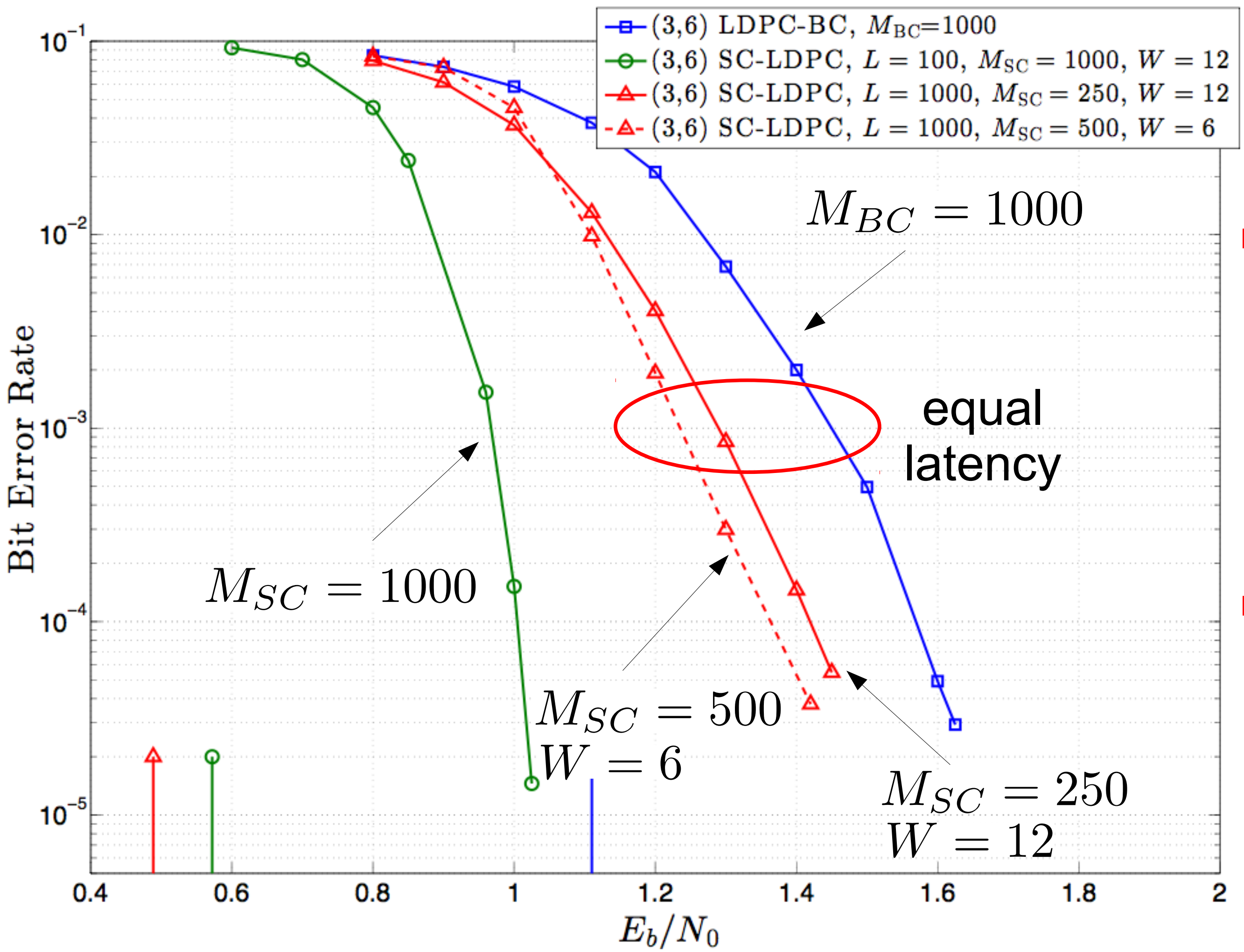

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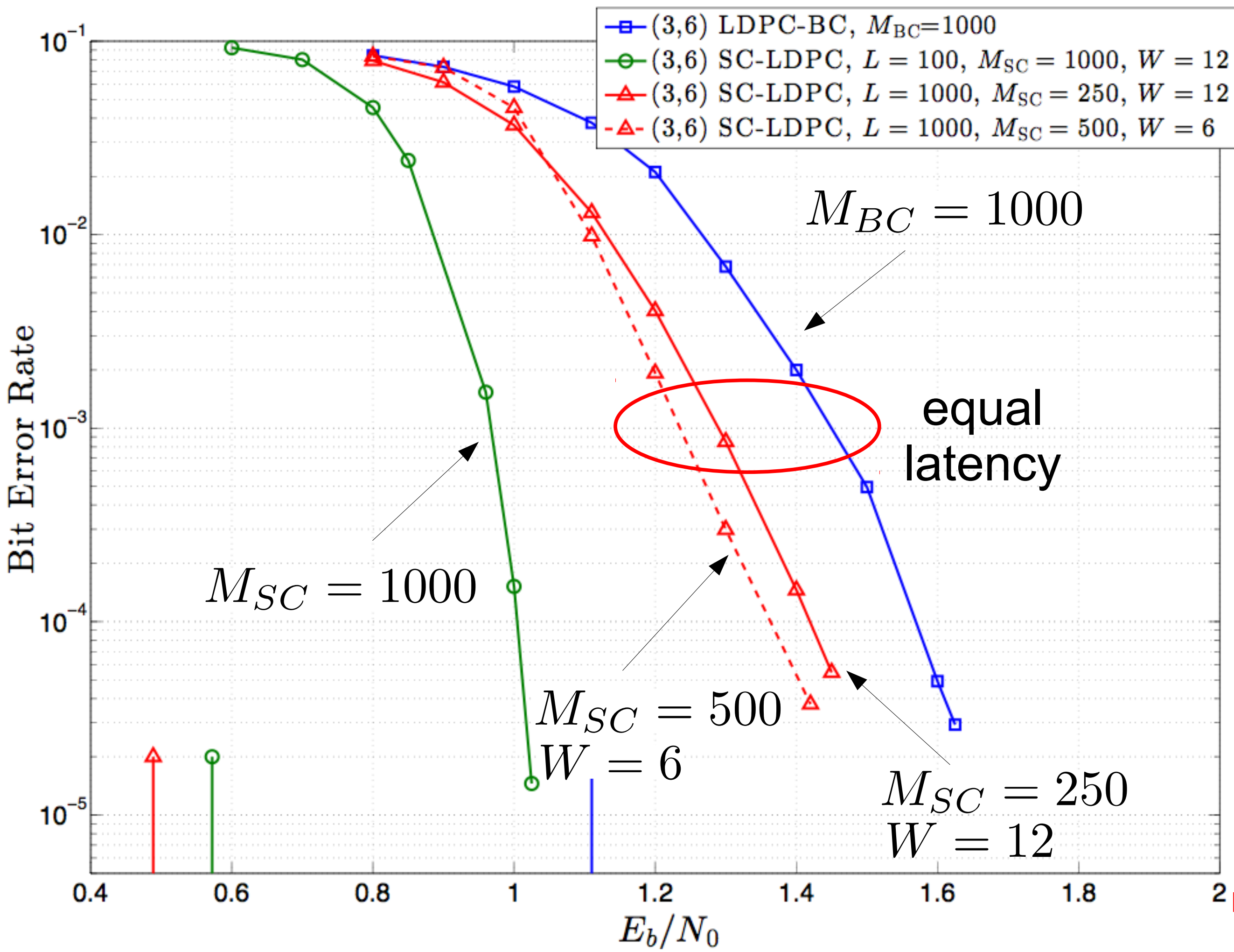

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- Trade-off in M vs W

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Equal Latency Comparison for (3,6)-Regular LDPC Codes

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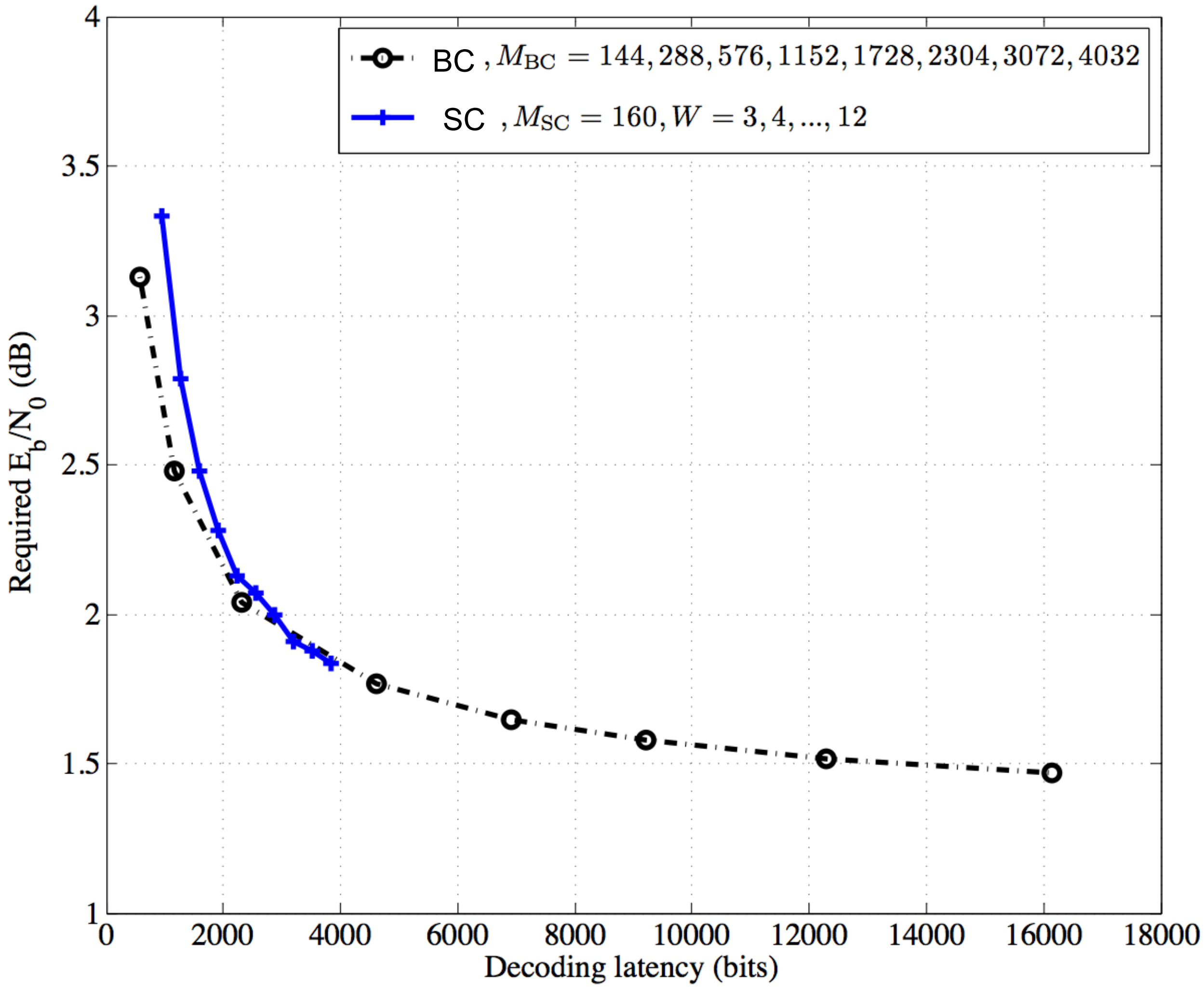
- Required E_b/N_0 to achieve a BER of 10^{-5} as a function of latency:



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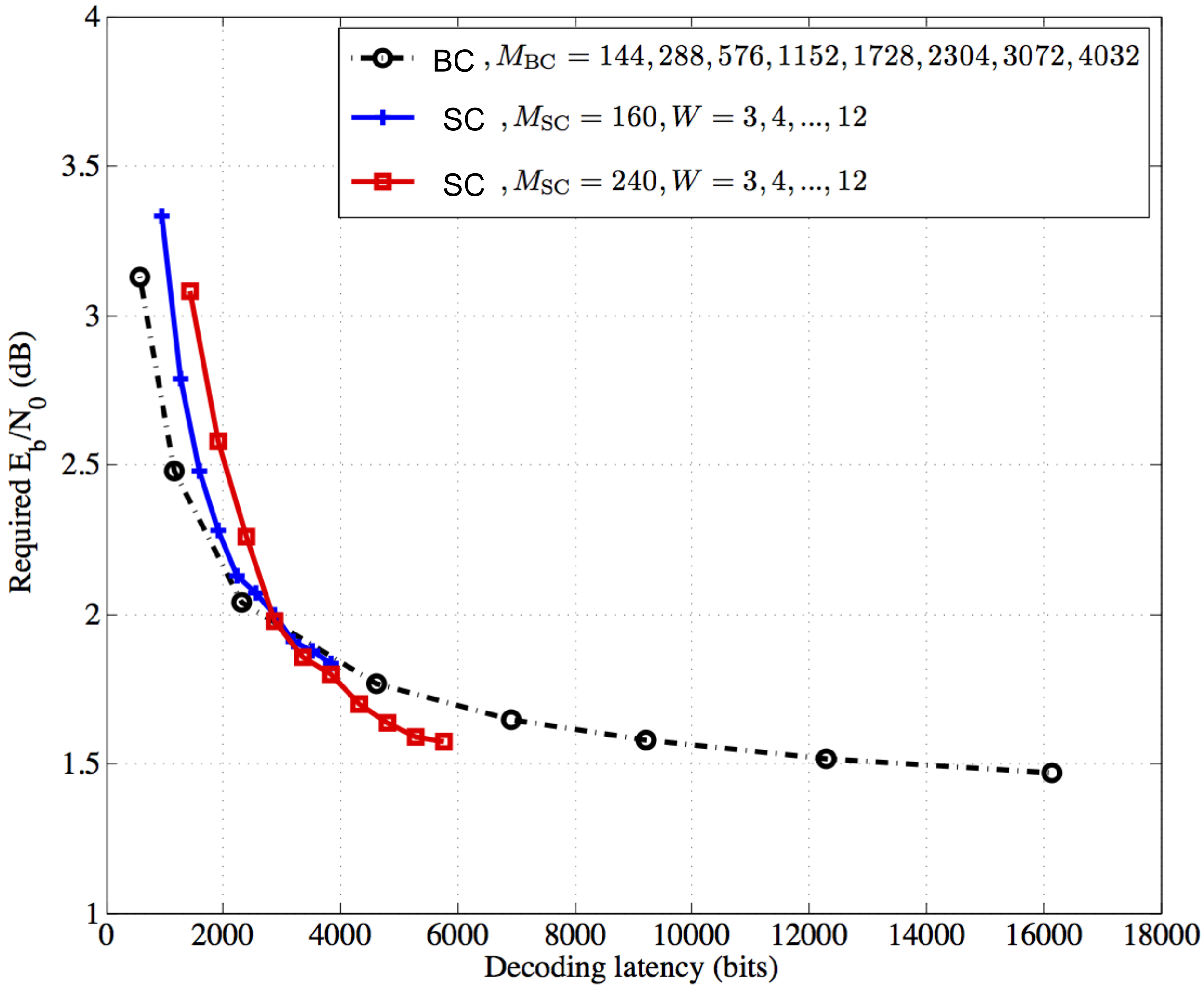

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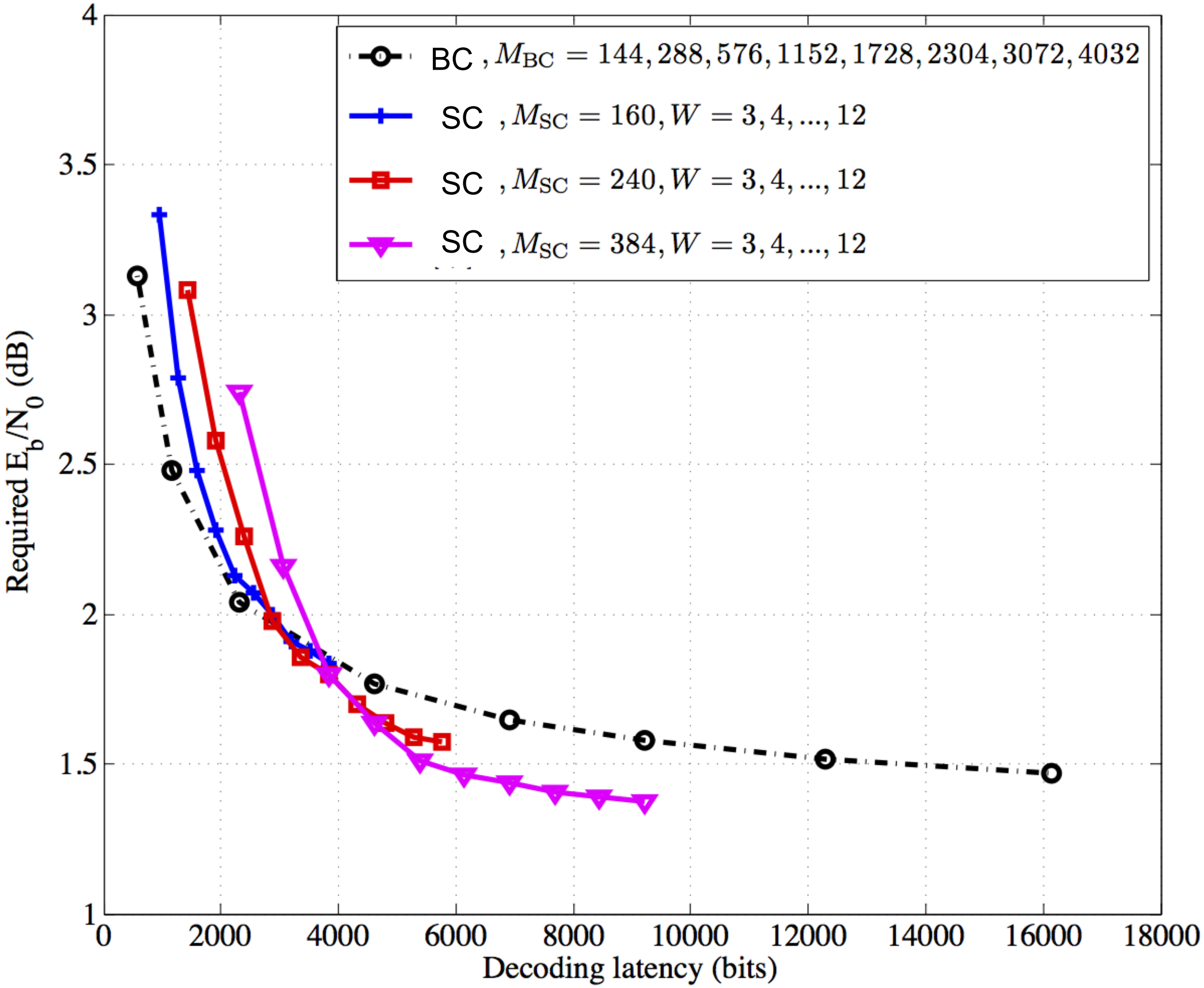

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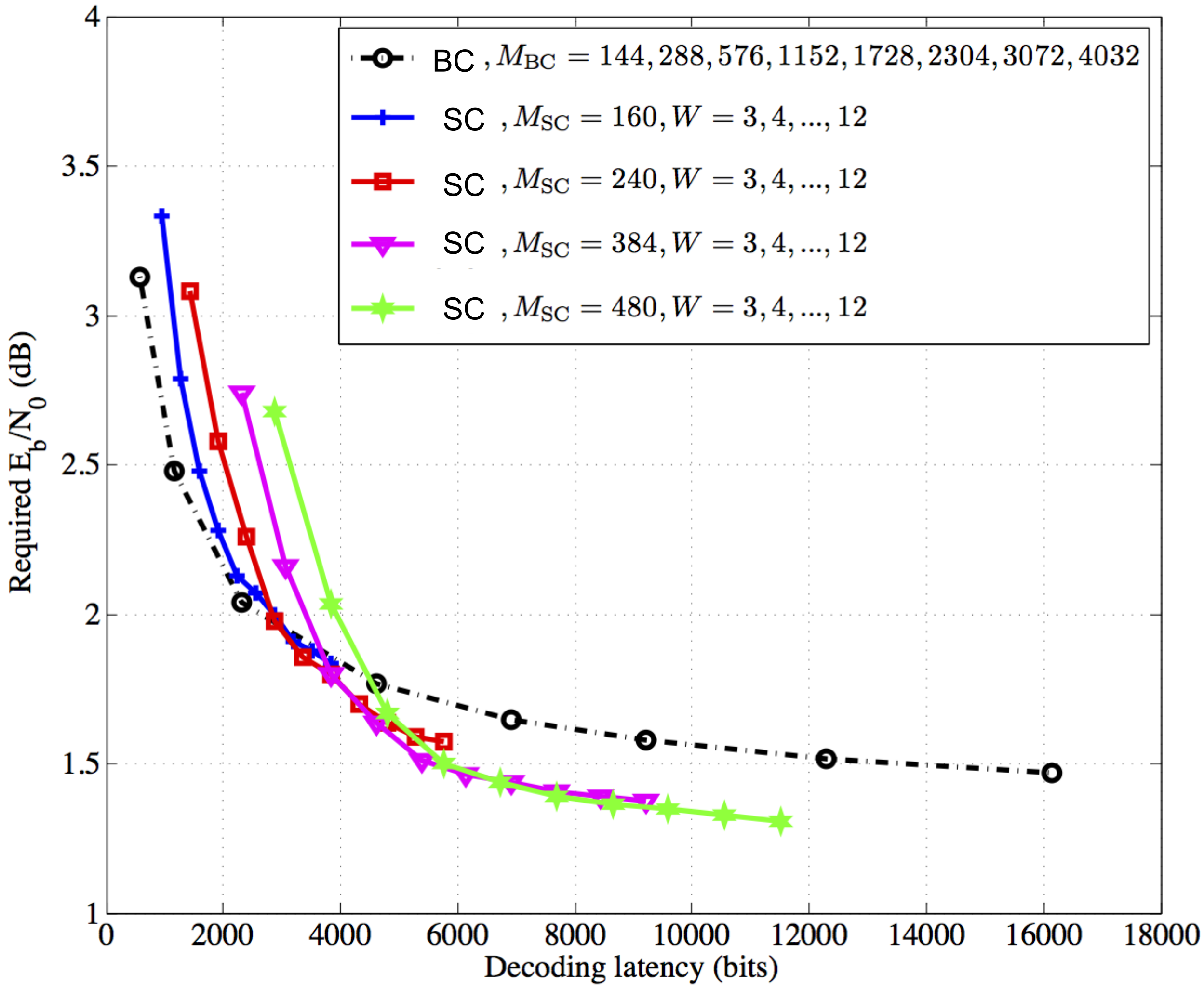

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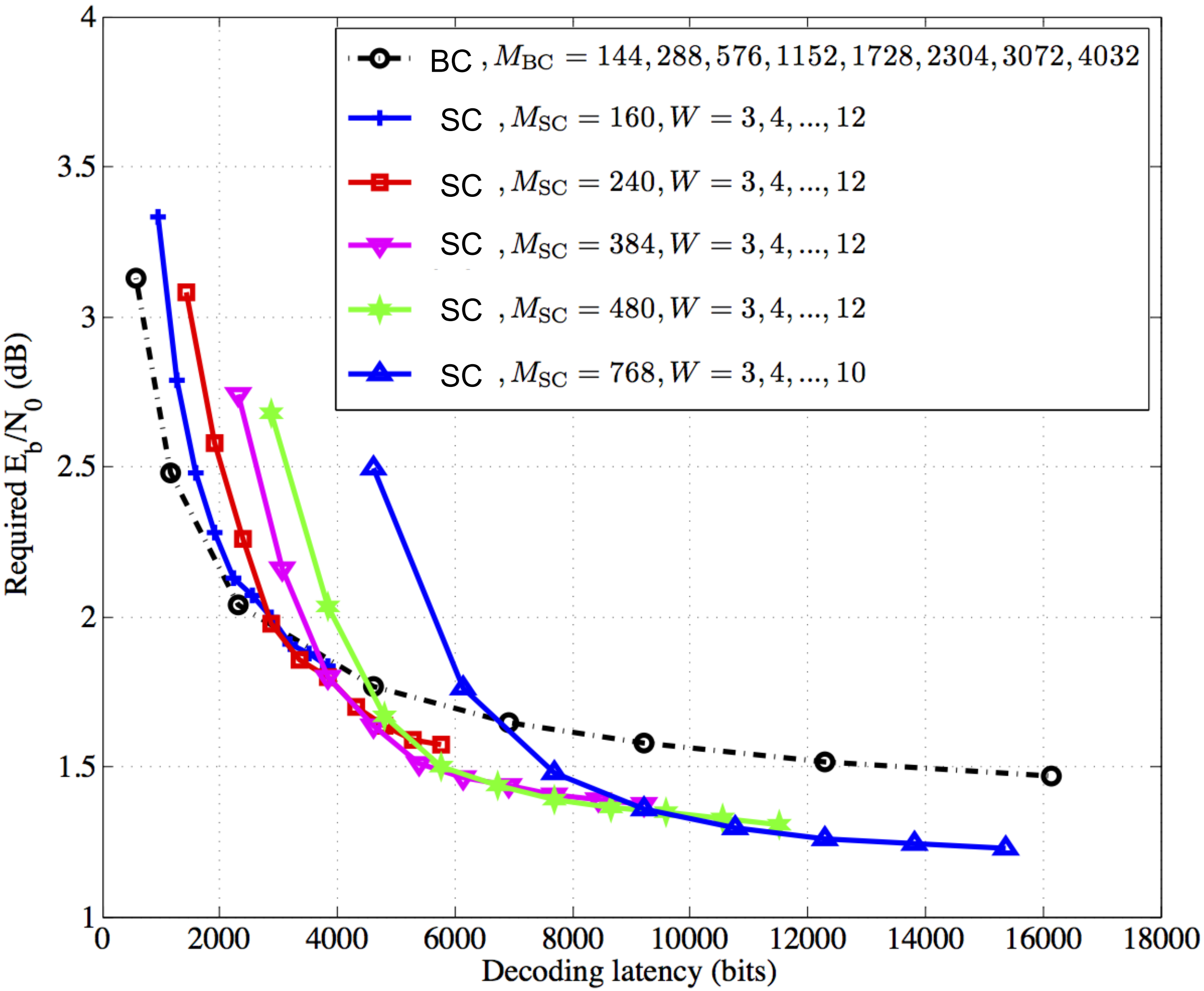

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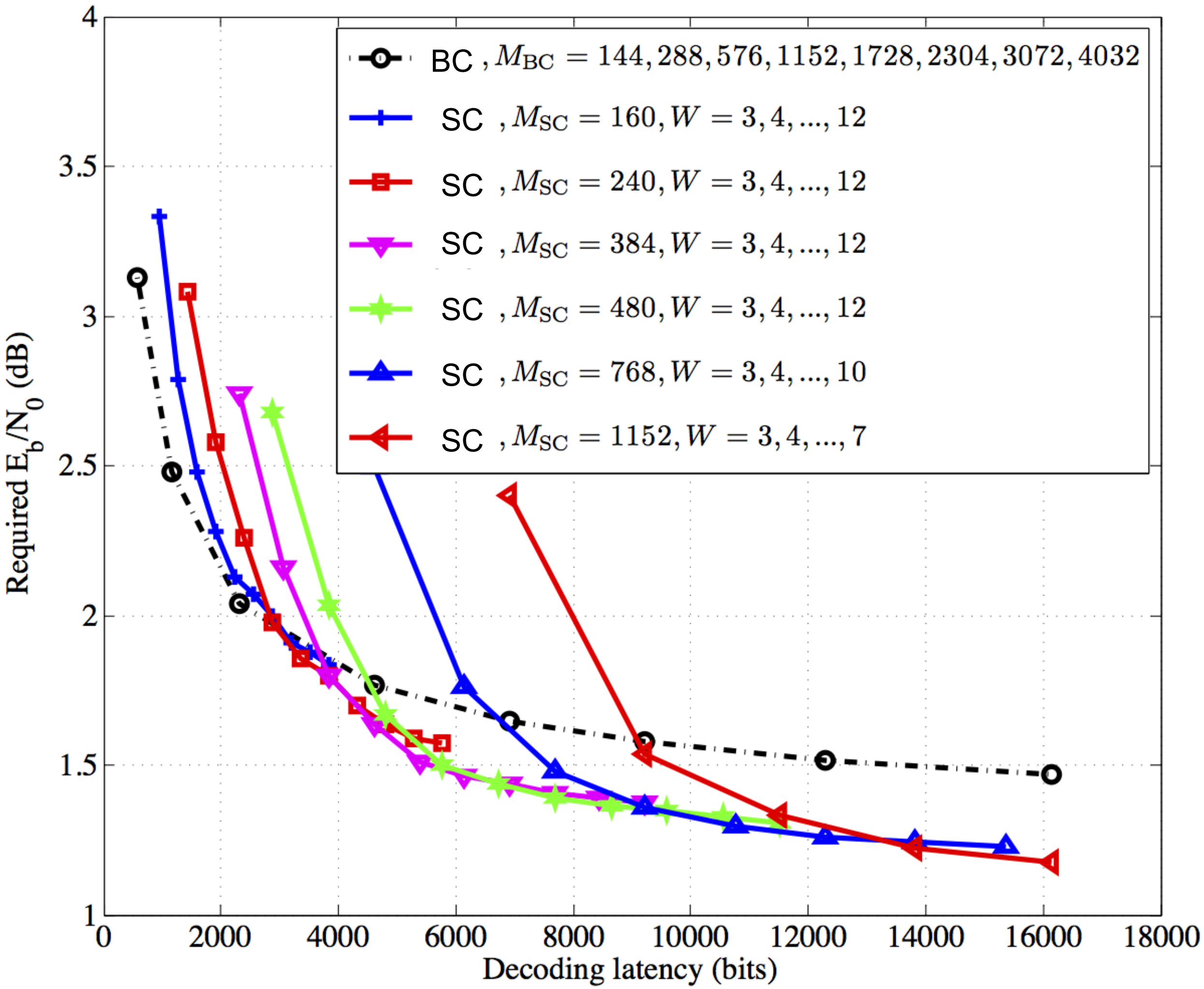

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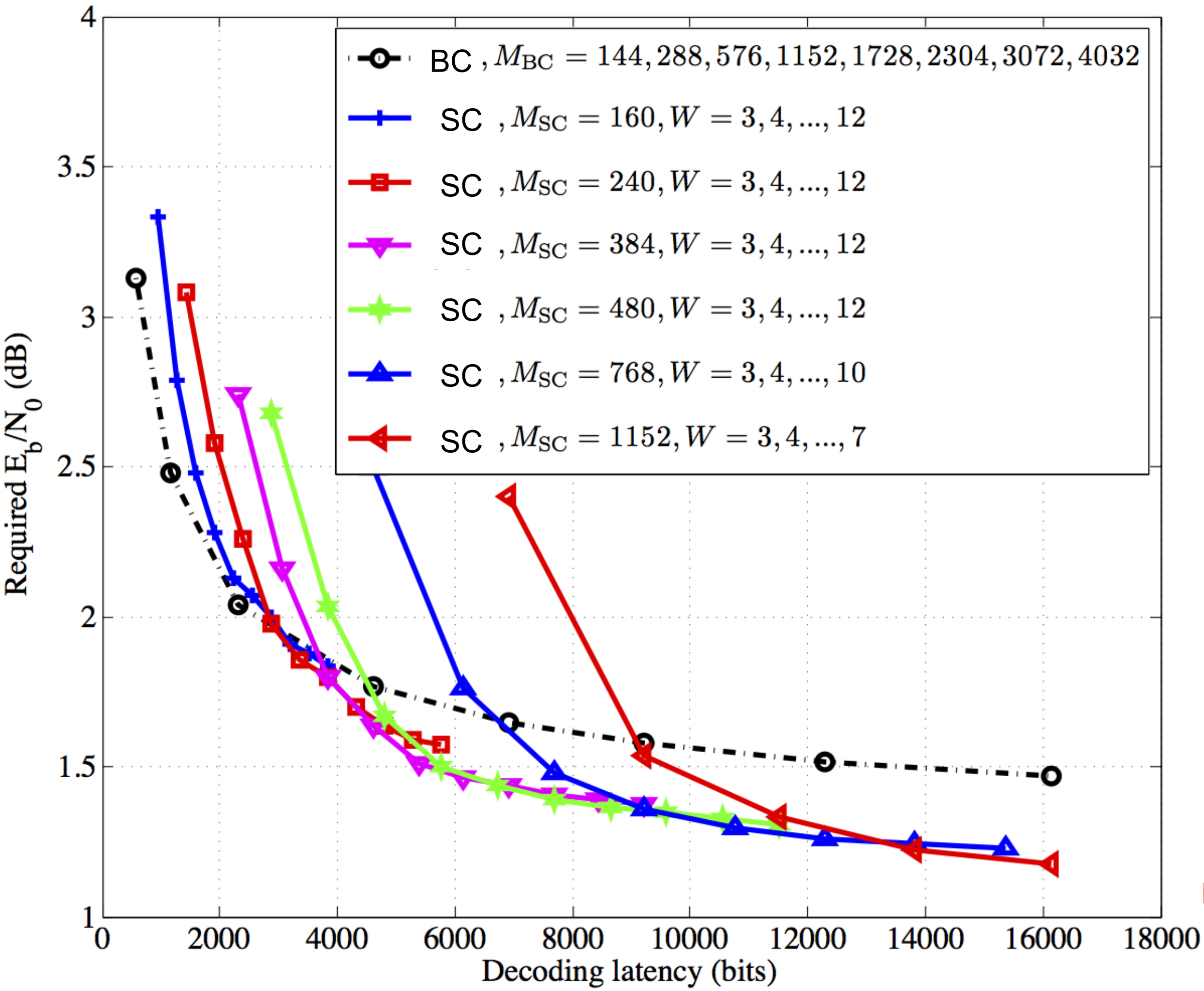

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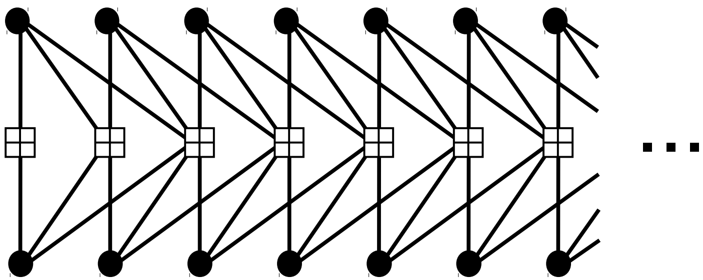
- When choosing parameters:

➡ large M_{SC} improves code performance.

➡ large W improves decoder performance.

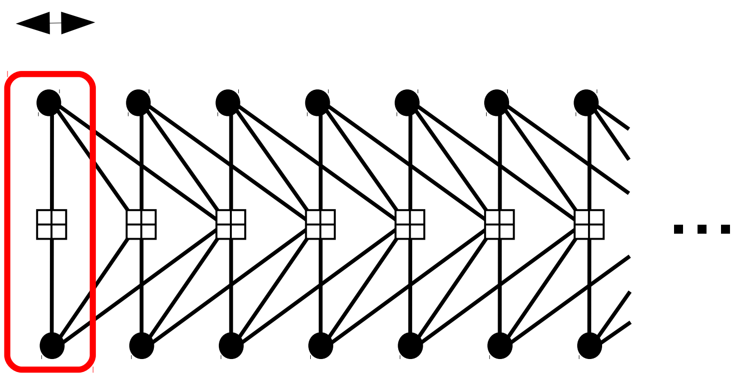
Protograph design



Protograph design

block size

$N_B = cM = 2M$



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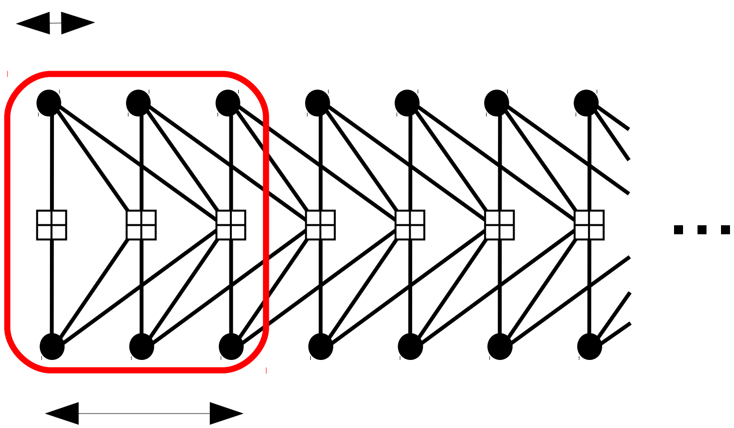

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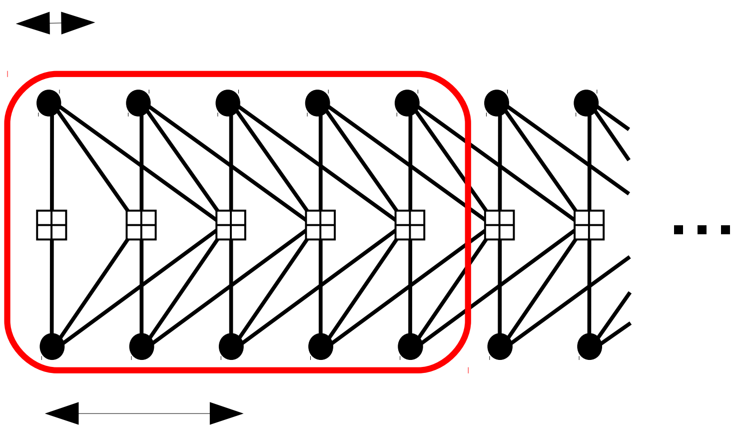

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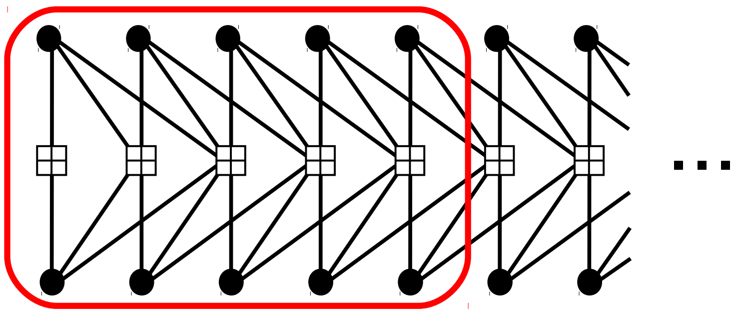

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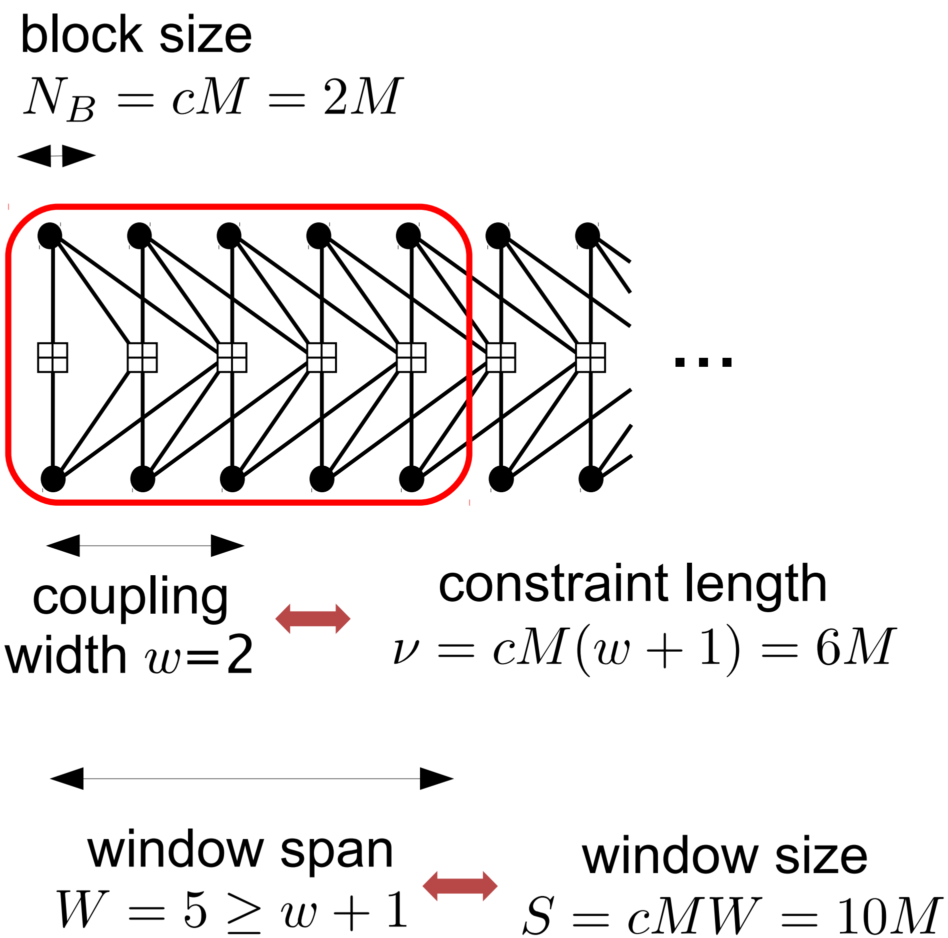

- The **strength** of a convolutional code grows with **constraint length** $\nu = cM(w+1)$
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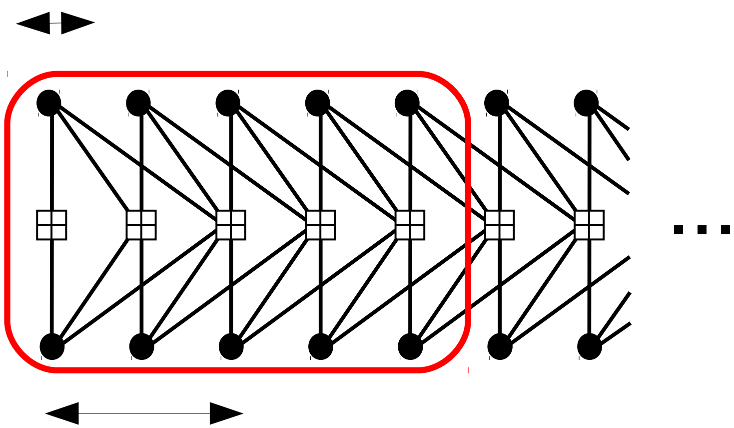

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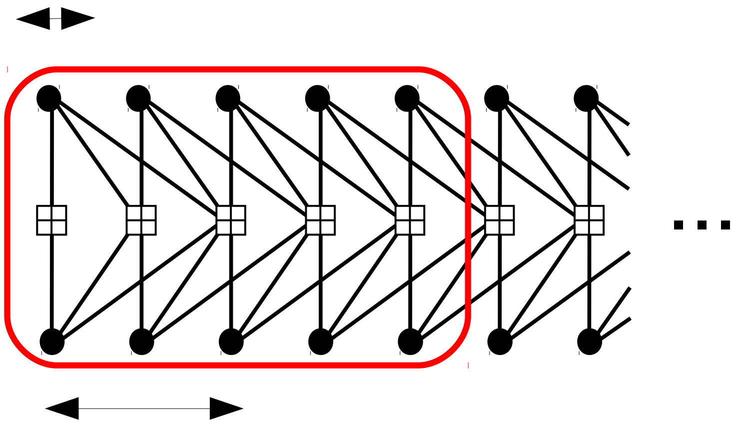

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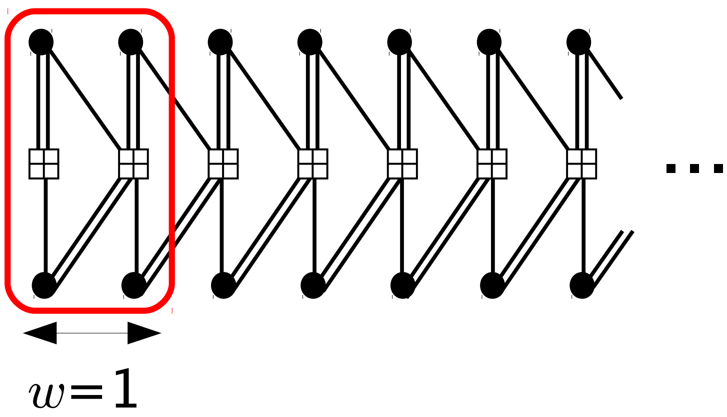
- Density evolution does not tell us how to choose these parameters to **optimize finite-length performance**

Examples: Protograph design

- We start from $\mathbf{B} = [3\ 3]$ and spread the edges such that $\sum_{i=0}^{w} \mathbf{B}_i = \mathbf{B}$

$$[\mathbf{B}_0;\ \mathbf{B}_1] = [2\,1;1\,2]$$



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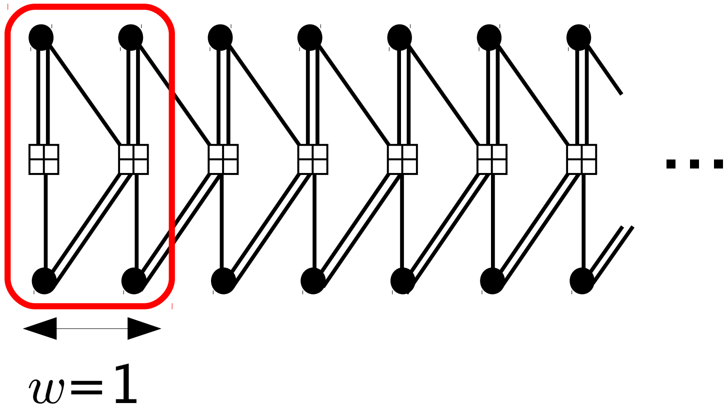

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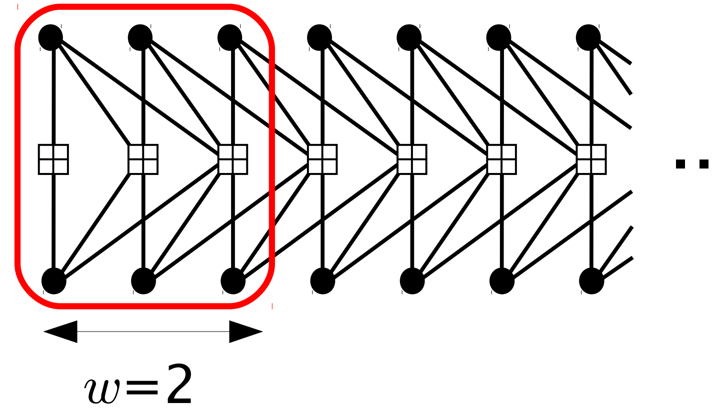

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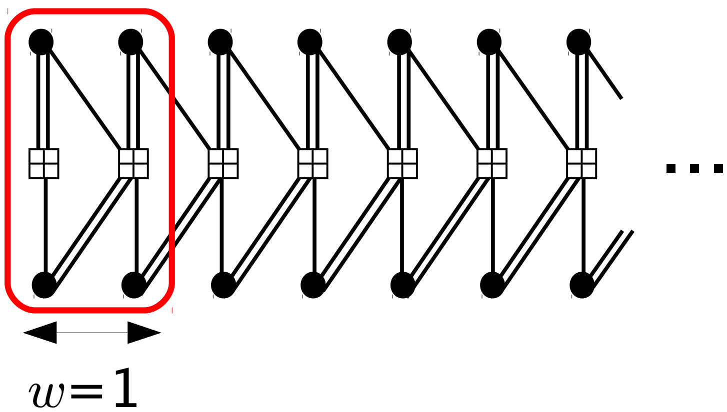

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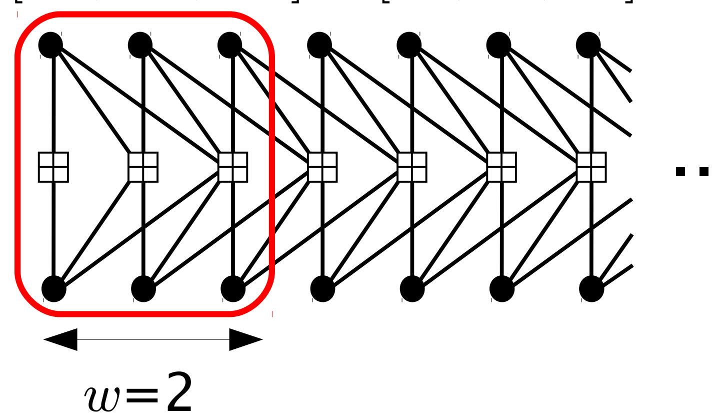

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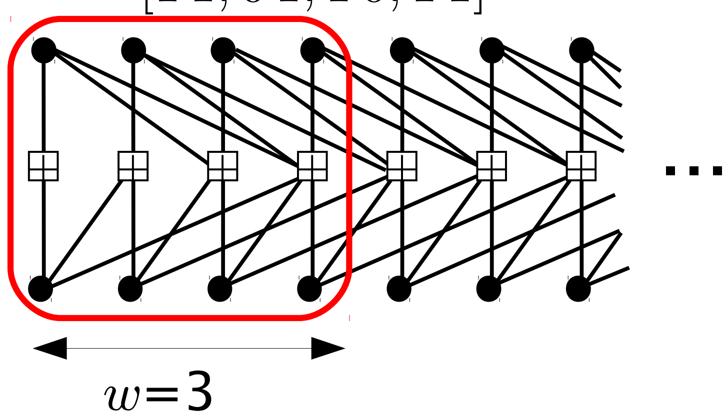

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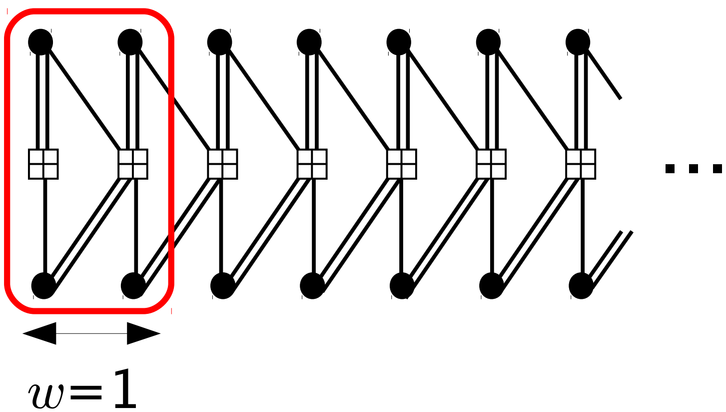

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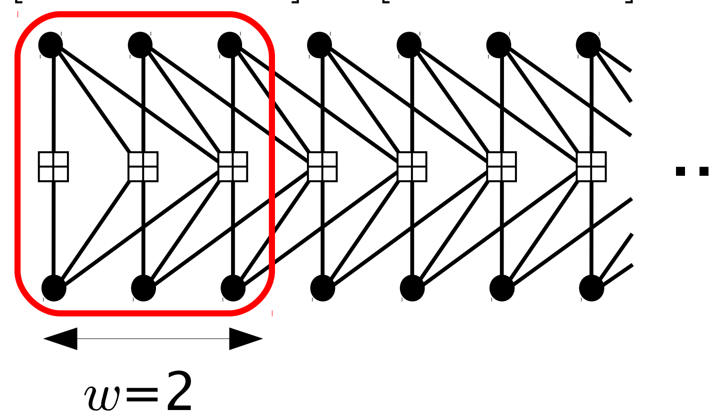

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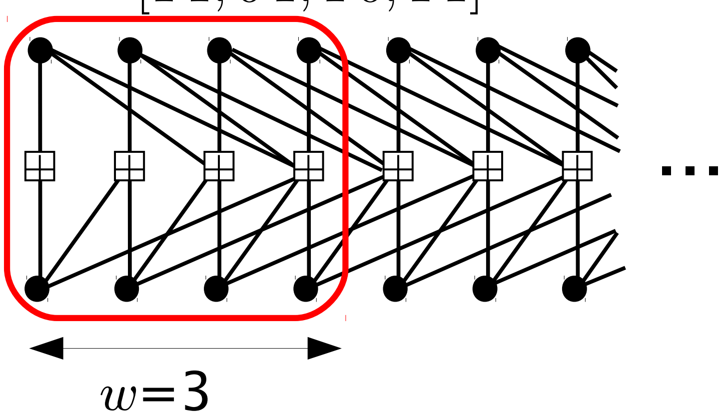

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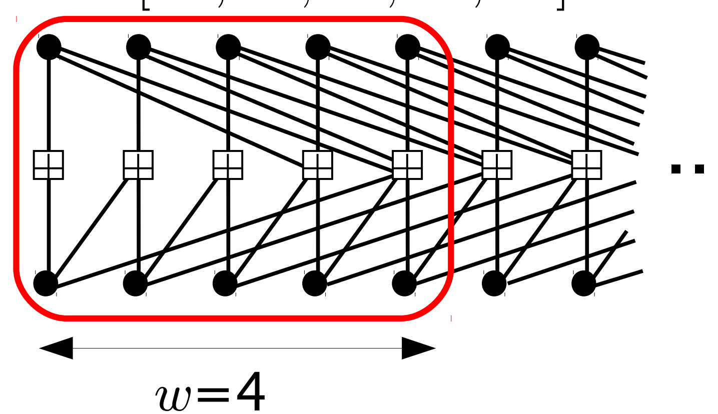

WD Scaling - AWGNC

- BER of the w = 2, [11;11;11] codes demonstrating the scaling with $\nu = 2M(w+1)$ for increasing M and fixed w

M	N_B	ν	S
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750	1500	4500	18,000
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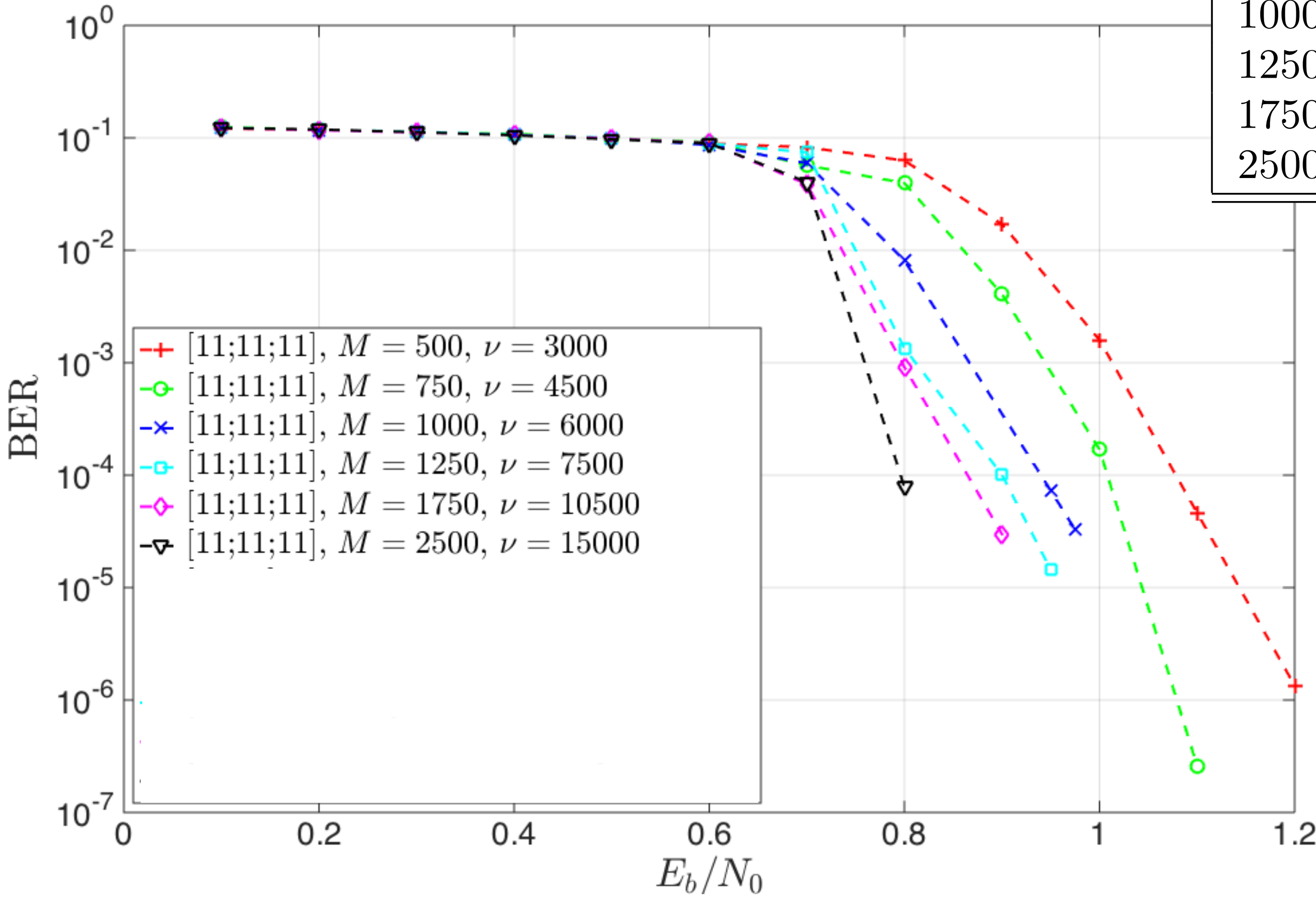

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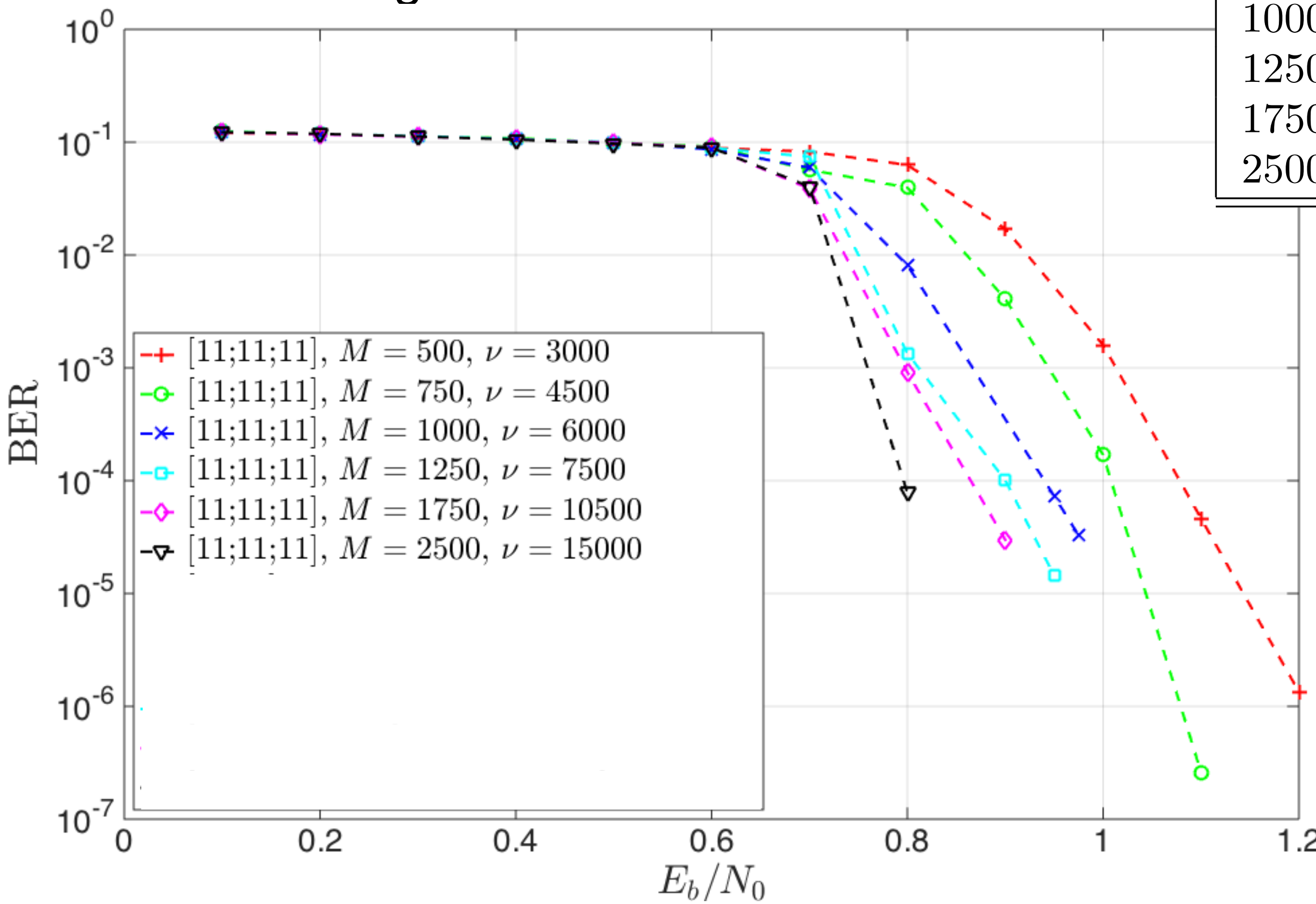

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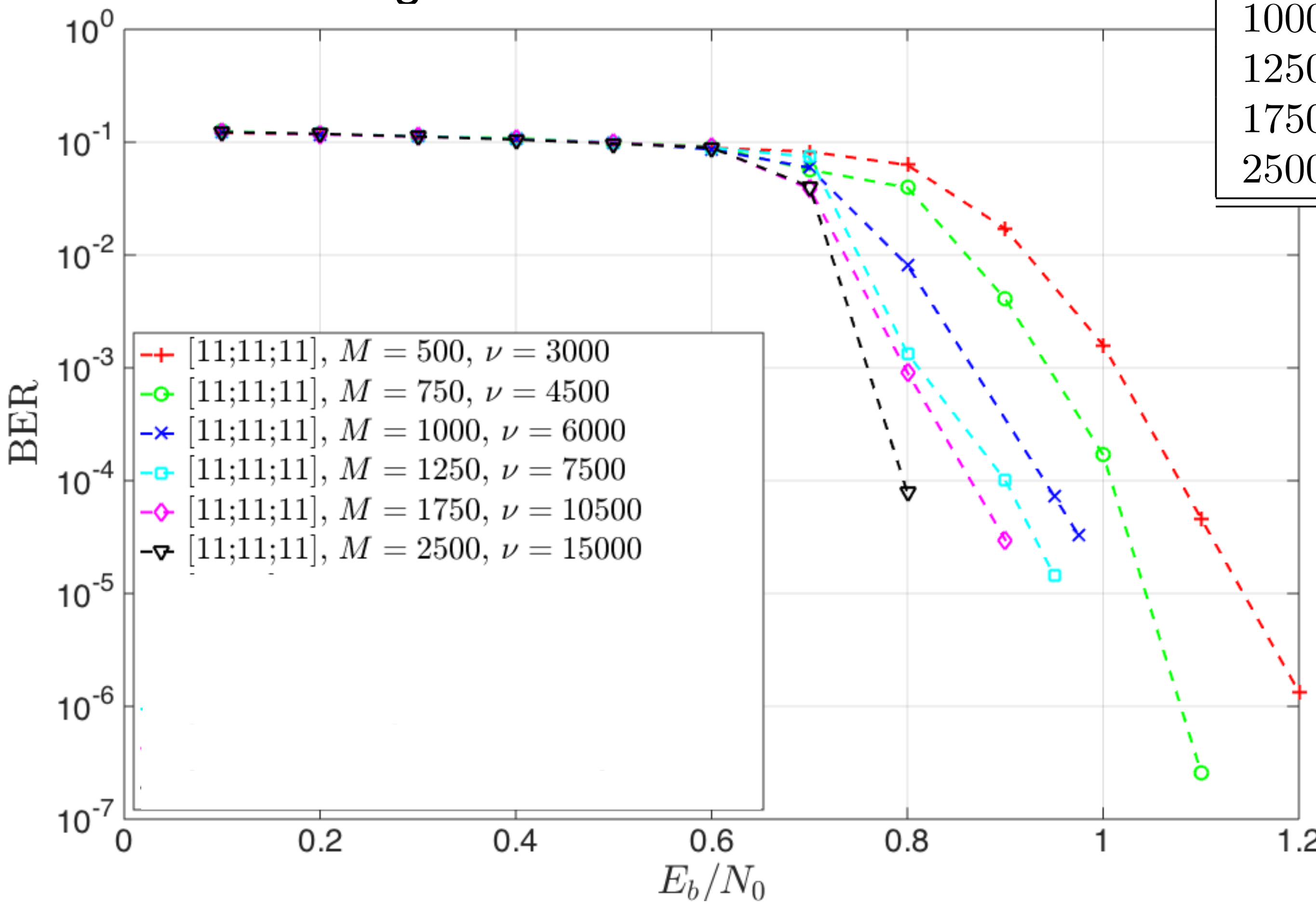

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- For fixed M = 750 (N_B = 1500) we observe improved scaling with w.

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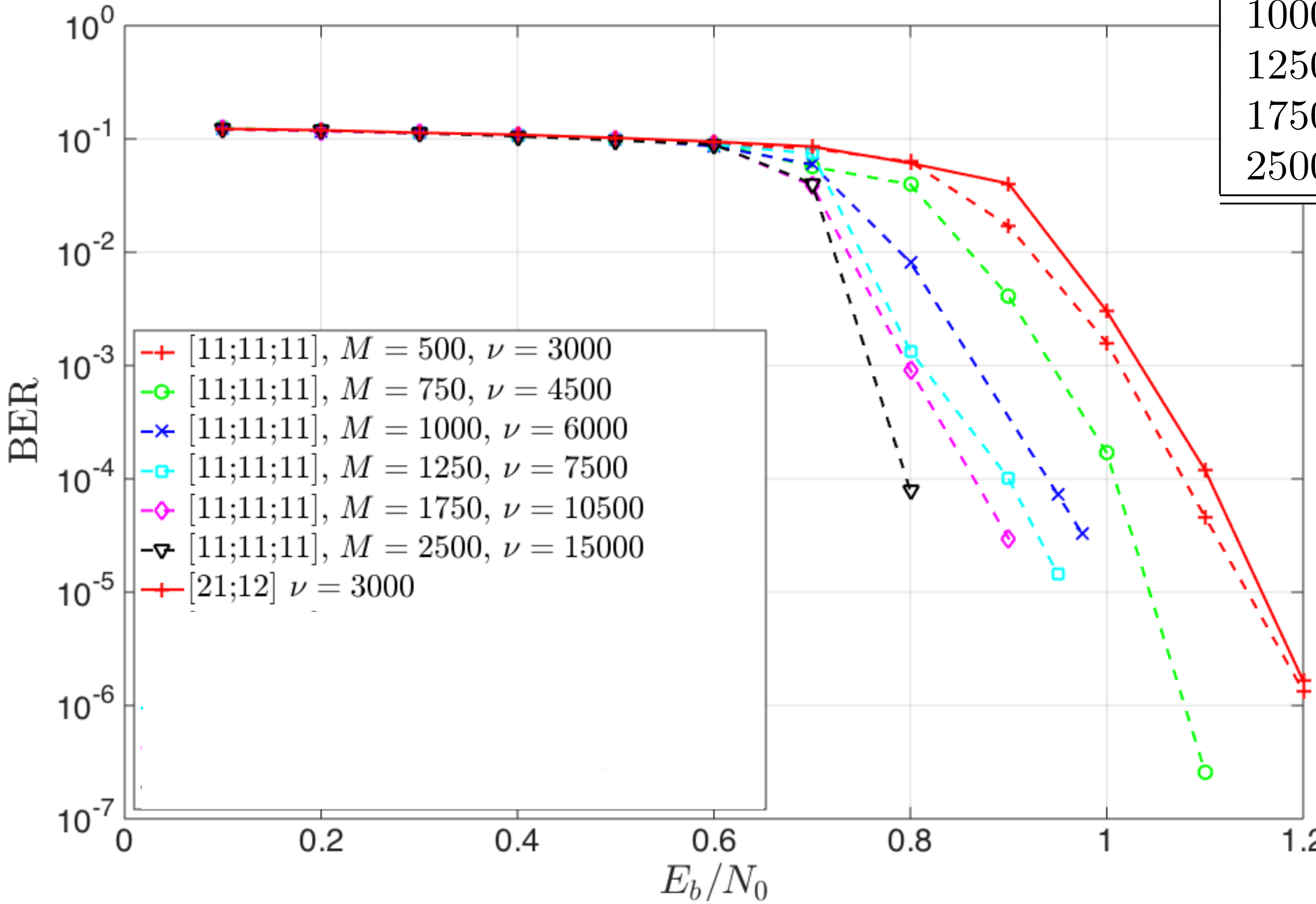

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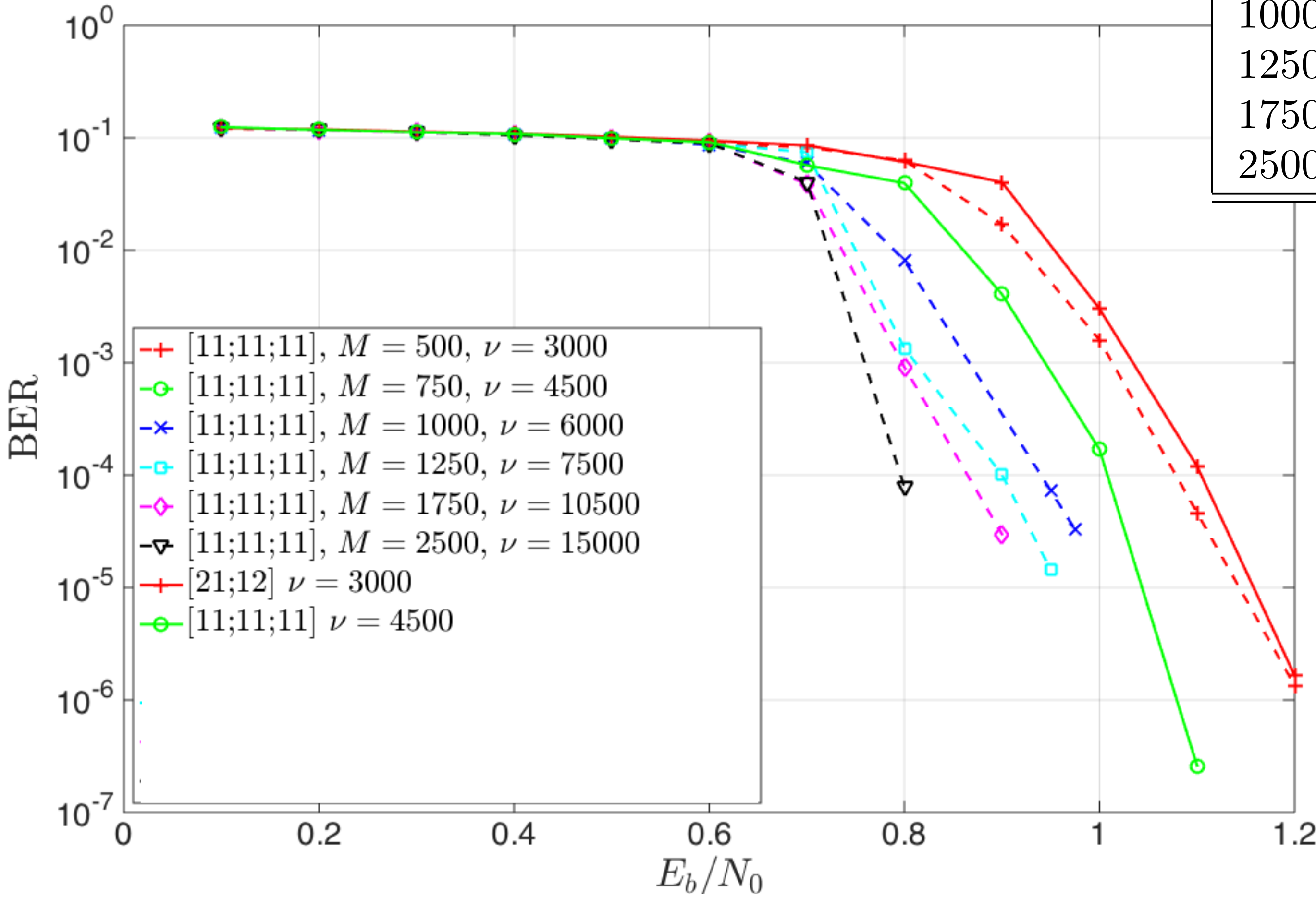
- BER of the w = 2, [11;11;11] codes demonstrating the scaling with $\nu = 2M(w+1)$ for increasing M and fixed w

M	N_B	ν	S
500	1000	3000	12, 000
750	1500	4500	18, 000
1000	2000	6000	24, 000
1250	2500	7500	30, 000
1750	3500	10500	42, 000
2500	5000	15000	60, 000



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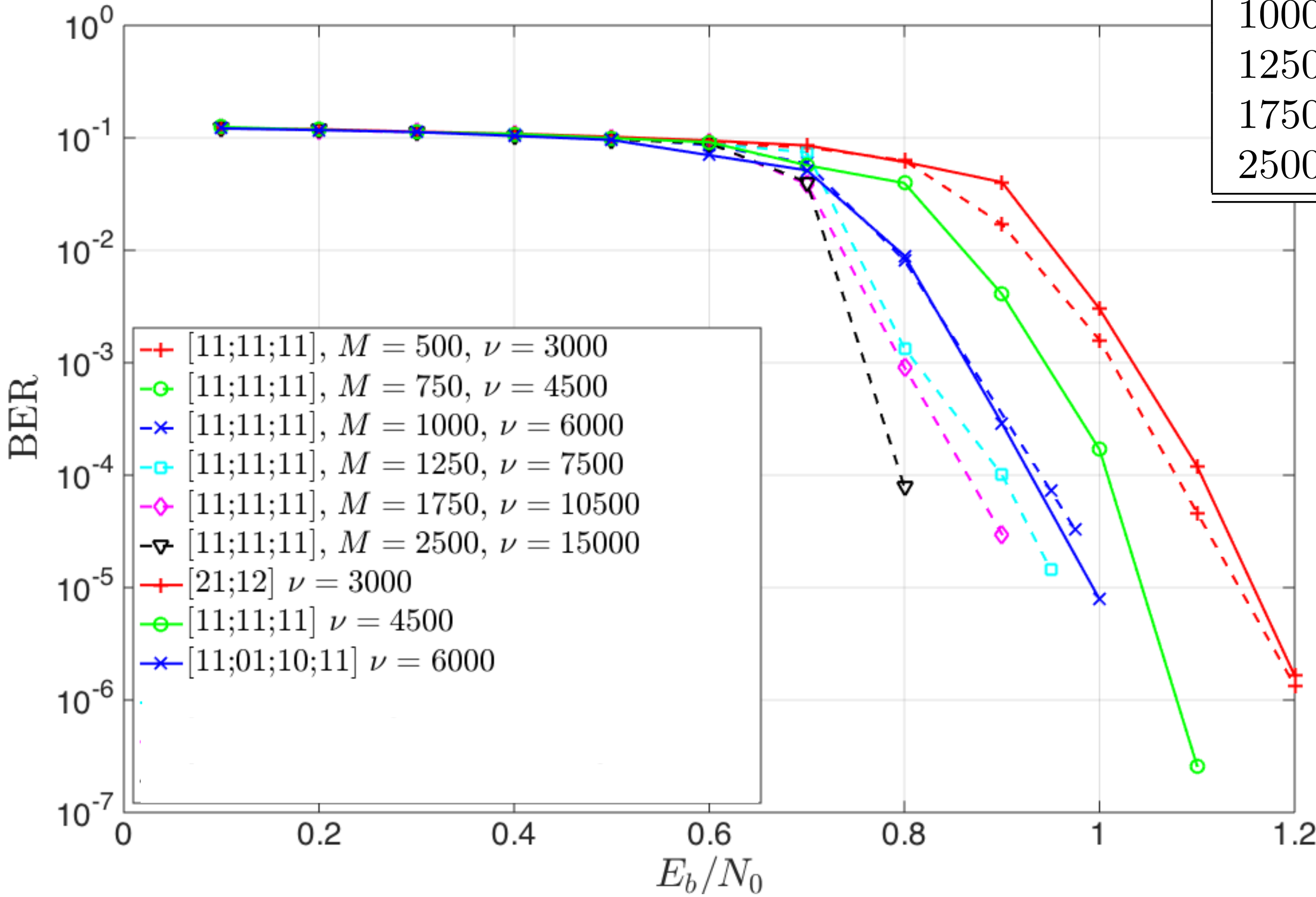

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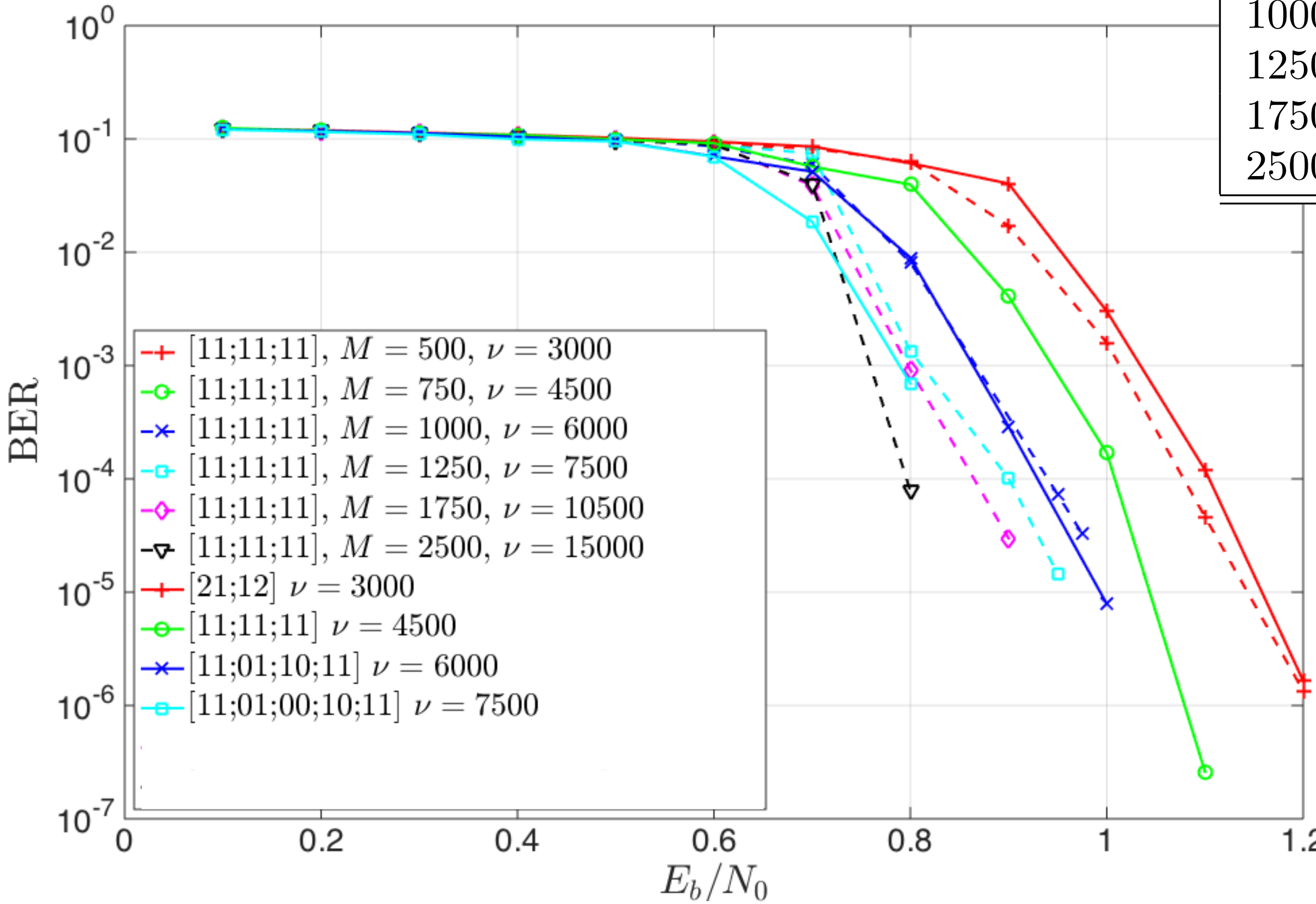

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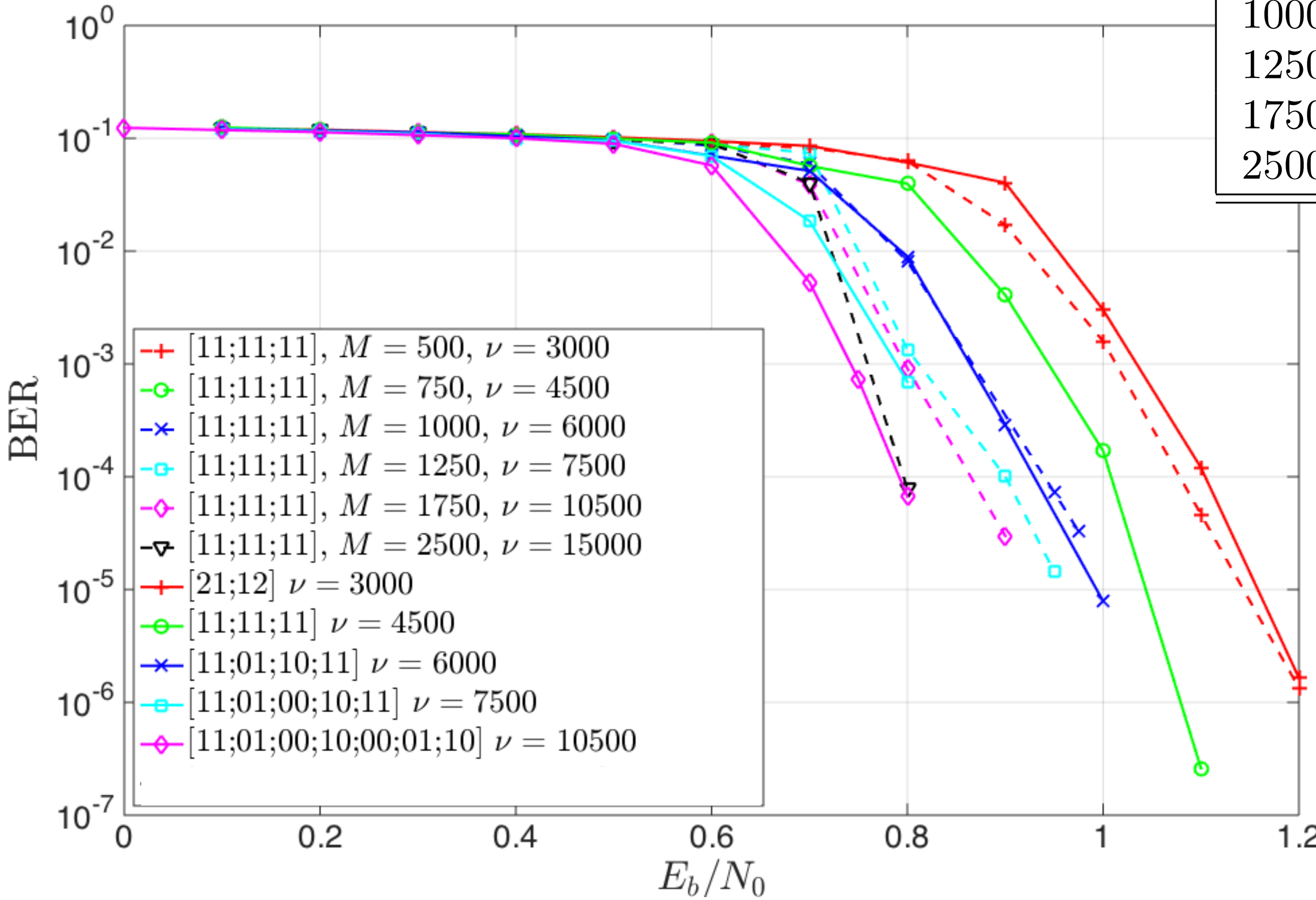

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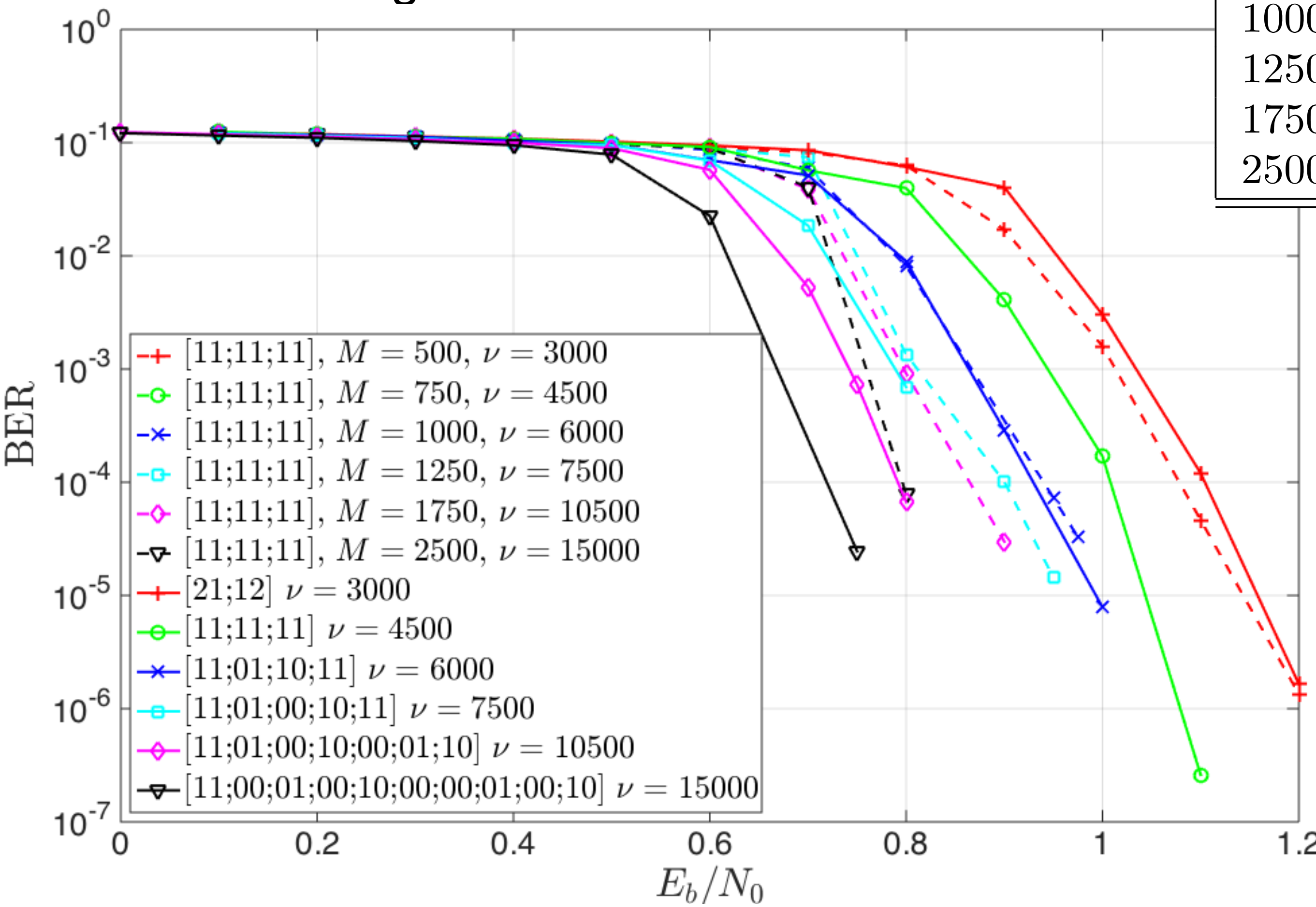

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Implementation Aspects

- As a result of their capacity approaching performance and simple structure, regular SC-LDPC codes may be attractive for future coding standards. Several key features will require further investigation:
  - ➡ Hardware advantages of QC designs obtained by circulant liftings
  - ➡ Hardware advantages of the 'asymptotically-regular' structure
  - ➡ Design advantages of flexible frame length and flexible rate obtained by varying M, L, and/or puncturing

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- Of particular importance for applications requiring extremely low decoded bit error rates (*e.g.*, optical communication, data storage) is an investigation of error floor issues related to **stopping sets**, **trapping sets**, and **absorbing sets**.

Conclusions

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- Spatially coupled LDPC code ensembles achieve **threshold saturation**, i.e., their iterative decoding thresholds (for large L and M) approach the MAP decoding thresholds of the underlying LDPC block code ensembles.
- The threshold saturation and linear minimum distance growth properties of (J,K)-regular SC-LDPC codes combine the best asymptotic features of both regular and irregular LDPC-BCs.
- With window decoding, SC-LDPC codes also compare favorably to LDPC-BCs in the finite-length regime, providing flexible tradeoffs between BER performance, decoding latency, and decoding complexity.
- Flexible frame lengths and rates can be obtained by varying M, L, and/or puncturing.