Asymptotics of the Error Probability in Quasi-Static Channels

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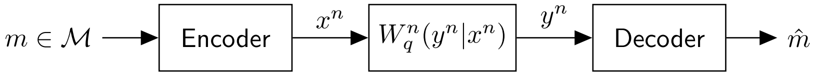
Random Coding



- Message uniformly from $\mathcal{M} = \{1, \ldots, M\}$
- State q known at the receiver, continuously differentiable p_Q
- Decoder outputs $\hat{m} = \arg\max_{m \in \mathcal{M}} W_q(y^n|x_m^n)$
- Error probability of a code $\epsilon_n(x_1^n, \ldots, x_M^n) = \mathbb{P}[\hat{m} \neq m]$
- Randomly generate codebook from p_{X^n} and compute the average error probability over all i.i.d. codebooks

$$\epsilon_n(R) = \mathbb{E}\left[\epsilon_n(X_1^n, \ldots, X_M^n)\right]$$

- Coding rate $R = \frac{1}{n} \log M$

Random Coding

	Ergodic	Nonergodic
Asymptotic	$R_n(\epsilon) \to C$ $\epsilon_n(R) = e^{-nE(R)}$	$R_n(\epsilon) \to C_\epsilon$ $\epsilon_n(R) \to P_{\text{out}}(R)$
Finitelength	$R_n(\epsilon) = C - \sqrt{\frac{V}{n}} Q^{-1}(\epsilon)$ $\epsilon_n(R) = Q\left(\frac{C-R}{\sqrt{V/n}}\right)$	$R_n(\epsilon) = C_\epsilon + O\left(\frac{\log n}{n}\right)$ $\epsilon_n(R) = P_{\text{out}}(R) + \frac{\log n}{n}\phi_{\log}(R) + \frac{1}{n}\phi_0(R)$

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- Polyanskiy, Poor & Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, 2010.
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- Biglieri, Proakis & Shamai (Shitz), "Fading channels: Information-theoretic and communications aspects," *IEEE Trans. Inf. Theory*, 1998.
- Malkamäki & Leib, "Coded diversity on block-fading channels", *IEEE Trans. Inf. Theory*, 1999.
- Kaplan & Shamai (Shitz), "Error probabilities for the block-fading Gaussian channel", *Archiv fur Elektronik und Ubertragungstechnik*, 1995.
- Yang, Durisi, Koch & Polyanskiy, "Quasi-static multiple-antenna fading channels at finite blocklength," *IEEE Trans. Inf. Theory*, 2014.
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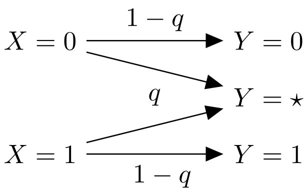
Random-Coding

- Under ML decoding and counting ties as error

$$
\begin{aligned}
\epsilon_n(R) &= \mathbb{E}_{X_1^n \cdots X_M^n Q}\left[\epsilon_n(X_1^n, \ldots, X_M^n)\right] \\
&= \mathbb{E}_{X_1^n \cdots X_M^n Q}\left[\frac{1}{M}\sum_{m=1}^{M} \mathbb{P}\left[\bigcup_{m' \neq m} \{W_Q^n(Y^n|X_{m'}^n) \geq W_Q^n(Y^n|X_m^n)\} \middle| X_m^n Q\right]\right] \\
&= \mathbb{E}_{X^n Q}\left[\mathbb{P}\left[\bigcup_{m' \neq m} \{W_Q^n(Y^n|X_{m'}^n) \geq W_Q^n(Y^n|X_m^n)\} \middle| X_m^n Q\right]\right] \\
&= \mathbb{E}_{X^n Y^n Q}\left[\mathbb{P}\left[\bigcup_{m' \neq m} \{W_Q^n(Y^n|X_{m'}^n) \geq W_Q^n(Y^n|X_m^n)\} \middle| X_m^n Y^n Q\right]\right] \\
&\leq \mathbb{E}_{X^n Y^n Q}\left[\min\left\{1, \sum_{m' \neq m} \mathbb{P}\left[W_Q^n(Y^n|X_{m'}^n) \geq W_Q^n(Y^n|X_m^n) \middle| X_m^n Y^n Q\right]\right\}\right] \\
&= \mathbb{E}_{X^n Y^n Q}\left[\min\left\{1, (M-1)\mathbb{P}\left[W_Q^n(Y^n|\overline{X}^n) \geq W_Q^n(Y^n|X^n) \middle| X_m^n Y^n Q\right]\right\}\right] \\
&= \mathbb{E}_{X^n Y^n Q}\left[\min\left\{1, (M-1)\mathrm{pep}_Q(X^n; Y^n)\right\}\right]
\end{aligned}
$$

▷ Polyanskiy, Poor & Verdú, “Channel coding rate in the finite blocklength regime,” *IEEE Trans. Inf. Theory*, 2010.

Binary Erasure Channel



- Channel transition probability

$$W(y|x) = \begin{cases} 1-q & x=y \\ q & y=\star \\ 0 & x\neq y \end{cases}$$

- Let t be the number of erasures

$$W_q^n(y^n|x^n) = \begin{cases} q^t(1-q)^{n-t} & x^n = y^n \text{ in the non-}\star\text{ bits} \\ 0 & \text{else} \end{cases}$$

Binary Erasure Channel

- Pairwise error probability

$$\begin{aligned}\mathrm{pep}_q(x^n;y^n) &= \mathbb{P}\left[W_q^n(y^n|\overline{X}^n) \geq W_q^n(y^n|x^n)\right] \\ &\leq \frac{\mathbb{E}\left[W_q^n(y^n|\overline{X}^n)\right]}{W_q^n(y^n|x^n)} \longleftarrow \mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a} \\ &= e^{-i_q^n(x^n;y^n)}\end{aligned}$$

- Information density

$$\begin{aligned}i_q^n(x^n;y^n) &= \log \frac{W_q^n(y^n|x^n)}{\mathbb{E}\left[W_q^n(y^n|\overline{X}^n)\right]} \\ &= \begin{cases}(n-t)\log 2 & x^n = y^n \text{ in the non-}\star\text{ bits} \\ -\infty & \text{else}\end{cases}\end{aligned}$$

- Tight!

$$\mathrm{pep}_q(x^n;y^n) = \mathbb{P}\left[W_q^n(y^n|\overline{X}^n) \geq W_q^n(y^n|x^n)\right] = \left(\frac{1}{2}\right)^{n-t} = e^{-i_q^n(t)}$$

- Random coding union bound

$$\mathrm{rcu}_n(R) = \mathbb{E}_{TQ}\left[\min\left\{1, (M-1)e^{-i_Q^n(T)}\right\}\right]$$

Binary Erasure Channel

- Random coding union bound

$$
\begin{aligned}
\mathrm{rcu}_n(R) &= \mathbb{E}\Big[\min\big\{1, (M-1)e^{-i_Q^n(T)}\big\}\Big] \\
&= \mathbb{P}\big[(M-1)e^{-i_Q^n(T)} \geq U\big] \longleftarrow \mathbb{E}[\min\{1, A\}] = \mathbb{P}[A \geq U] \\
&= \mathbb{E}\big[\mathbb{1}\{nR - i_q^n(T) - \log U \geq 0\}\big] \\
&= \mathbb{E}\left[\mathbb{1}\left\{R - I(Q) + \frac{1}{\sqrt{n}}\left(\sqrt{n}I(Q) - \frac{1}{\sqrt{n}} i_q^n(T)\right) - \frac{1}{n}\log U \geq 0\right\}\right] \\
\text{Taylor} \longrightarrow &\simeq \mathbb{E}\big[\mathbb{1}\{R - I(Q) \geq 0\}\big] + \frac{1}{\sqrt{n}}\mathbb{E}\left[\delta\big(R - I(Q)\big)\left(\sqrt{n}I(Q) - \frac{1}{\sqrt{n}} i_Q^n(T)\right)\right] \\
&\quad + \frac{1}{2n}\mathbb{E}\left[\delta'\big(R - I(Q)\big)\left(\sqrt{n}I(Q) - \frac{1}{\sqrt{n}} i_Q^n(T)\right)^2\right] - \frac{1}{n}\mathbb{E}\left[\delta\big(R - I(Q)\big)\log U\right] \\
&= P_{\mathrm{out}}(R) + \frac{1}{\sqrt{n}}0 - \frac{1}{2n} p'_{\frac{R - I(Q)}{\sqrt{V(Q)}}}(0) + \frac{1}{n} p_{R - I(Q)}(0)
\end{aligned}
$$

$$\Downarrow$$

$$\boxed{\mathrm{rcu}_n(R) \simeq P_{\mathrm{out}}(R) + \frac{1}{n}\phi_0(R)}$$

$$\Downarrow$$

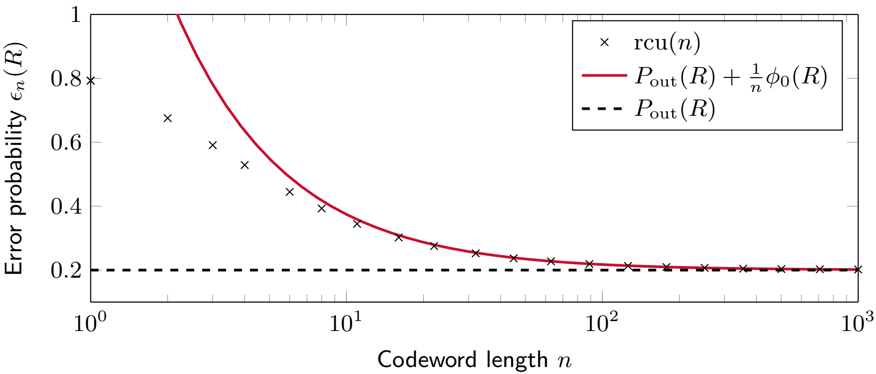
$$P_{\mathrm{out}}(R) = \mathbb{P}[I(Q) < R] \quad \text{and} \quad \phi_0(R) = -\frac{1}{2} p'_{\frac{R - I(Q)}{\sqrt{V(Q)}}}(0) + p_{R - I(Q)}(0)$$

Binary Erasure Channel

- Uniform erasure probability $q \in (0,1)$ at $R = \frac{1}{5}\log 2$

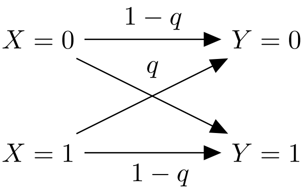
$$P_{\text{out}}(R) = \frac{R}{\log 2}$$

$$\phi_0(R) = \frac{1}{\log 2} - \frac{R}{\log 2} + \frac{1}{2}$$



▷ Font-Segura, Martinez & Guillén i Fàbregas, "Refined error probability approximations in quasi-static erasure channels", *IZS*, 2016

Binary Symmetric Channel



- Channel transition probability

$$W(y|x) = \begin{cases} 1 - q & x = y \\ q & x \neq y \end{cases}$$

- Let d be the Hamming distance between x^n and y^n

$$\begin{aligned} W_q^n(y^n|x^n) &= \prod_{i=1}^{n} W_q(y_i|x_i) \\ &= q^d (1-q)^{n-d} \end{aligned}$$

Binary Symmetric Channel

- Information density lies in a lattice of span γ

$$i_q^n(x^n; y^n) = n \log(2 - 2q) - d \log \frac{1-q}{q}$$

- Pairwise error probability

$$\begin{aligned}
\mathrm{pep}_q(x^n; y^n) &= \mathbb{P}\left[W_q^n(y^n|\overline{X}^n) \geq W_q^n(y^n|x^n)\right] \\
&= \mathbb{P}\left[i_q^n(\overline{X}^n; y^n) \geq i_q^n(x^n; y^n)\right] \\
\kappa(\tau) = \log \mathbb{E}[e^{\tau i_q(\overline{X}^n; y^n)}] \longrightarrow &= \sum_{\ell=0}^{d} \frac{\gamma}{2\pi j} \int_{\hat{\tau} - j\frac{\pi}{\gamma}}^{\hat{\tau} + j\frac{\pi}{\gamma}} e^{\kappa(\tau) - \tau i_q^n(\ell)} \, \mathrm{d}\tau \\
\text{Taylor} \longrightarrow &\simeq \sum_{\ell=0}^{d} \frac{\gamma}{2\pi j} \int_{\hat{\tau} - j\infty}^{\hat{\tau} + j\infty} e^{\kappa(\hat{\tau}) + \kappa'(\hat{\tau})(\tau - \hat{\tau}) + \frac{1}{2}\kappa''(\hat{\tau})(\tau - \hat{\tau})^2 - \tau i_q^n(\ell)} \, \mathrm{d}\tau \\
&= \frac{e^{\kappa(1) - i_q^n(d)}}{\sqrt{2\pi\kappa''(1)}} \sum_{\ell=0}^{d} \gamma \, e^{-(i_q^n(\ell) - i_q^n(d)) - \frac{(i_q^n(\ell) - \kappa'(1))^2}{2\kappa''(1)}} \\
&= \frac{k_q^n(d)}{\sqrt{n}} e^{-i_q^n(d)}
\end{aligned}$$

▷ G. Doetsch, *Introduction to the Theory and Applications of the Laplace Transformation*. Springer, 1974

Binary Symmetric Channel

- Random coding union bound

$$\begin{aligned}\mathrm{rcu}_n(R) &= \mathbb{E}_{DQ}\left[\min\left\{1, (M-1)\frac{k_Q(D)}{\sqrt{n}}e^{-i_Q^n(T)}\right\}\right]\\ &= \mathbb{E}\left[\mathbb{1}\left\{R - I(Q) + \frac{1}{\sqrt{n}}\left(\sqrt{n}I(Q) - \frac{i_Q^n(D)}{\sqrt{n}}\right) + \frac{1}{n}\left(\log k_Q^n(D) - \frac{\log n}{2} - \log U\right) \geq 0\right\}\right]\\ &\simeq P_{\mathrm{out}}(R) + \frac{1}{\sqrt{n}}0 - \frac{1}{2n}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) + \frac{1}{n}p_{R-I(Q)}(0)\left(\mathbb{E}\big[\log k_{q_0}^n(D)\big] - \frac{\log n}{2} + 1\right)\end{aligned}$$

Binary Symmetric Channel

$$k_q^n(d) = \frac{1}{\sqrt{2\pi V}} \sum_{\ell=0}^{d} \underbrace{\gamma e^{-\gamma(d-\ell) - \frac{\gamma^2 (nq-\ell)^2}{2nV}}}_{f(\ell)}$$

- Using the Euler-Maclaurin summation formula...

$$k_q^n(d) = \frac{1}{\sqrt{2\pi V}} \left(\int_0^d f(\ell)\,\mathrm{d}\ell + \sum_{m=1}^{\infty} \frac{B_m^+}{m!} \left(f^{(m-1)}(d) - f^{(m-1)}(0) \right) \right) \longleftarrow$$

Bernoulli numbers $B_m^+ = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \ldots$

$$= \frac{1}{\sqrt{2\pi V}} \left(\sqrt{2\pi n V} e^{\frac{nV}{2}} e^{-(d-nq)\gamma} \left(Q\left(\frac{nq\gamma + nV}{\sqrt{nV}} \right) - Q\left(\frac{nq\gamma + nV - d\gamma}{\sqrt{nV}} \right) \right) + \sum_{m=1}^{\infty} \frac{B_m^+}{m!} (\ldots) \right)$$

$$\simeq \frac{e^{-\frac{a^2\gamma^2}{2V}}}{\sqrt{2\pi V}} \sum_{m=0}^{\infty} \frac{B_m^+}{m!} \gamma^m \longleftarrow a = \frac{d-nq}{\sqrt{n}}$$

$$= \frac{e^{-\frac{a^2\gamma^2}{2V}}}{\sqrt{2\pi V}} \frac{\gamma}{1-e^{-\gamma}}$$

- Hence

$$\mathbb{E}[\log k_q^n(D)] \simeq \log \frac{\gamma (1-e^{-\gamma})^{-1}}{\sqrt{2\pi e V}}$$

Binary Symmetric Channel

- Random coding union bound

$$\mathrm{rcu}_n(R) = \mathbb{E}_{DQ}\left[\min\left\{1, (M-1)\frac{k_Q(D)}{\sqrt{n}}e^{-i_Q^n(T)}\right\}\right]$$

$$= \mathbb{E}\left[\mathbb{1}\left\{R - I(Q) + \frac{1}{\sqrt{n}}\left(\sqrt{n}I(Q) - \frac{i_Q^n(D)}{\sqrt{n}}\right) + \frac{1}{n}\left(\log k_Q^n(D) - \frac{\log n}{2} - \log U\right) \geq 0\right\}\right]$$

$$\simeq P_{\mathrm{out}}(R) + \frac{1}{\sqrt{n}}0 - \frac{1}{2n}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) + \frac{1}{n}p_{R-I(Q)}(0)\left(\mathbb{E}\left[\log k_{q_0}^n(D)\right] - \frac{\log n}{2} + 1\right)$$

$$\simeq P_{\mathrm{out}}(R) + \frac{1}{\sqrt{n}}0 - \frac{1}{2n}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) + \frac{1}{n}p_{R-I(Q)}(0)\left(\log\frac{\gamma(1-e^{-\gamma})^{-1}}{\sqrt{2\pi e V(q_0)}} - \frac{\log n}{2} + 1\right)$$

$$\Downarrow$$

$$\boxed{\mathrm{rcu}_n(R) \simeq P_{\mathrm{out}}(R) + \frac{\log n}{n}\phi_{\log}(R) + \frac{1}{n}\phi_0(R)}$$

$$\Downarrow$$

$$P_{\mathrm{out}}(R) = \mathbb{P}[I(Q) < R]$$

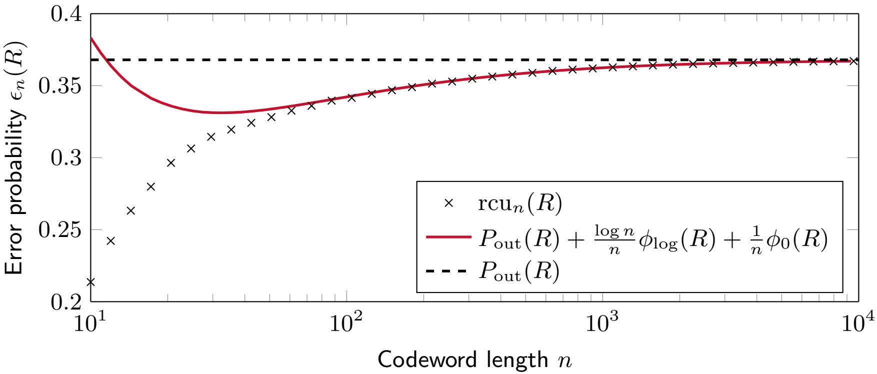
$$\phi_{\log}(R) = -\frac{1}{2}p_{R-I(Q)}(0)$$

$$\phi_0(R) = -\frac{1}{2}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) + p_{R-I(Q)}(0)\left(1 + \log\frac{\gamma(1-e^{-\gamma})^{-1}}{\sqrt{2\pi e V(q_0)}}\right)$$

Binary Symmetric Channel

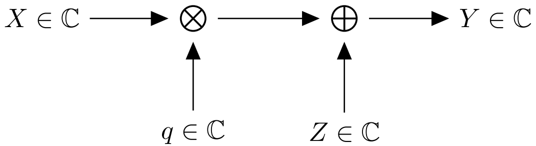
- Uniformly distributed crossover probability $q \in (0, \frac{1}{2})$ at rate $R = \frac{1}{10}\log 2$, $q_0 = 0.3160$

$$P_{\text{out}}(R) = 1 - 2q_0$$
$$\phi_{\log}(R) = -\frac{1}{\log \frac{1-q_0}{q_0}}$$
$$\phi_0(R) = 2q_0 + \frac{2 - 2\log(1-2q_0) - \log 2\pi}{\log \frac{1-q_0}{q_0}}$$



▷ Font-Segura, Martinez & Guillén i Fàbregas, "Asymptotics of the error probability in quasi-static binary symmetric channels", *ISIT*, 2017

Fading Channel



- Channel transition probability

$$W_q(y|x) = \frac{1}{\pi\sigma^2} e^{-\frac{|y-qx|^2}{\sigma^2}}$$

- and

$$\begin{aligned} W_q^n(y^n|x^n) &= \prod_{i=1}^{n} W_q(y_i|x_i) \\ &= \frac{1}{(\pi\sigma^2)^n} e^{-\frac{||y^n-qx^n||^2}{\sigma^2}} \end{aligned}$$

Fading Channel

- Information density

$$i_q^n(x^n;y^n) = n\log\left(1+|q|^2\rho\right) + \frac{\|y^n\|^2}{1+|q|^2\rho} - \|y^n - qx^n\|^2.$$

- Pairwise error probability

$$\begin{aligned}
\mathrm{pep}_q(x^n;y^n) &= \mathbb{P}\left[W_q^n(y^n|\overline{X}^n) \geq W_q^n(y^n|x^n)\right] \\
&= \mathbb{P}\left[i_q^n(\overline{X}^n;y^n) \geq i_q^n(x^n;y^n)\right] \\
\kappa(\tau) = \log\mathbb{E}[e^{\tau X}] \longrightarrow &= \int_{i_q^n(x^n;y^n)}^{\infty} \frac{1}{2\pi j}\int_{\hat{\tau}-j\infty}^{\hat{\tau}+j\infty} e^{\kappa(\tau)-\tau z}\,\mathrm{d}\tau\,\mathrm{d}z \\
\text{Taylor} \longrightarrow &\simeq \int_{i_q^n(x^n;y^n)}^{\infty} \frac{1}{2\pi j}\int_{\hat{\tau}-j\infty}^{\hat{\tau}+j\infty} e^{\kappa(\hat{\tau})+\kappa'(\hat{\tau})(\tau-\hat{\tau})+\frac{1}{2}\kappa''(\hat{\tau})(\tau-\hat{\tau})^2-\tau z}\,\mathrm{d}\tau\,\mathrm{d}z \\
&= e^{\kappa(1)-i_q^n(x^n;y^n)} e^{\frac{\kappa''(1)}{2}-\kappa'(1)+i_q^n(x^n;y^n)} Q\left(\frac{\kappa''(1)-\kappa'(1)+i_q^n(x^n;y^n)}{\sqrt{\kappa''(1)}}\right) \\
&= \frac{k_q^n(x^n;y^n)}{\sqrt{n}} e^{-i_q^n(x^n;y^n)}
\end{aligned}$$

Fading Channel

- Random coding union bound

$$
\begin{aligned}
\mathrm{rcu}_n(R) &= \mathbb{E}_{X^nY^nQ}\left[\min\left\{1, (M-1)\frac{k_Q^n(X^n;Y^n)}{\sqrt{n}}e^{-i_Q^n(X^n;Y^n)}\right\}\right] \\
&\simeq P_{\mathrm{out}}(R) - \frac{1}{2n}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) + \frac{1}{n}p_{R-I(Q)}(0)\left(\mathbb{E}\big[\log k_{q_0}^n(D)\big] - \frac{\log n}{2} + 1\right) \\
&\simeq P_{\mathrm{out}}(R) + \frac{1}{\sqrt{n}}0 - \frac{1}{2n}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) + \frac{1}{n}p_{R-I(Q)}(0)\left(\log\frac{1}{\sqrt{2\pi e\nu(q_0)}} - \frac{\log n}{2} + 1\right)
\end{aligned}
$$

$$\Downarrow$$

$$\boxed{\mathrm{rcu}_n(R) \simeq P_{\mathrm{out}}(R) + \frac{\log n}{n}\phi_{\log}(R) + \frac{1}{n}\phi_0(R)}$$

$$\Downarrow$$

$$P_{\mathrm{out}}(R) = \mathbb{P}[I(Q) < R]$$

$$\phi_{\log}(R) = -\frac{1}{2}p_{R-I(Q)}(0)$$

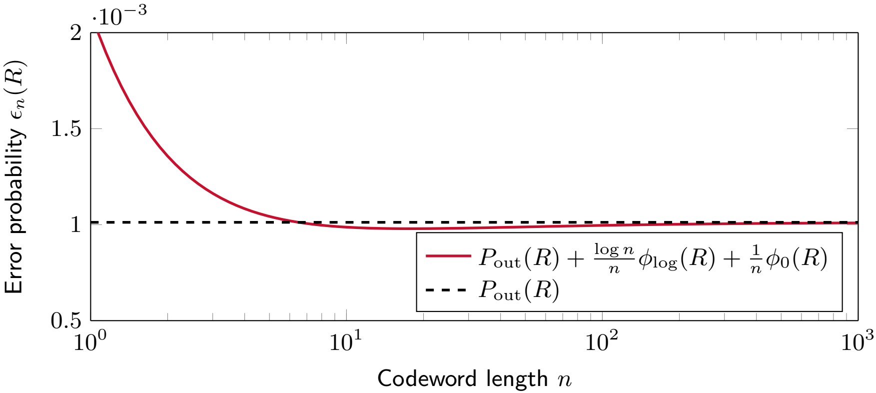
$$\phi_0(R) = -\frac{1}{2}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) + p_{R-I(Q)}(0)\left(1 + \log\frac{1}{\sqrt{2\pi e\nu(q_0)}}\right)$$

Fading Channel

- For Rayleigh fading and i.i.d. Gaussian input, rate $R = 2$ and SNR $\rho = 38$ dB

$$P_{\mathrm{out}}(R) = 1 - \exp\left(-\frac{e^R - 1}{\rho}\right) \qquad \phi_{\log}(R) = -\frac{e^R}{2\rho}\exp\left(-\frac{e^R-1}{\rho}\right)$$

$$\phi_0(R) = \frac{e^R}{\rho}\exp\left(-\frac{e^R-1}{\rho}\right)\left(2 - \frac{e^R-1}{\rho} + \log\frac{1}{\sqrt{2\pi e(1-e^{-2R})}}\right)$$



▷ Font-Segura, Martinez & Guillén i Fàbregas, "Asymptotics of the random-coding union bound in quasi-static fading channels", *ITW*, 2016

Rate Expansions

- For the BEC, we saw...

$$\mathrm{rcu}_n(R) \simeq P_{\mathrm{out}}(R) + \frac{1}{n}\phi_0(R)$$

...but we know that $\lim_{n\to\infty} R_n(\epsilon) = C_\epsilon$. Then,

$$\epsilon \simeq P_{\mathrm{out}}(C_\epsilon) + \frac{1}{n}\phi_0(C_\epsilon) + (R - C_\epsilon)\left(P'_{\mathrm{out}}(C_\epsilon) + \frac{1}{n}\phi'_0(C_\epsilon)\right)$$

$$\Downarrow$$

$$R_n(\epsilon) \simeq C_\epsilon - \frac{1}{n}\frac{\phi_0(C_\epsilon)}{p_{C_\epsilon - I(Q)}(0)}$$

- Similarly for the BSC and the fading channel...

$$\mathrm{rcu}_n(R) \simeq P_{\mathrm{out}}(R) + \frac{\log n}{n}\phi_{\log}(R) + \frac{1}{n}\phi_0(R)$$

$$\Downarrow$$

$$R_n(\epsilon) \simeq C_\epsilon + \frac{1}{2}\frac{\log n}{n} - \frac{1}{n}\frac{\phi_0(C_\epsilon)}{p_{C_\epsilon - I(Q)}(0)}$$

In summary

- Expansion to the error probability based on saddlepoint approximation to the pairwise error probability and Taylor expansion of the RCU
- Special case of BEC $\epsilon_n(R) \simeq P_{\text{out}}(R) + \frac{1}{n}\phi_0(R)$
- BSC and fading channel $\epsilon_n(R) \simeq P_{\text{out}}(R) + \frac{\log n}{n}\phi_{\log}(R) + \frac{1}{n}\phi_0(R)$
- Relation to the rate $R_n(\epsilon) \simeq C_{\text{out}}(\epsilon) + \frac{1}{2n}\log n + O\left(\frac{1}{n}\right)$
- Useful tool to approximate any tail probability!

Thank you for your attention!