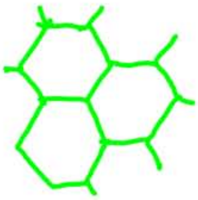
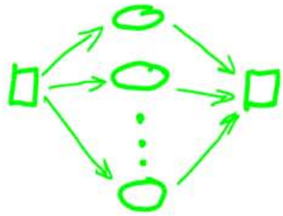
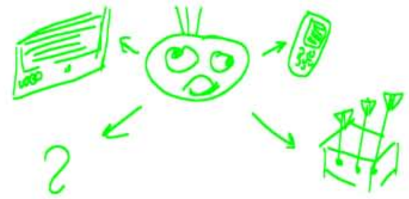
Lattice Coding for Signals & Networks







Rami Zamir

Information Theory School @ UniCamp

January 2015

Lattice: Definition

Lattice = discrete subgroup of Euclidean space

$$\Lambda = \{ \underline{G} \cdot \underline{i} \;:\; \underline{i} = \text{vector of integers } (0, \pm 1, \pm 2, \ldots) \}$$

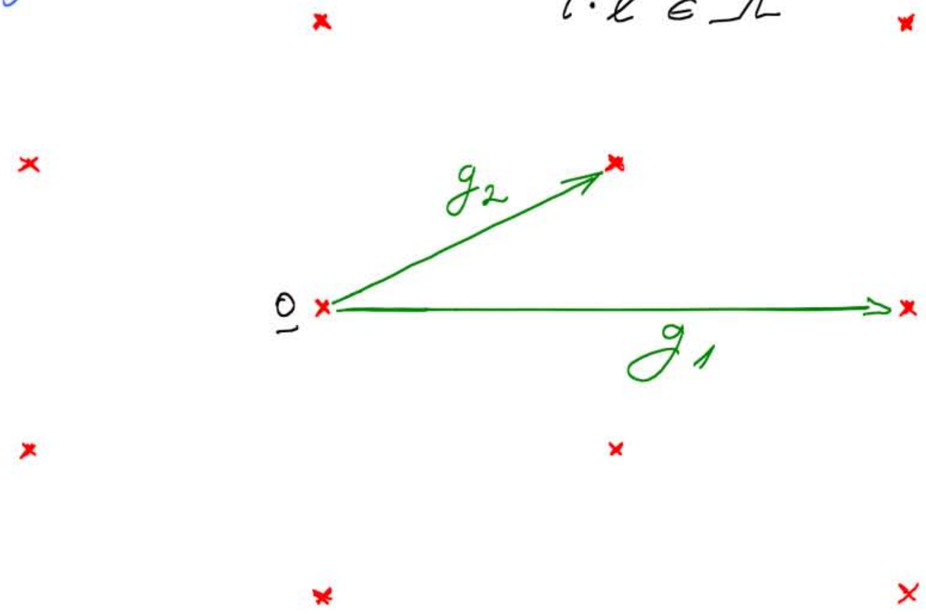
Λ: Lattice in $\mathbb{R}^n$

$\underline{G}$: Generator Matrix $n \times n$

Closed under reflection & addition:

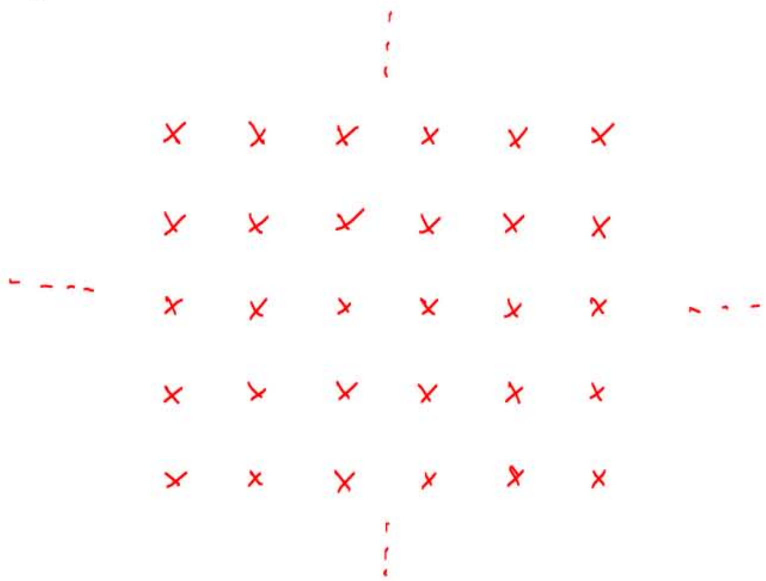
linearity: $l_1, l_2 \in \Lambda \Rightarrow l_1 + l_2 \in \Lambda$

$i \cdot l \in \Lambda$

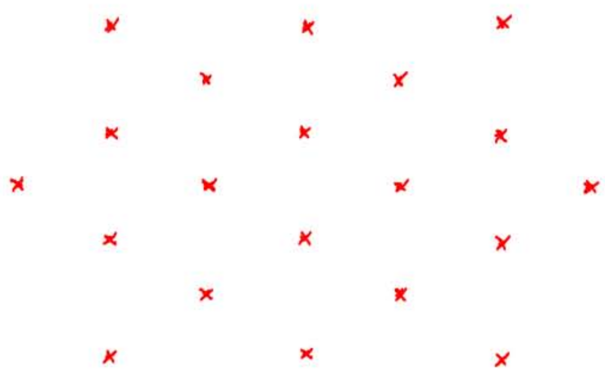


Lattice Codes in Signal Space

square ($\mathbb{Z}$)-lattice $\Rightarrow$ uncoded constellation



More "interesting" lattice $\Rightarrow$ coded constellation



What a Lattice Means?...

What a Lattice Means?...

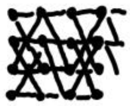
For my 8-year old kid: 

-//- a physicist/crystallographer: 

-//- a mathematician: 

-//- a Computer Scientist: 

-//- a coding theorist: Λ_8, Λ_{24}, ...

 ...

-//- an Information Theorist:

$$n \to \infty$$

People Who Influenced ...

Hermann Minkowski
(1864 – 1909)

Neil Sloane

John Conway

Dave Forney

also, Rudi de Buda, Gregory Poltyrev ...

Why Lattices in Communication?

1

2

3

4

Why Lattices in Communication?

1. a bridge from $n=1$  to $n=\infty$

   = non-asymptotic analysis per dimension

2. Algebraic (low complexity) Binning

   = structured coding schemes for networks

3. bridge from Analog  - to - Digital

   = Robust joint source - channel coding

4. Better than Random - Coding !

   in distributed side - information problems

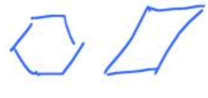
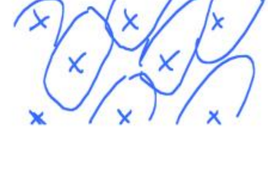
Lattices do things differently:

Randomness $\longrightarrow$

Typicality $\longrightarrow$

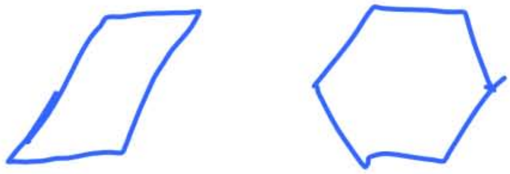
Binning $\longrightarrow$

Special features:

* **Dither** - why? how? is it critical?
* **MMSE** - estimation or decoding? linear or not?
* **Volume & noise** → "soft" sphere packing & covering
* **Lattices in high dimensions**
  → white Gaussian noise?
  → non-Gaussian noise?
* **Nesting, cosets & binning**
  * AWGN + dither 
  * Voronoi or Parallepiped? 
  * Mixed dimensions n_1, n_2
  * noise-matched decoding 

1. Definitions: Partition, Construction

$\mathrm{Vol}(\Lambda)$



modulo Λ

Lattice: Definition

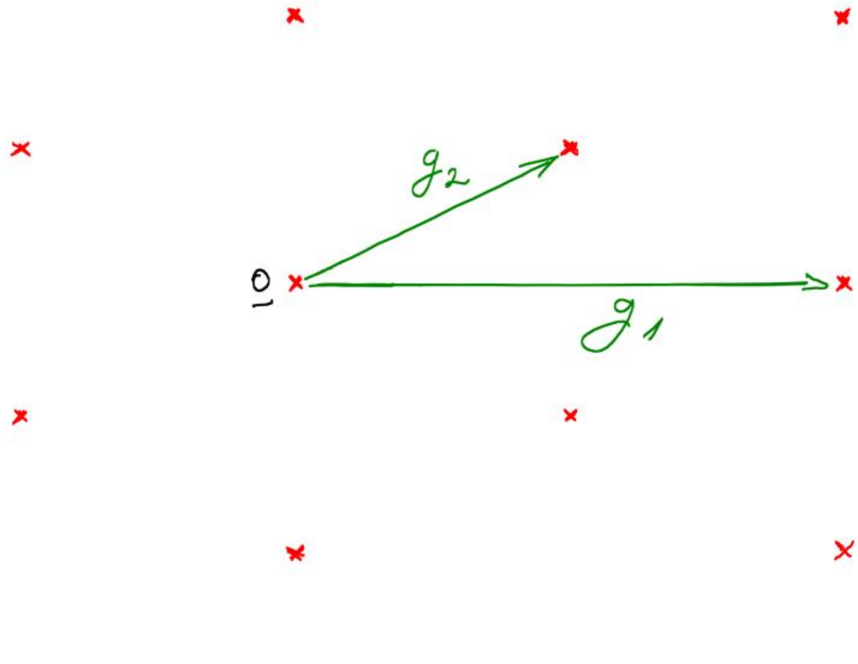
Let $\underline{g}_1, \dots, \underline{g}_n$ - linearly independent vectors in $\mathbb{R}^n$

$$\underline{\underline{G}} = \left(\underline{g}_1 \middle| \dots \middle| \underline{g}_n \right) = \text{generator matrix}$$

$$\Lambda(G) = \left\{ i_1 \cdot \underline{g}_1 + \dots + i_n \cdot \underline{g}_n : i_1 \dots i_n \in \mathbb{Z} \right\}$$

$$= \left\{ \underline{\underline{G}} \cdot \underline{i} : \underline{i} \in \mathbb{Z}^n \right\}$$

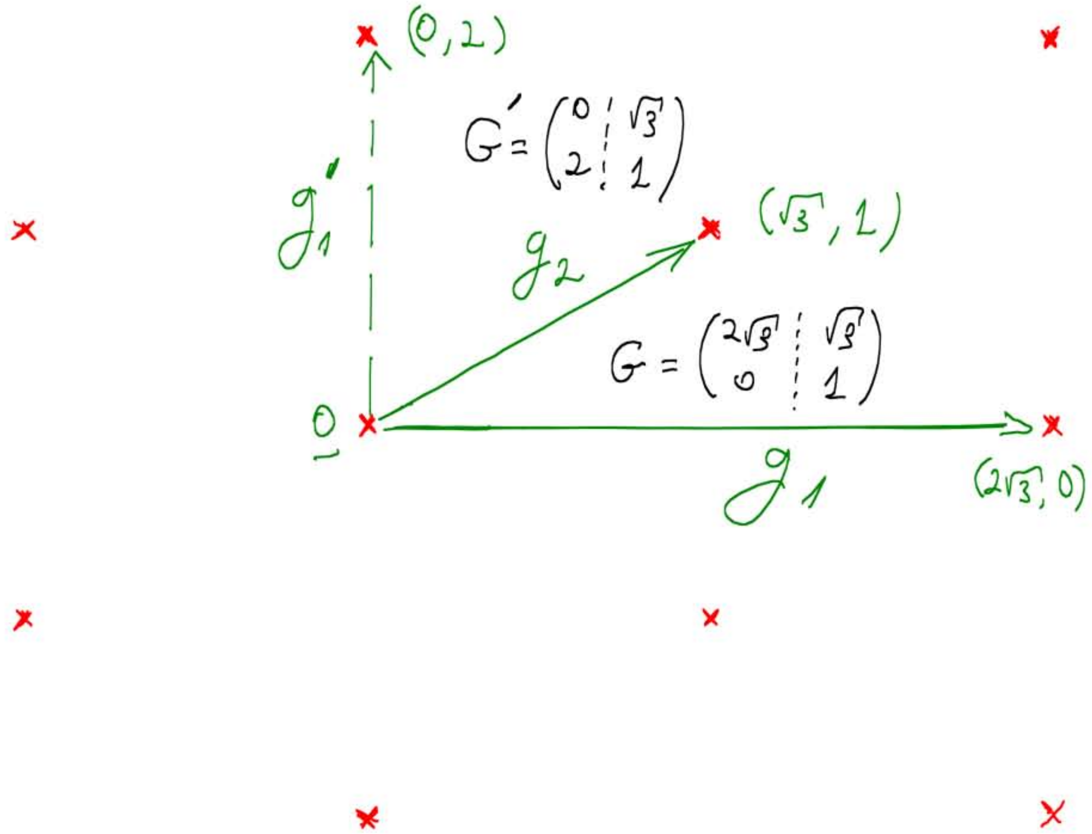
$$= \underline{\underline{G}} \cdot \mathbb{Z}^n$$



Lattice: Equivalent Representations

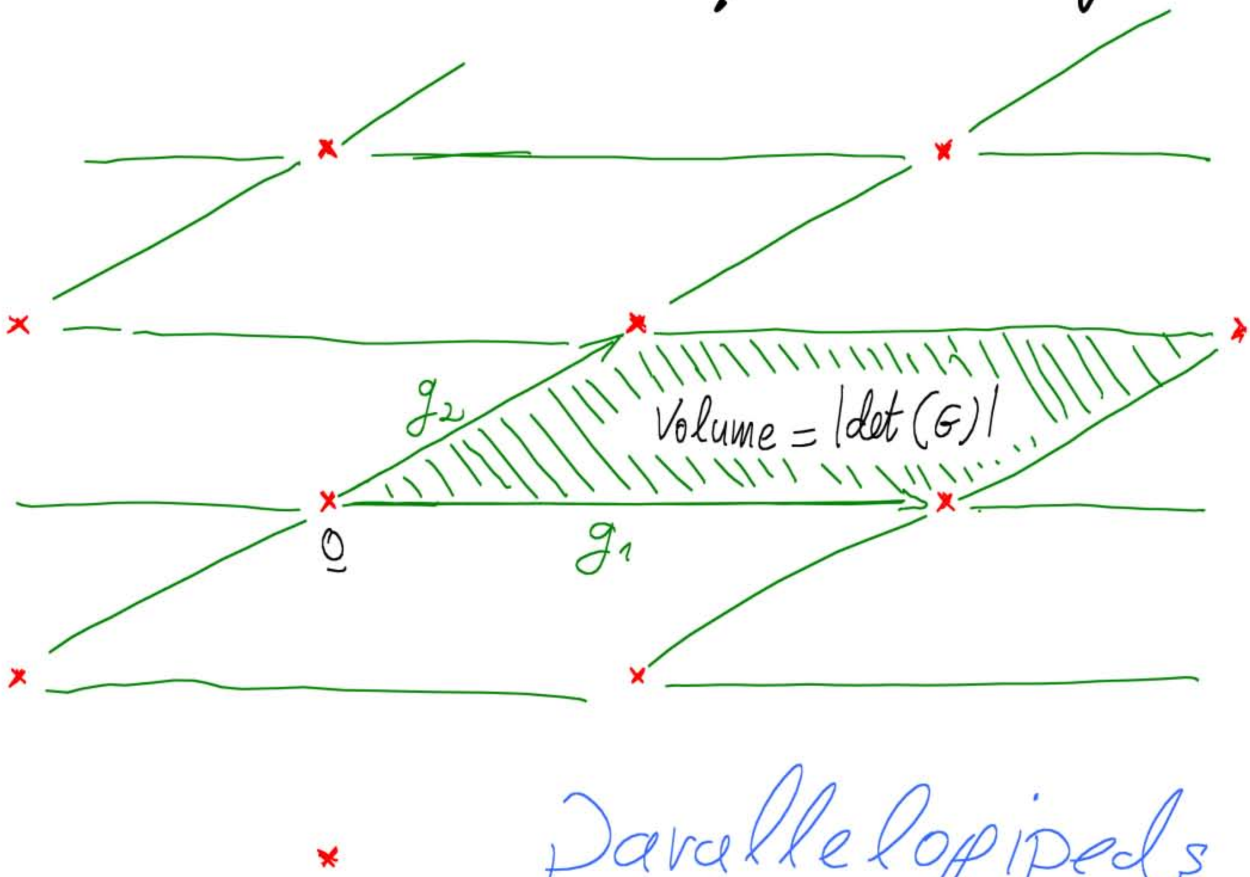
T = unimodular matrix
(integer elements, $\det(T) = \pm 1$)

$$\Rightarrow \quad \Lambda(G \cdot T) = \Lambda(G)$$



Lattice Partition:

Quantization / Decision Regions

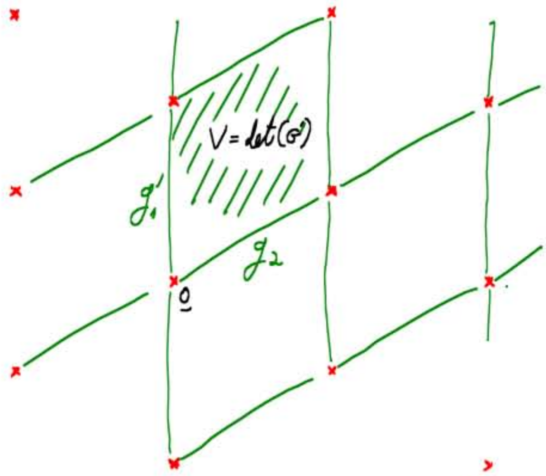


parallelopipeds

$$P_0 = \{ \alpha_1 g_1 + \alpha_2 g_2 : \quad 0 \le \alpha_1, \alpha_2 \le 1 \}$$

$$\Lambda + P_0 = \mathbb{R}^k$$

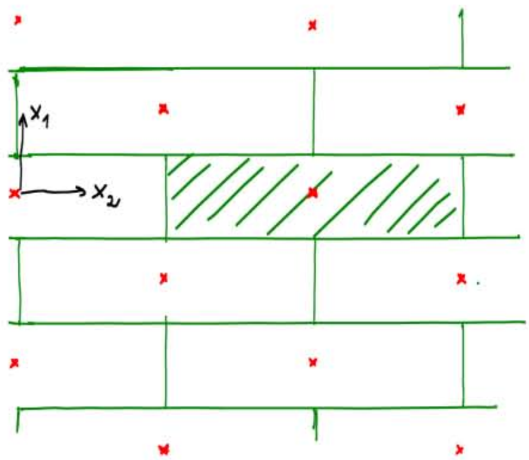
Partitions, Fundamental Cells

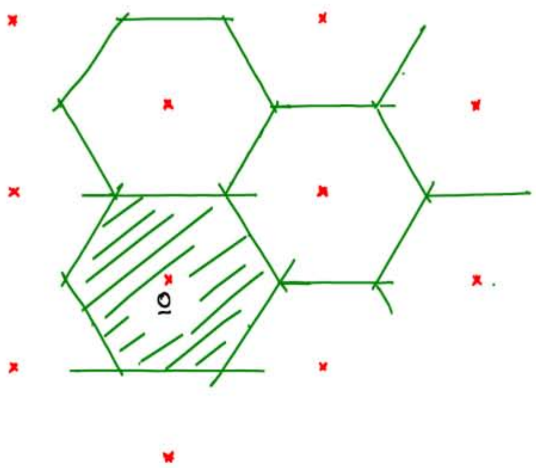


Other Basis ⇒ other Parallelepiped

⇒ Cell Volume V is invariant of partition

Sequential Quantization





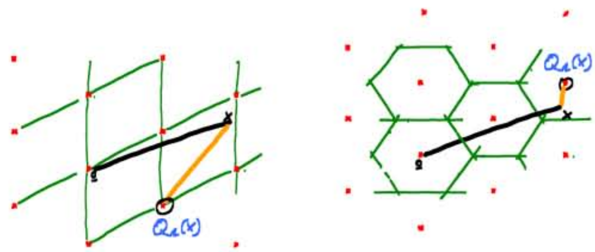
Voronoi Partition

$$P_0 = \{ x : \|x\| \le \|x - l_i\| \} \quad \forall l_i \in \Lambda$$

Lattice Quantization, Modulo Lattice

$$\boxed{\begin{aligned} Q_{\Lambda,P_0}(x) &= \lambda \quad \text{if} \quad x \in (\lambda + P_0) \\ x \bmod_{P_0} \Lambda &= x - Q_{\Lambda,P_0}(x) \end{aligned}}$$

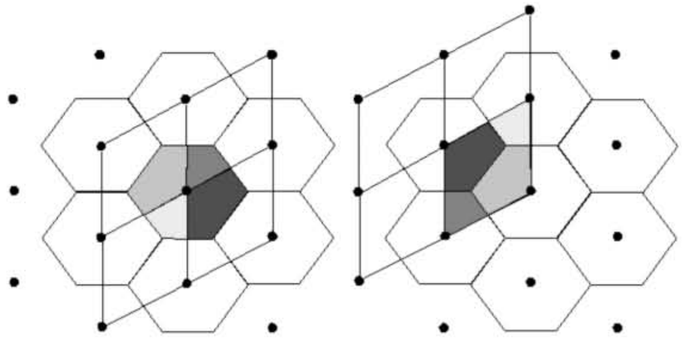
$\Rightarrow$ $x \in \mathbb{R}^n$ uniquely written as $\underbrace{Q_\Lambda(x)}_{\text{quantization}} + (\underbrace{x \text{ modulo } \Lambda}_{\text{error}})$



Modulo Laws:

* $a \bmod \Lambda = a + \lambda(a), \quad \lambda(a) \in \Lambda$
* $(a + \lambda) \bmod \Lambda = a \bmod \Lambda, \quad \forall \lambda \in \Lambda$
* $[(a \bmod \Lambda) + b] \bmod \Lambda = (a+b) \bmod \Lambda$
* $(a \bmod_{P_0} \Lambda) \bmod_{Q_0} \Lambda = a \bmod_{Q_0} \Lambda$

Fundamental Cells & Cosets



all fundamental cells are "modulo equivalent"!

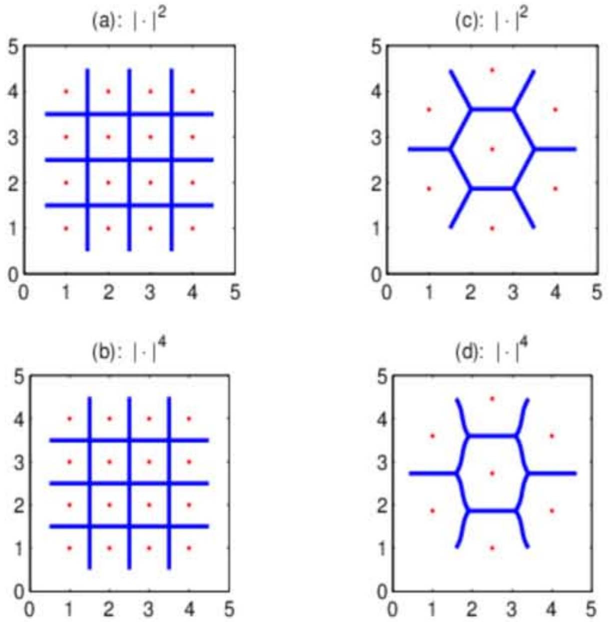
Coset: shift of Λ = points identical modulo Λ

$$\Lambda_v = v + \Lambda$$

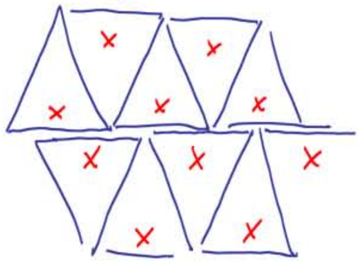
$$= \{ x \in \mathbb{R}^n : x \bmod \mathcal{P}_0 = v \} \quad , \quad v \in \mathcal{P}_0$$

∴ **Fundamental cell:** a complete set of coset representatives

Non-Euclidean Voronoi partition



Tiling & Tranformation

 $\neq \Lambda$

$A \cdot$  $=$ 

$$\text{parallelepiped}(A \cdot \Lambda) = A \cdot \text{parallelepiped}(\Lambda)$$

But ...

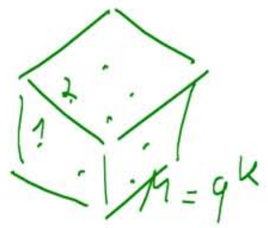
$$\text{Voronoi}(A \cdot \Lambda) \neq A \cdot \text{Voronoi}(\Lambda) \quad !$$

because $\|x\| > \|y\| \not\Longrightarrow \|A \cdot \underline{x}\| > \|A \cdot \underline{y}\|$

Lattices from Linear Codes ("Construction A")

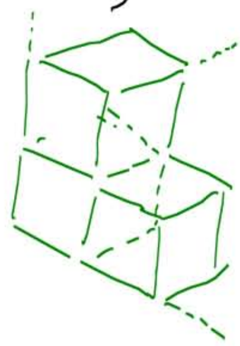
Let $\mathcal{C} = q$-ary (n,k) liner code over $\mathbb{Z}_q = \{0, \dots q-1\}$

$$= \{ \underset{n \times k}{\underline{\underline{G}}} \cdot \underline{i} \; : \; \underline{i} \in \mathbb{Z}_q^k \}$$



Let $\Lambda_{\mathcal{C}}$ = modulo-q lattice

$$= \{ \lambda \in \mathbb{R}^n : \lambda \bmod q \in \mathcal{C} \}$$



e.g., E_8 (a lattice in $\mathbb{R}^8$)

is a modulo-2 lattice

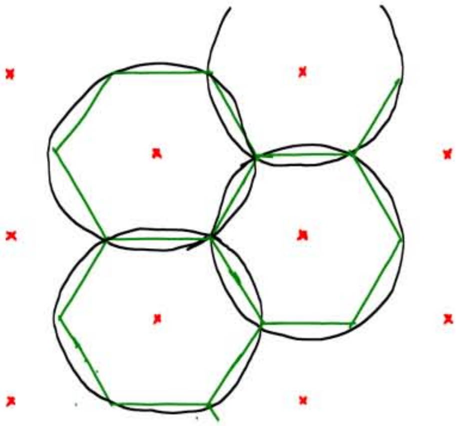
$\Lambda_{\text{Hamming Code } (8,4,4)}$

2. Figures of merit

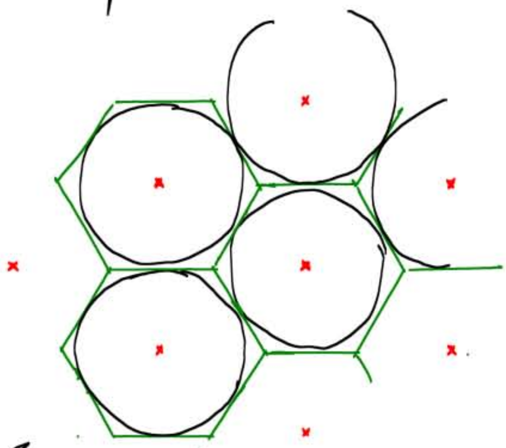
$G(\Lambda)$, $\mu(\Lambda, pe)$

Covering, Packing, kissing Number & More ....

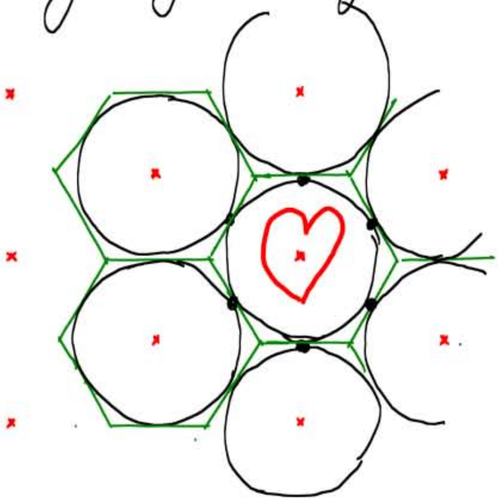
Covering $\mathbb{R}^n$ with (few) spheres



packing (many) spheres in $\mathbb{R}^n$



kissing by (many) spheres

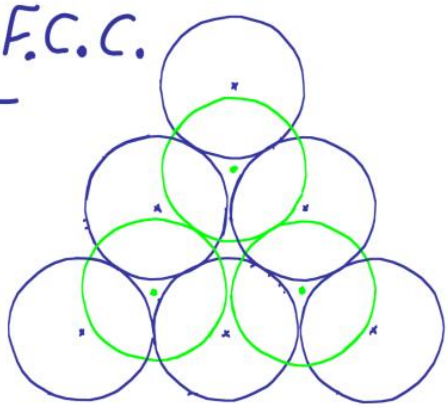


&

good arrangements for quantization and AWGN channel coding

Not an "All-Purpose" Lattice!

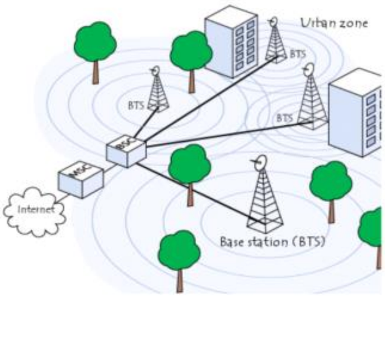
*** Best 3-dim packing: F.C.C.**

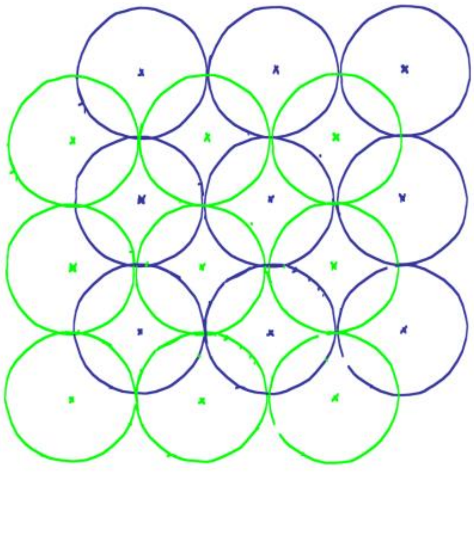


each layer = hexagonal Λ
layers are staggered

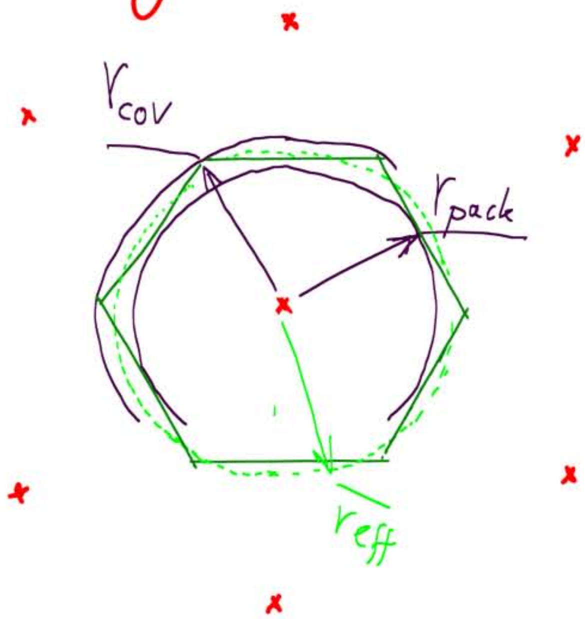
*** Best 3-dim Covering: B.C.C.**

each layer = cubic Λ
layers are staggered





Figures of Merit



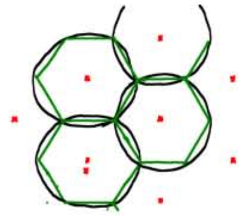
Radiuses:

r_{cov} = min sphere containing V_0

r_{pack} = max sphere contained in V_0
= $d_{min} / 2$

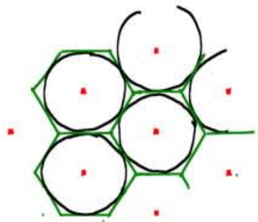
r_{eff} = Sphere with same volume

- Covering efficiency:



$$\rho_{cov}(\Lambda) = \frac{r_{cov}}{r_{eff}} > 1$$

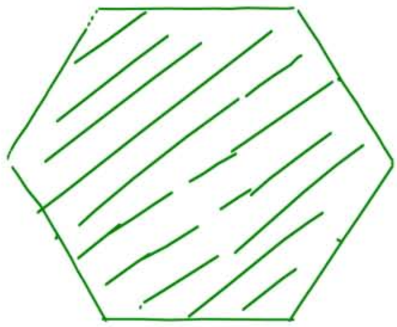
- Packing efficiency:



$$\rho_{pack}(\Lambda) = \frac{r_{pack}}{r_{eff}} < 1$$

Figures of Merit (Continued)

- Quantization efficiency:



$\underline{X} \sim \text{Uniform}(\mathcal{V}_0)$

$\sigma^2(\Lambda) \triangleq \frac{1}{n} E\|X\|^2$

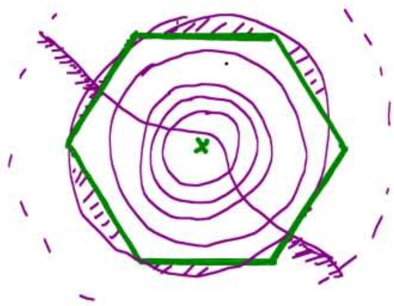
$G(\Lambda) \triangleq \dfrac{\sigma^2(\Lambda)}{V^{2/n}}$ = normalized second moment

Figures of Merit (Continued)

- AWGN coding efficiency: $\underline{Z} \sim$ AWGN $N(0, \sigma^2)$

$$\mu(\Lambda, \sigma^2) \triangleq \frac{V^{2/n}}{\sigma^2} = \text{Volume-to-Noise Ratio}$$

$$P_e \triangleq Pr\{\underline{Z} \notin V_0\}$$



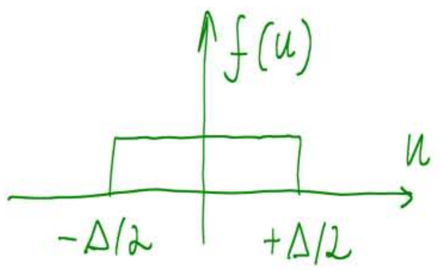
$$\mu(\Lambda, P_e) \triangleq \left.\frac{V^{2/n}}{\sigma^2}\right|_{@ P_e}$$

Example: One dimensional lattice

(Voronoi cell = interval)

1. NSM

U = dither ~ uniform on Voronoi cell $= (-\Delta/2, +\Delta/2)$



$$V(\Lambda) = \Delta$$

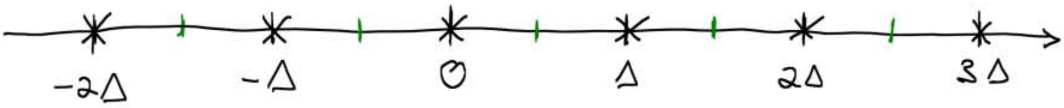
$$EU^2 = \frac{\Delta^2}{12}$$

$$\Rightarrow \quad G(\square) = \frac{EU^2}{V^2(\Lambda)} = \frac{\Delta^2/12}{\Delta^2} = \frac{1}{12}$$

invariant of Δ

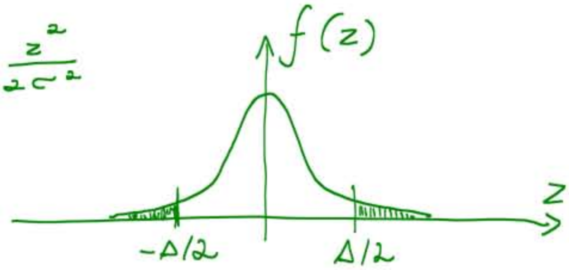
Example: One dimensional lattice

(Voronoi cell = interval)



2. NVNR

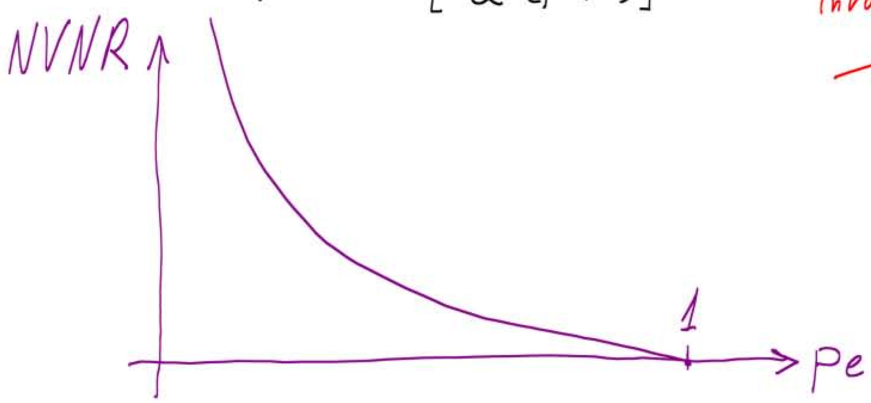
$Z \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$



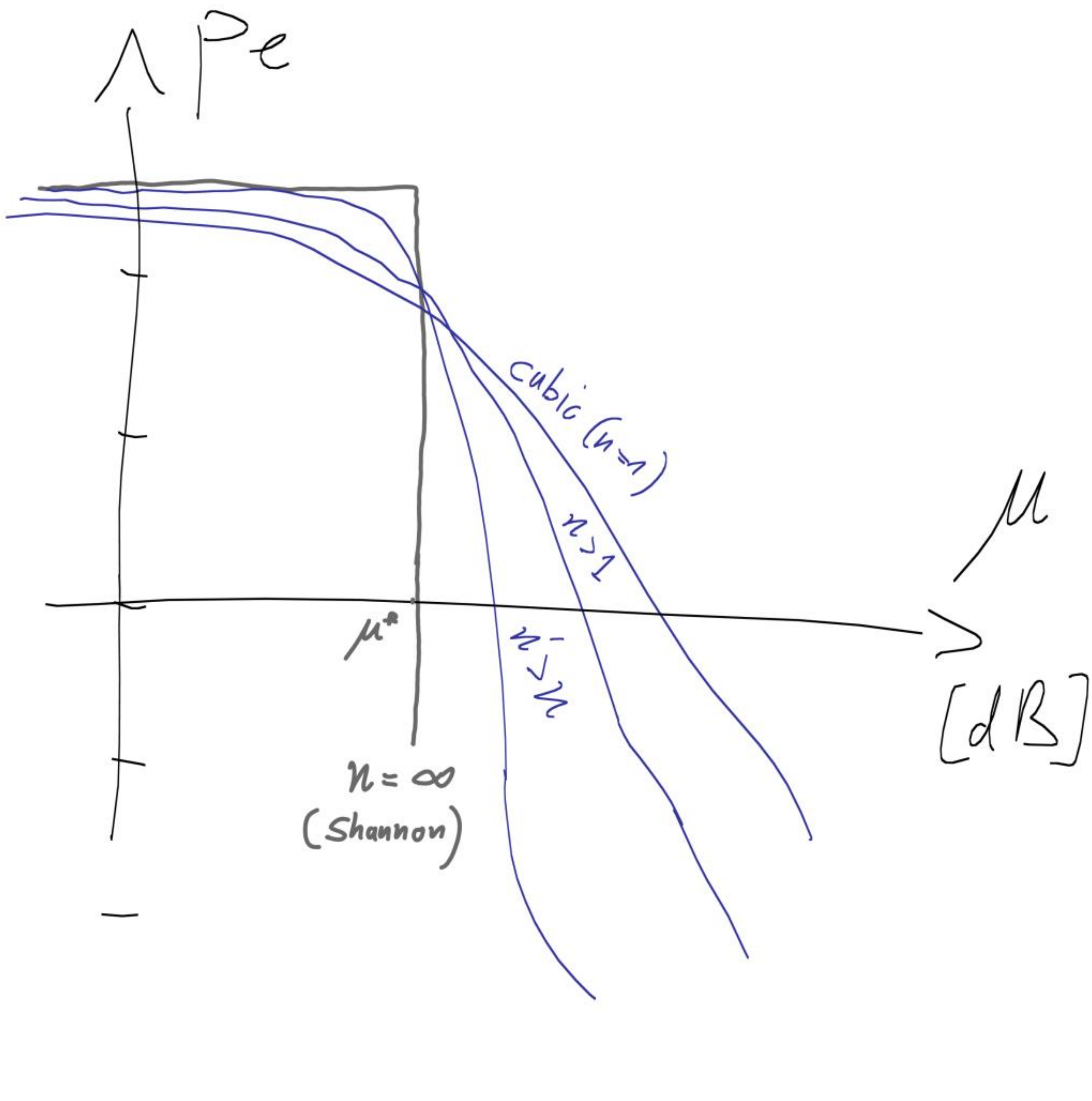
$$P_e = \Pr\left\{ |Z| > \frac{\Delta}{2} \right\} = 2 \cdot Q\left(\frac{\Delta/2}{\sigma} \right)$$

$$\Rightarrow \mu(\Lambda, P_e) = \frac{V^2(\Lambda)}{\sigma^2_{P_e}} = \left[\frac{\Delta}{\frac{\Delta/2}{Q^{-1}(P_e/2)}} \right]^2 = \left[2 \cdot Q^{-1}\left(\frac{P_e}{2} \right) \right]^2$$

invariant of Δ



Pe versus V.N.R.



$G(\Lambda_n)$ and $\mu(\Lambda_n, p_e)$ as a function of n

n.

[Conway & Sloane Book 1988]

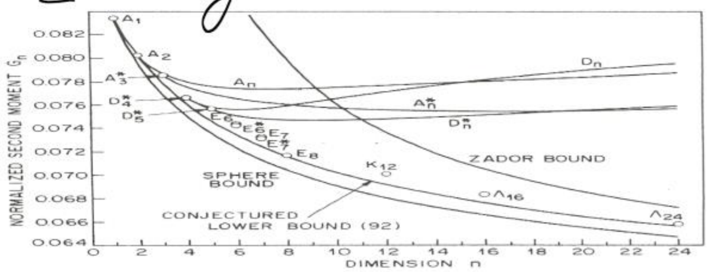
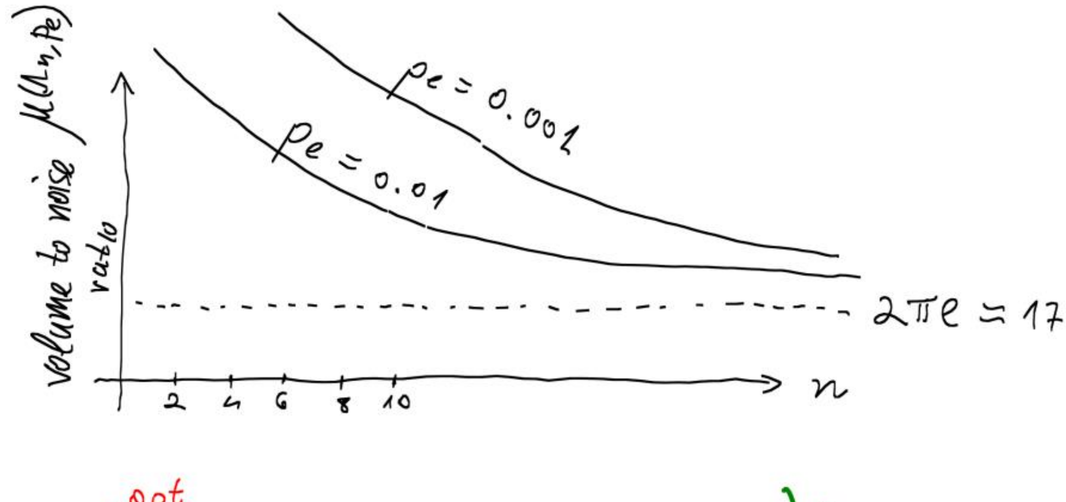


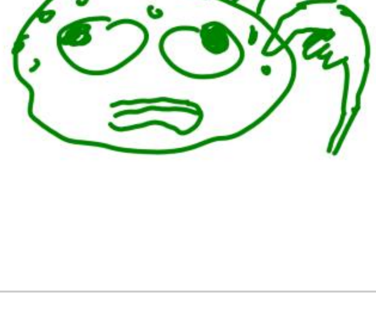
Figure 2.9. The best quantizers known in dimensions $n \leqslant 24$. - - - - - - $\frac{1}{2\pi e} \simeq 0.058$



$\Lambda_k^{opt} \longrightarrow$

$G_k \longrightarrow$?

$\mu_k \longrightarrow$



Vector Quantization Gain of Λ_n, for $n=1,2,3,\ldots$

Dimension	Lattice		Γ_q [dB]	Sphere Bound
1	$\mathbb{Z}$	integer	0	0
2	A_2	hexagonal	0.17	0.20
3	A_3	FCC	0.24	0.34
3	A_3^*	BCC	0.26	0.34
4	D_4	(Example 2.4.2)	0.36	0.45
5	D_5^*		0.42	0.54
6	E_6^*		0.50	0.61
7	E_7^*		0.57	0.67
8	E_8^*	Gosset*	0.65	0.72
12	K_{12}		0.75	0.87
16	BW_{16}	Barnes-Wall	0.86	0.97
24	Λ_{24}^*	Leech*	1.03	1.10
∞	?	?	1.53	1.53

Coding Gain of Λ_n, for $n=1,2,3,\ldots$

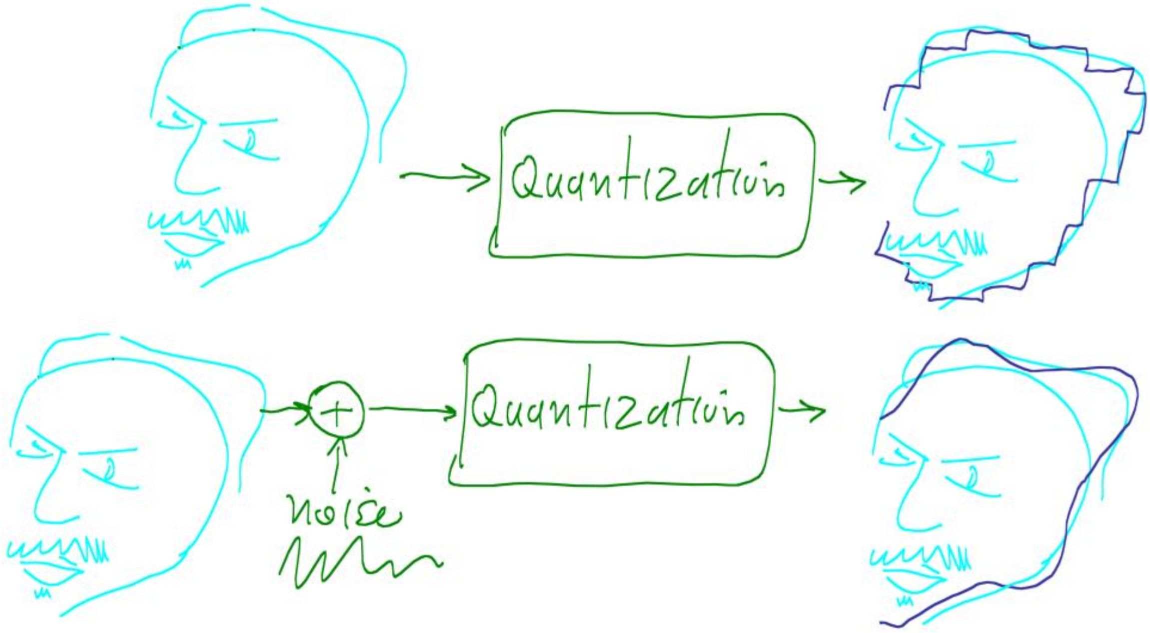
SER		10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Dim.	Lattice					
1	$\mathbb{Z}^1$	0	0	0	0	0
2	A_2	0.14 (0.16)	0.27 (0.33)	0.33 (0.45)	0.42 (0.54)	0.46 (0.6)
3	A_3	0.20 (0.27)	0.42 (0.56)	0.55 (0.78)	0.65 (0.93)	0.72 (1.05)
	A_3^*	0.20 (0.27)	0.40 (0.56)	0.52 (0.78)	0.59 (0.93)	0.61 (1.05)
4	D_4	0.29 (0.36)	0.60 (0.75)	0.82 (1.03)	0.95 (1.24)	1.00 (1.40)
8	E_8	0.50 (0.56)	1.08 (1.2)	1.49 (1.68)	1.80 (2.04)	2.00 (2.30)
16	BW_{16}	0.63 (0.75)	1.47 (1.63)	2.09 (2.32)	2.52 (2.83)	2.80 (3.22)
24	Λ_{24}	0.75 (0.84)	1.76 (1.85)	2.51 (2.65)	3.08 (3.25)	3.50 (3.71)
∞	?	-2.0	1.9	4.0	5.5	6.6

3. Dither & estimation

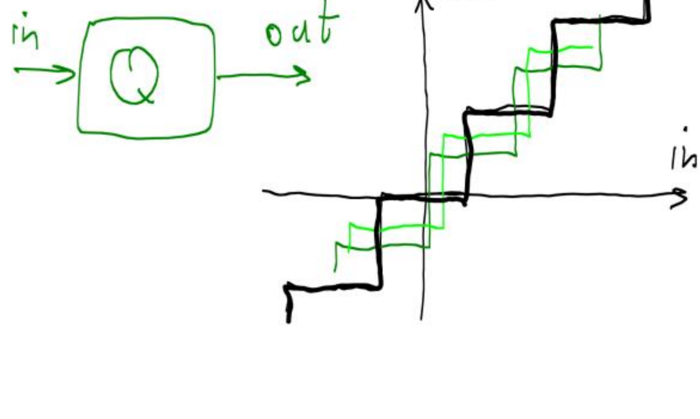
noise ( )

Dithered Quantization

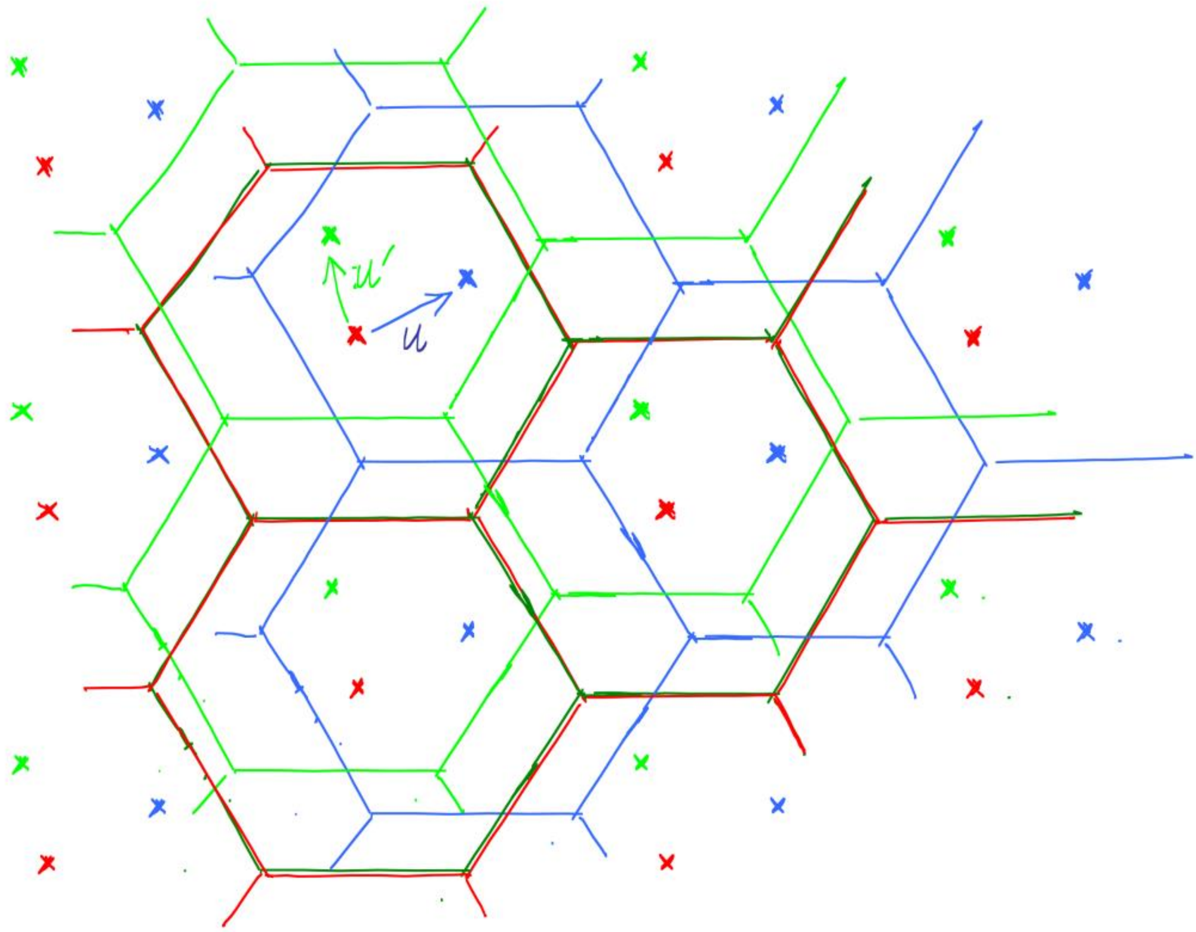
- dither for perceptual reasons:



- dither for analytical reasons:



$$Q_\Lambda(x+u)-u$$



$\Rightarrow$ Random shift of the lattice quantizer

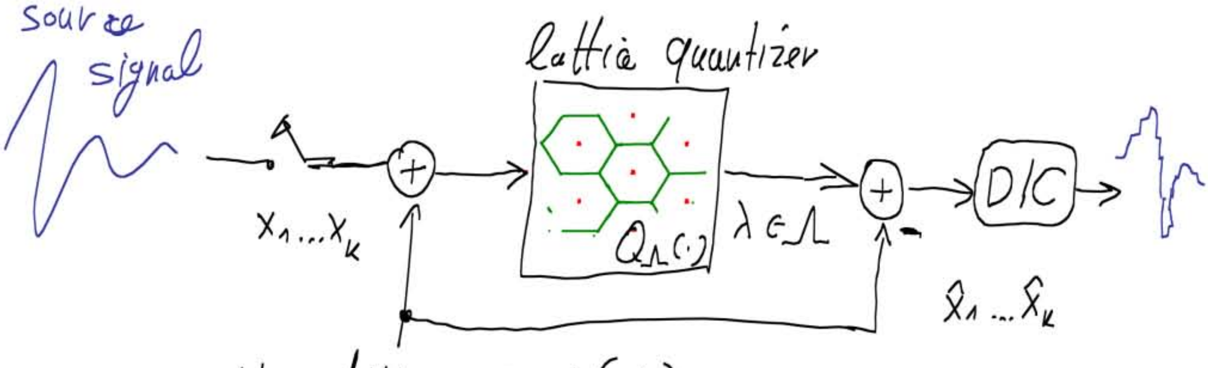
The Crypto-Lemma

Let $\boxed{x \bmod \Lambda \triangleq x - Q_\Lambda(x)}$

$$\boxed{\begin{array}{l}\text{If } \quad U \sim \text{unif}(\mathcal{P}_0), \quad \text{then} \\ (x+U) \bmod \Lambda \sim \text{unif}(\mathcal{P}_0), \quad \forall x\end{array}}$$

Proof: View as a modulo-additive noise channel, with a uniform noise.

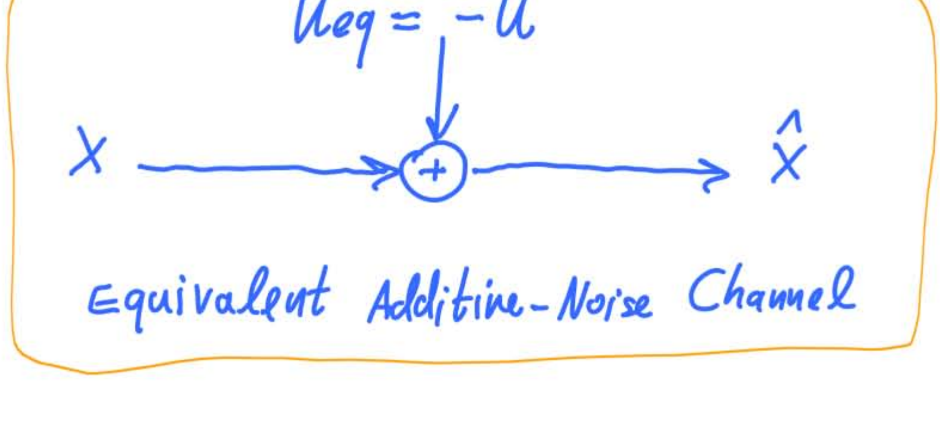
Dithered Quantization Error



u = dither ~ Unif$(\mathcal{V}_0)$

Crypto Lemma ⇒

Thm. 1: quantization error $Q(x+u) - x - u$
is independent of input x, and
uniform over (reflection of) lattice cell:



Generalized Dither

Def. $\mathcal{U}$ is G.D. if $\boxed{(s+\mathcal{U}) \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)}$ $\forall s$

Necessary condition on $f_{\mathcal{U}}(\cdot)$ for G.D. ?

Generalized Dither

Def. $\mathcal{U}$ is G.D. if $\boxed{(s+\mathcal{U}) \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)}$ $\forall s$

Necessary condition for G.D. ?

1. $\mathcal{U}$ is G.D. iff $\mathcal{U} \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)$

2. $\mathcal{U}$ is G.D. iff $f_{\mathcal{U}_{rep}}(x) = \text{constant}$

   where,

   $f_{rep}(x) \triangleq$ periodic replication $f(x) \triangleq \sum_{\lambda \in \Lambda} f(x-\lambda)$

3. $\mathcal{U}$ is G.D. iff its characteristic function is zero on the dual lattice:

$$\mathcal{F}\{f_{\mathcal{U}}(\cdot)\} = 0 \quad \text{on } \Lambda^* \setminus \underline{0}$$

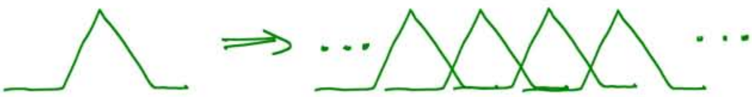
where Λ^* = dual lattice = $\Lambda(G^{-t})$

Generalized Dither

Def. $\mathcal{U}$ is G.D. if $\boxed{(s+\mathcal{U}) \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)}$ $\forall s$

Necessary condition for G.D.?

$f_{rep}(x) \triangleq$ periodic replication $f(x) \triangleq \sum_{\lambda \in \Lambda} f(x-\lambda)$



Claims

1. $f_{rep}(x)$ is periodic-Λ in space
2. If $X \sim f(x)$, and $\mathcal{P}_0$ = fundmental cell of Λ, then

$$f_{X \bmod \Lambda}(x) = \begin{cases} f_{rep}(x), & x \in \mathcal{P}_0 \\ 0, & \text{o.w.} \end{cases}$$

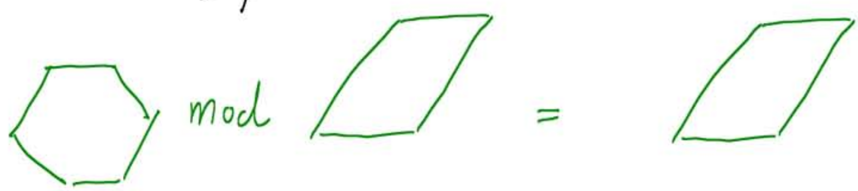
3. $X \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)$ iff $f_{rep}(x) = \text{constant}$
4. $\mathcal{U}$ is generalized dither iff $f_{\mathcal{U}rep}(x) = \text{constant}$

Generalized Dither: Examples

1. Uniform over _any_ fundmental cell

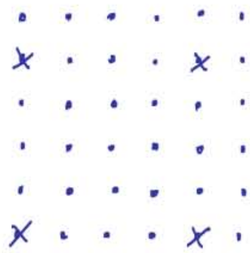
$$\mathrm{Unif}(Q_0) \bmod_{P_0} \Lambda \sim \mathrm{Unif}(P_0)$$

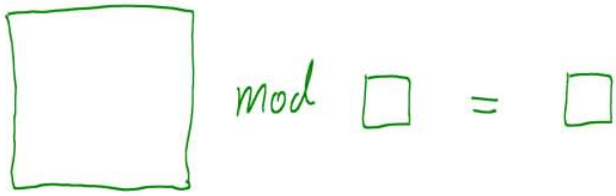
where Q_0, P_0 = fundmental cells of Λ.



2. Uniform over a nested coarse lattice cell

Q_0 = fundmental cell of $\Lambda_C \subset \Lambda$





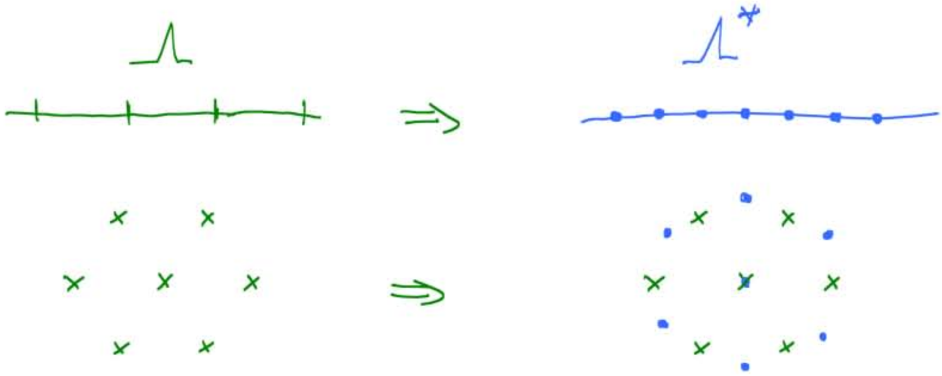
3. Spreading

$$\left\{ f_u(\cdot) \right\}_{rep} = \text{constant} \Rightarrow \left\{ f_u(\cdot) * \tilde{f}(\cdot) \right\}_{rep} = \text{constant}$$

Generalized Dither $\Rightarrow$ Zeroes on Dual Lattice

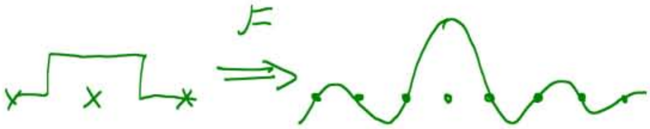
Def. Λ^* = dual lattice of $\Lambda(G)$

$$= \Lambda(G^{-t})$$



Claim: U is G.D. iff its characteristic function is zero on the dual lattice:

$$\mathcal{F}\{f_U(\cdot)\} = 0 \quad \text{on } \Lambda^* \setminus \underline{0}$$



Good lattice ⇒ white dither

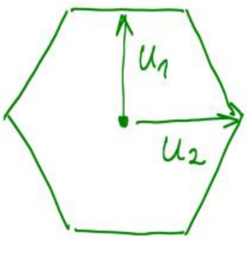
$\underline{\underline{R}}_Q \triangleq$ dither auto-correlation matrix $= E\{\underline{u}\cdot\underline{u}^t\}$

$M_u \triangleq \frac{1}{n}\,\mathrm{trace}\{\underline{\underline{R}}_Q\} \geq \sigma^2(\Lambda)$

equality if Voronoi cells

Thm.: If Λ is an optimal lattice quantizer in $\mathbb{R}^n$ (minimizes N.S.M. $G(\Lambda)$), then $\underline{u}$ is white:

$$\underline{\underline{R}}_Q = \sigma^2(\Lambda)\cdot \underline{\underline{I}}_n$$



u_1 and u_2 are dependent

but $Var(u_1) = Var(u_2)$

$E\{u_1\cdot u_2\} = 0$

Proof:

1. $\Lambda, V_0 \rightarrow$ whitenning (orthonormal) transformation $\rightarrow \Lambda', P_0'$

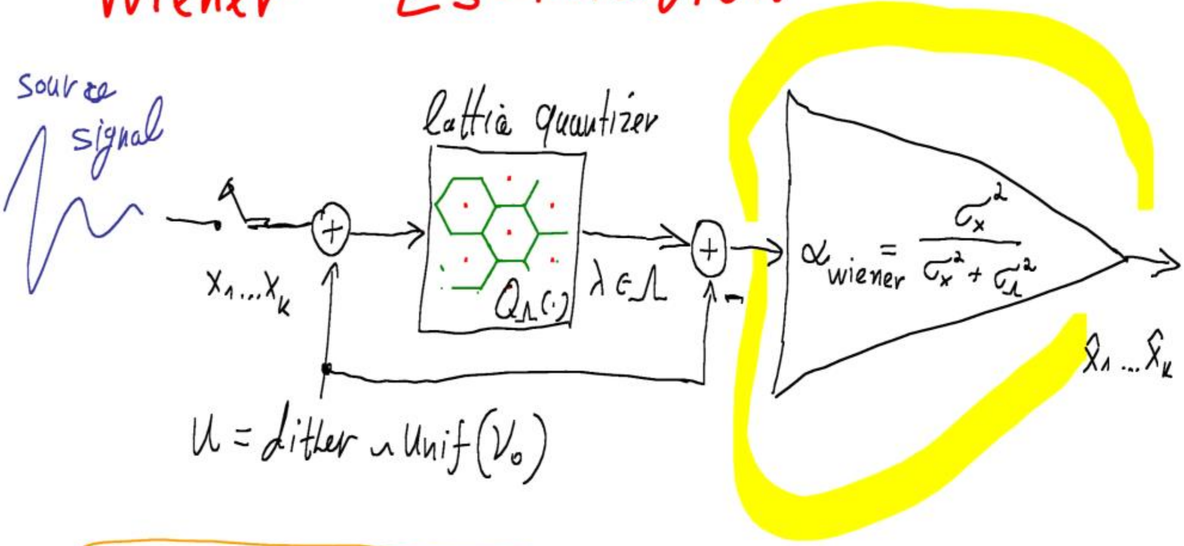
2. $\Lambda', P_0' \rightarrow$ Voronoi Partition $\rightarrow \Lambda', V_0'$

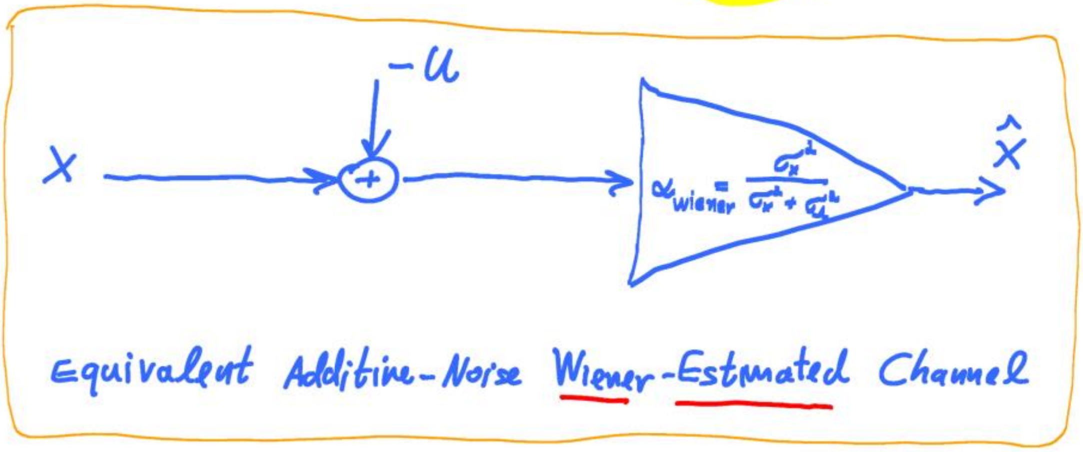
and repeat ...

$\Rightarrow \quad G(\Lambda) \geq G(\Lambda') \geq G(\Lambda'') \geq \dots$

w. equality iff Λ is white ?

Wiener Estimation





$\Rightarrow$ distortion : $\sigma_\Lambda^2 \longrightarrow \dfrac{\sigma_x^2 \, \sigma_\Lambda^2}{\sigma_x^2 + \sigma_\Lambda^2}$

Over-Sampling for A/D and MD

Saturday, May 06, 2017 3:25 PM

Over Sampling for

Analog-to-Digital conversion

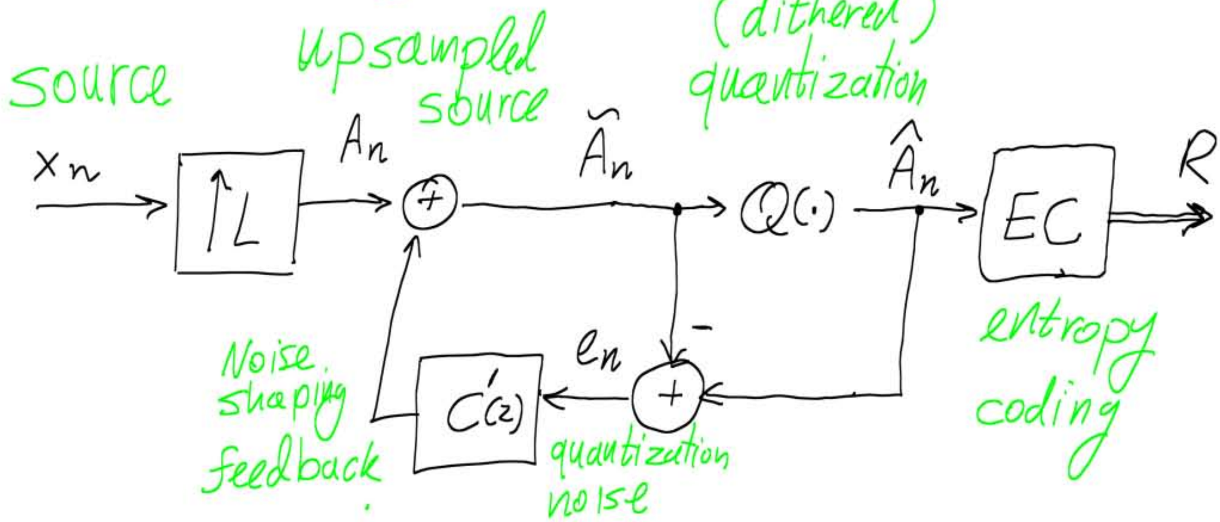
and

Multiple Descriptions

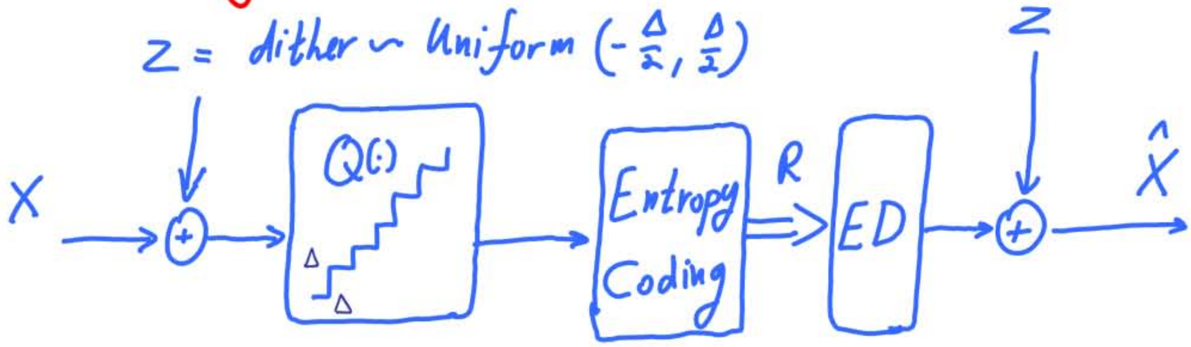
Ram Zamir

Madrid School of Info-Theory

Delta-Sigma Quantization



Entropy-Coded Dithered Quantization



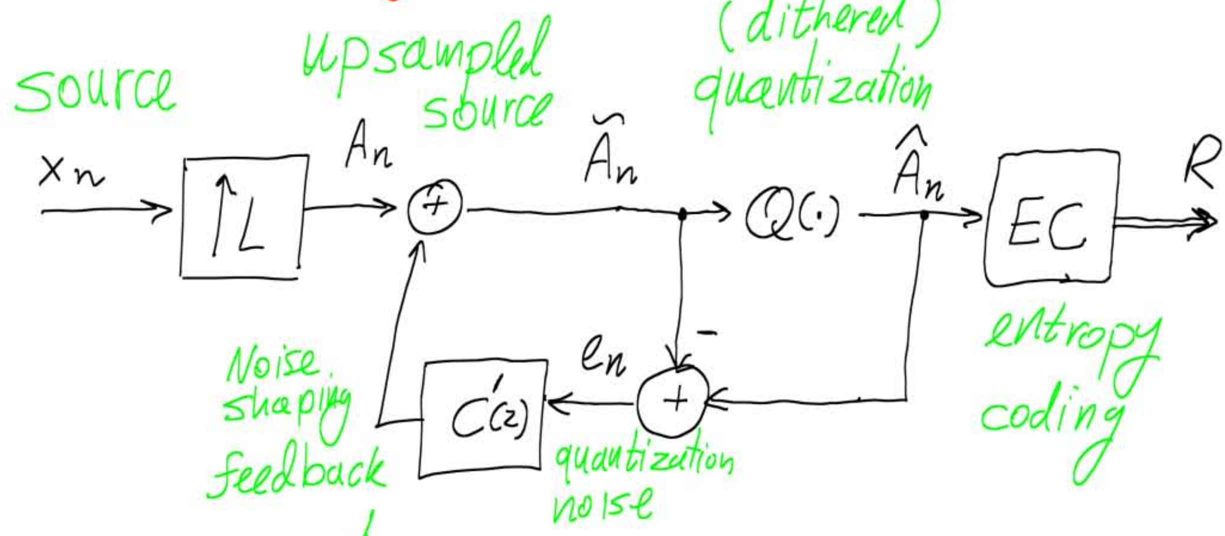
$$\hat{X} = Q(X+Z) - Z$$

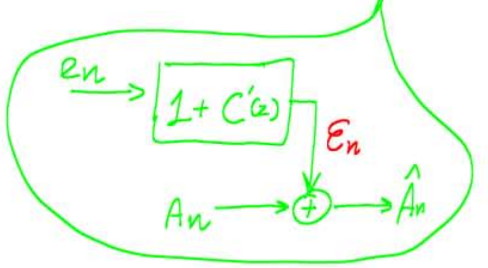
$$R = H\left(Q(X+Z) \mid Z\right)$$

Observations

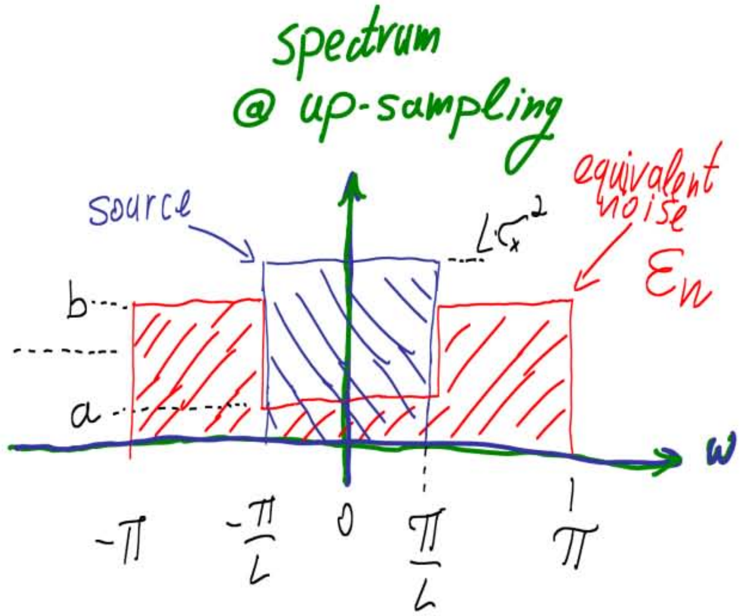
- $E \triangleq \hat{S} - S \sim$ Uniforom $(-\frac{\Delta}{2}, \frac{\Delta}{2}) \perp\!\!\!\perp S$
- $R = I(S;\ S + E)$
- Can be exteded to vector lattice quantizer
- Dither $\longrightarrow$ Gaussian as dimension $\to \infty$ ("good")
- Joint entropy coding, feedback...

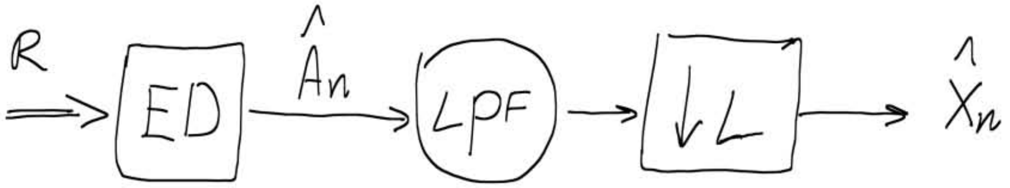
Delta-Sigma Quantization





$$\sigma_e^2 = \sqrt[L]{a \cdot b^{L-1}}$$





Single-Description DSQ analysis

*Distortion: (Wiener$\{\hat{A}_n\}$) $D = \frac{\sigma_x^2 \cdot a/L}{\sigma_x^2 + a/L}$

⟹ independent of HPF b and oversampling L !
(for a fixed a/L)

* Rate: I. with joint entropy coding

$$R_{joint} = L \cdot \bar{I}(A_n ; A_n + \varepsilon_n)$$

$$= L \cdot \bar{I}(A_n ; A_n + \varepsilon_{inband} + \cancel{\varepsilon_{out\text{-}of\text{-}band}})$$

Gaussian dither

$$= \tfrac{1}{2} \cdot \log\left(\frac{\sigma_x^2 + a/L}{a/L}\right)$$

Gaussian dither

⟹ independent of b and L !

Single-Description DSQ analysis

* Rate: II. with memoryless entropy coding

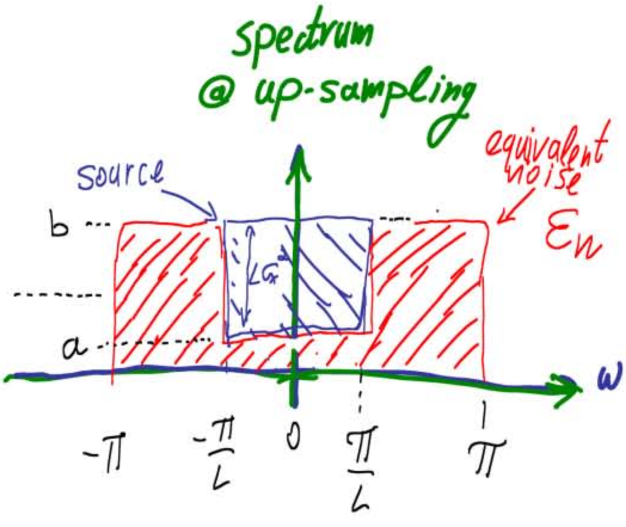
$$R_{memoryless} = L \cdot I(\tilde{A}_n ; \tilde{A}_n + e_n)$$

$$\left(\overset{\text{for Gaussian dither}}{=} L \cdot \frac{1}{2} \log \left(\frac{\sigma_x^2 + \frac{a}{L} + b\frac{L-1}{L}}{\sqrt[L]{a \cdot b^{L-1}}} \right) \right)$$

$$\geqslant L \cdot I(A_n ; A_n + \mathcal{E}_n) = R_{joint}$$

equality iff
source + noise = white

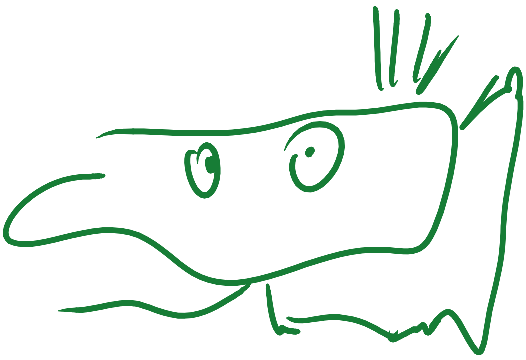
$\therefore b = a + L\sigma_x^2$



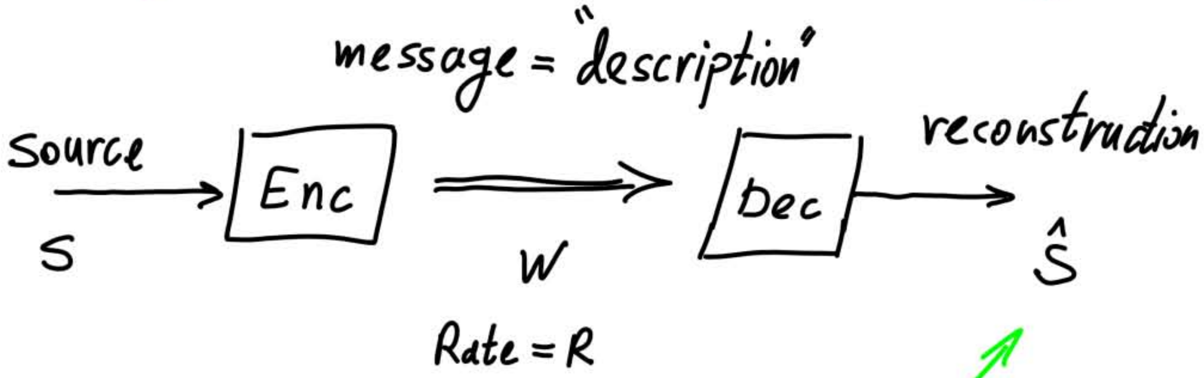
Source coding w multiple descriptions

Saturday, May 06, 2017 6:10 PM

Source Coding
with
Multiple Descriptions



Single-Description Source Coding



$$\text{Distortion} = E\{(\hat{S}-S)^2\}$$

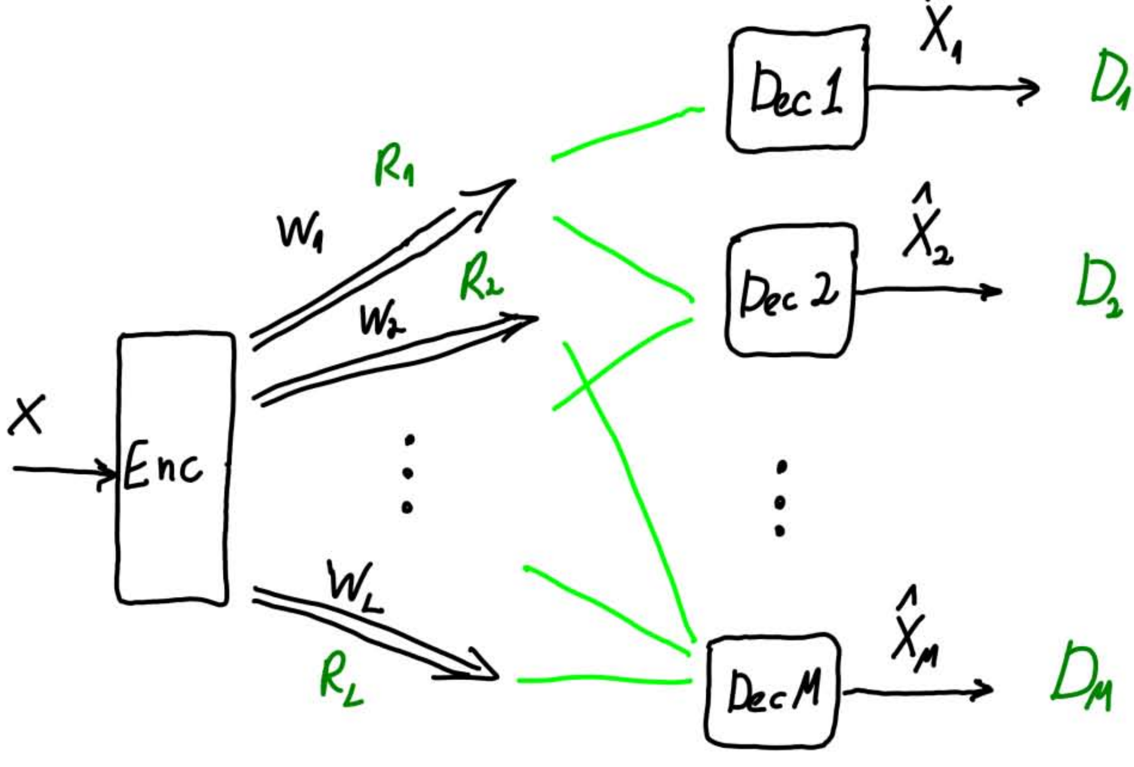
$$R \geq R^*(D) = \frac{1}{2}\log\left(\frac{\sigma_x^2}{D}\right)$$

$$\text{Source} \sim \mathcal{N}(0, \sigma_x^2)$$

Multiple-Description Source Coding

L descriptions

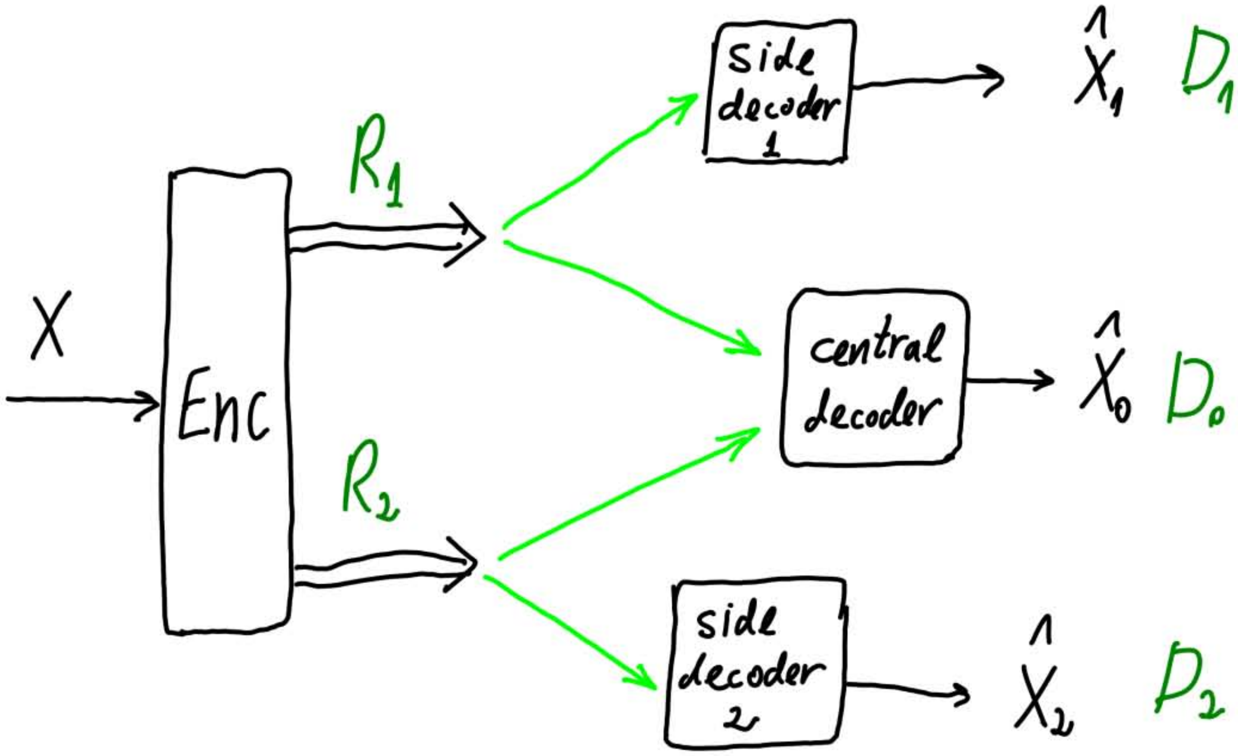
$M = 2^L - 1$ receivers = all subsets of $\{1, \dots, L\}$



parameters: $(R_1, \dots, R_L; D_1, \dots, D_M)$

2 Descriptions

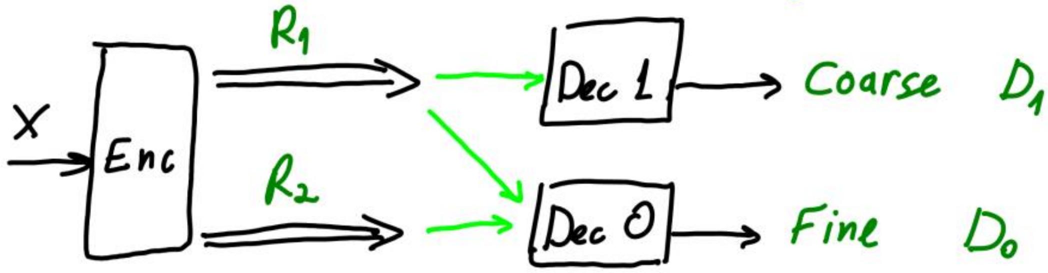
$L=2$, $M=3$ receivers



5 Parameters : $(R_1, R_2, D_1, D_2, D_0)$

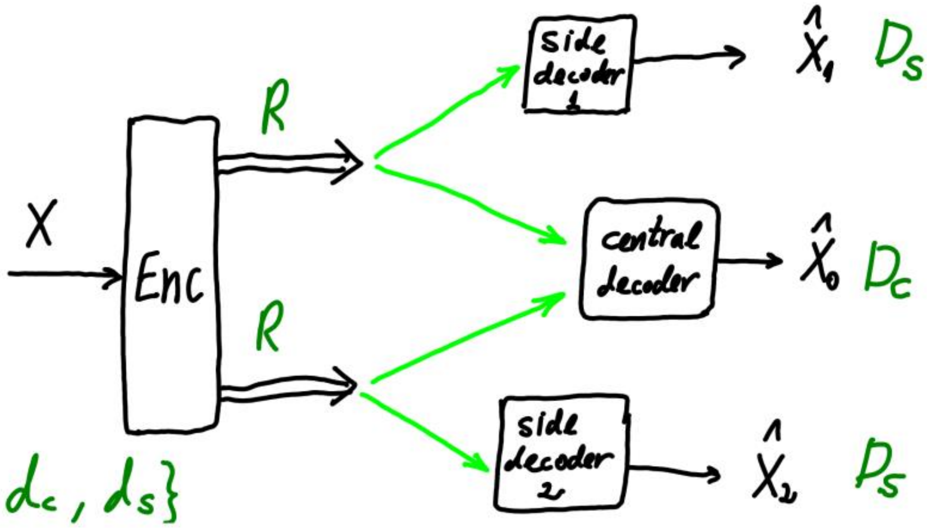
Important Special Cases

1. Successive refinement (totally asymmetric)



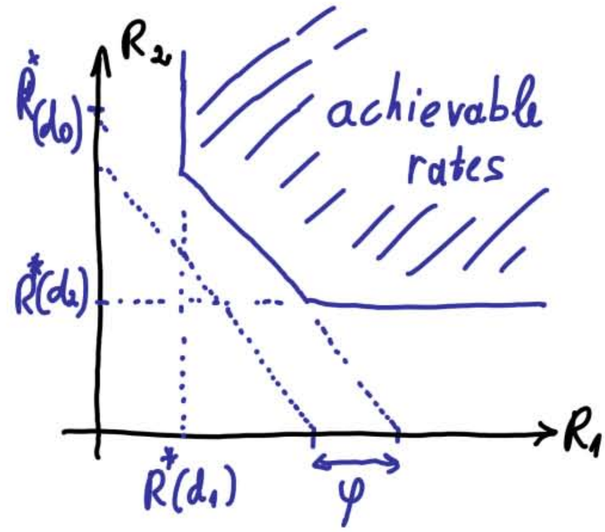
$D_2 = \sigma_x^2 =$ don't care

2. symmetric descriptions: $R_1 = R_2 \triangleq R$, $D_1 = D_2 \triangleq D_s$



$\{R, d_c, d_s\}$

Rate-Distortion Region



$$R_1 \geq R^*(d_1) \quad , \quad R_2 \geq R^*(d_2)$$

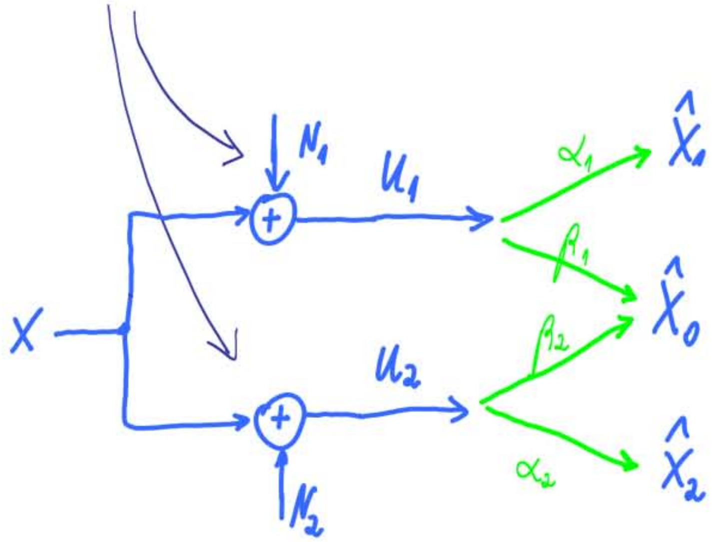
$$R_1 + R_2 \geq R^*(d_0) + \underbrace{\varphi(d_0, d_1, d_2)}_{\text{excess sum rate}}$$

$$\varphi = \begin{cases} 0 & \text{(no excess sum rate)} \quad \text{if } d_0 \leq d_{0min} \\ \frac{1}{2}\log\left(\frac{\sigma_x^2 d_0}{d_1 d_2}\right) & \text{(no excess marginal rate)} \quad \text{if } d_0 \geq d_{0max} \\ \frac{1}{2}\log\left(\frac{(\sigma_x - d_0)^2}{\left[\sqrt{(d_2-d_0)(\sigma_x^2-d_1)} + \sqrt{(d_1-d_0)(\sigma_x^2-d_0)}\right]^2}\right) & \text{o.w.} \end{cases}$$

$d_{0min} \triangleq d_1 + d_2 - \sigma_x^2$ (can be negative) $\qquad d_{0max} \triangleq \left(\frac{1}{d_1} + \frac{1}{d_2} - \frac{1}{\sigma_x^2}\right)^{-1}$

Optimal "Test Channel"

$$(N_1, N_2) \sim \mathcal{N}\left(\underline{0}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$



$I(X;U_1) = R^*(d_1)$

$I(X;U_2) = R^*(d_2)$

$I(X;U_1U_2) = R^*(d_0)$

$I(U_1;U_2) =$ excess sum-rate (central)

Noise correlation ρ is <u>negative</u> $\rho^* \leq \rho \leq 0$

- $\rho = 0$ $\quad N_1 \perp\!\!\!\perp N_2$, no excess marginal, high excess central
- $\rho < 0$ $\quad d_0$ is reduced
- $\rho = \rho^*$ (most negative) – no excess central

Practical MD coding schemes

- MD quantization with index assignment [Vaishampayan 1993], [V & Sloane 2001]
- Even / odd speach coding [Jayant 1981]
- Correlating transforms [Wang, Orchard, Reiman 1997]
- Channel coding Un Equal Error Protection [Witsenhausen 1982, Puri Ramchandran 1999]

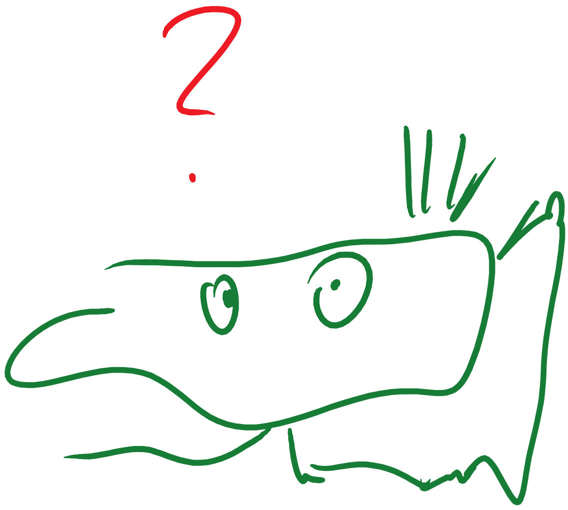
DSQ for MD?

Saturday, May 06, 2017 6:10 PM

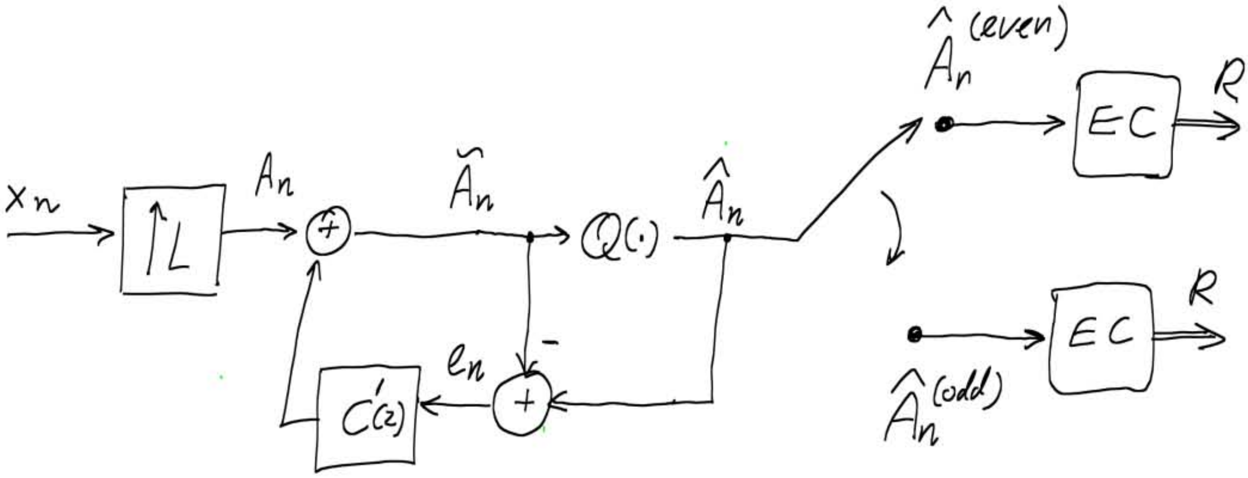
How can Δ-Σ-Q

be used for

multiple descriptions



Two-Description DSQ $(L=2)$

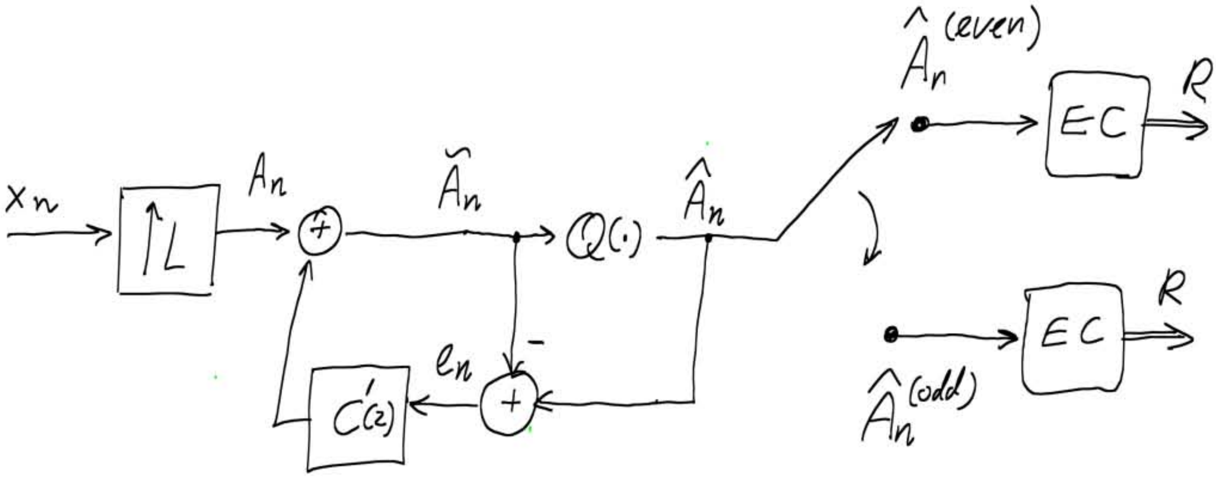


* Entropy rate

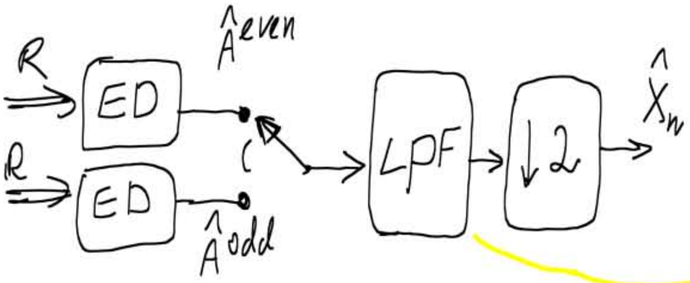
$$R = R_{memoryless} = \frac{1}{2}\log\left(\frac{Var(\hat{A}_n)}{\sigma_e^2}\right) = \frac{1}{2}\log\left(\frac{\sigma_x^2 + \frac{a+b}{2}}{\sqrt{ab}}\right)$$

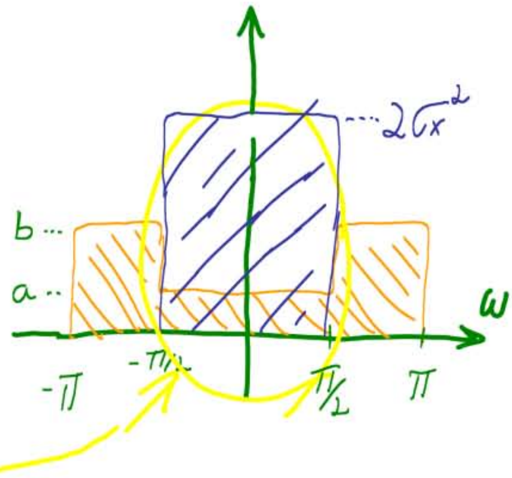
Gaussian dither

Two-Description DSQ $(L=2)$



* Central Decoder

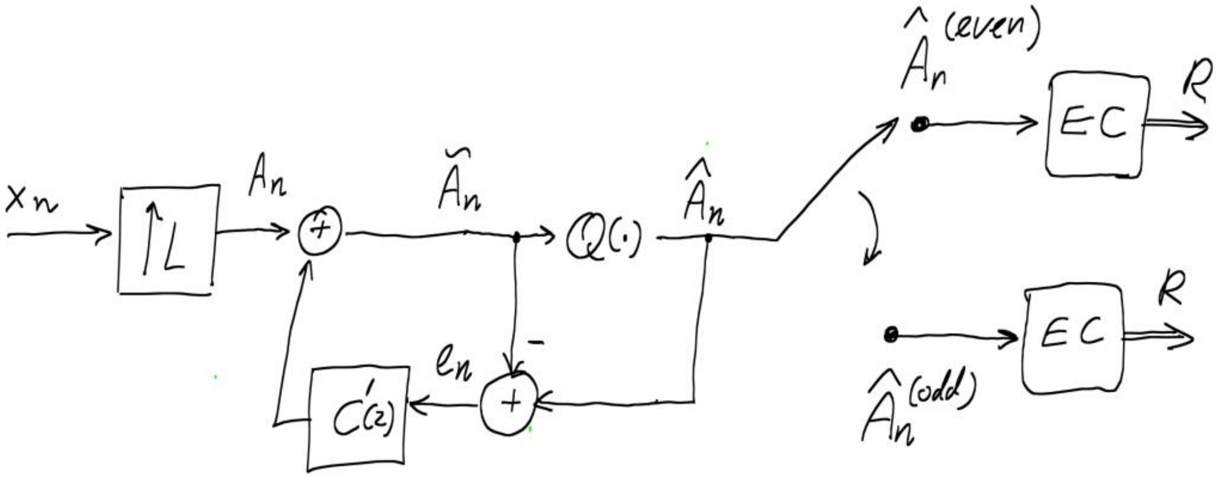




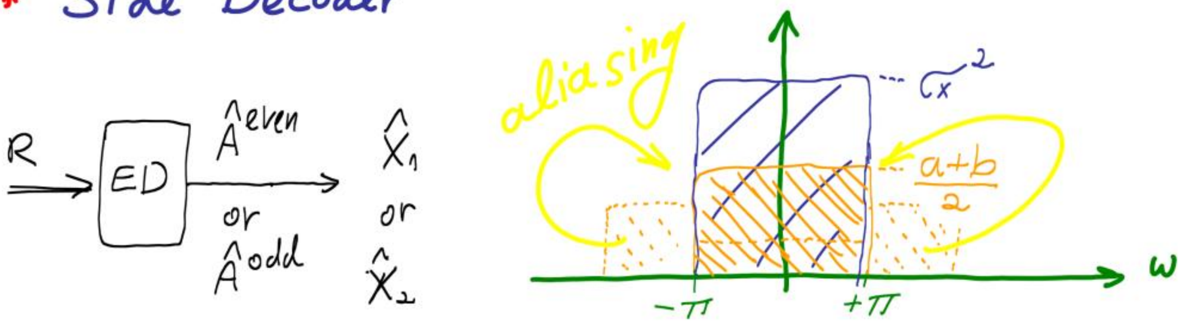
$$\therefore \; d_c = \frac{1}{2}\cdot\frac{2\sigma_x^2 \cdot a}{2\sigma_x^2 + a} \approx a/2$$

HR $\sigma_x^2 \gg a$

Two-Description DSQ $(L=2)$



* Side Decoder



$$\therefore \; d_s = \frac{\sigma_x^2 \cdot \frac{a+b}{2}}{\sigma_x^2 + \frac{a+b}{2}} \simeq \frac{a+b}{2}$$

HR , $\sigma_x^2 \gg a, b$

Optimality of Symmeric MD-DSQ @ $L=2$

[Ostergaard - Zamir 2009]

$\{R, d_c, d_s\}$ = Ozarow's symmetric RD region

@ Gaussian dither

* Simple characterization @ high resolution:

$$R = \frac{1}{2}\log\left(\frac{\sigma_x^2}{\sqrt{ab}}\right)$$

geometric average

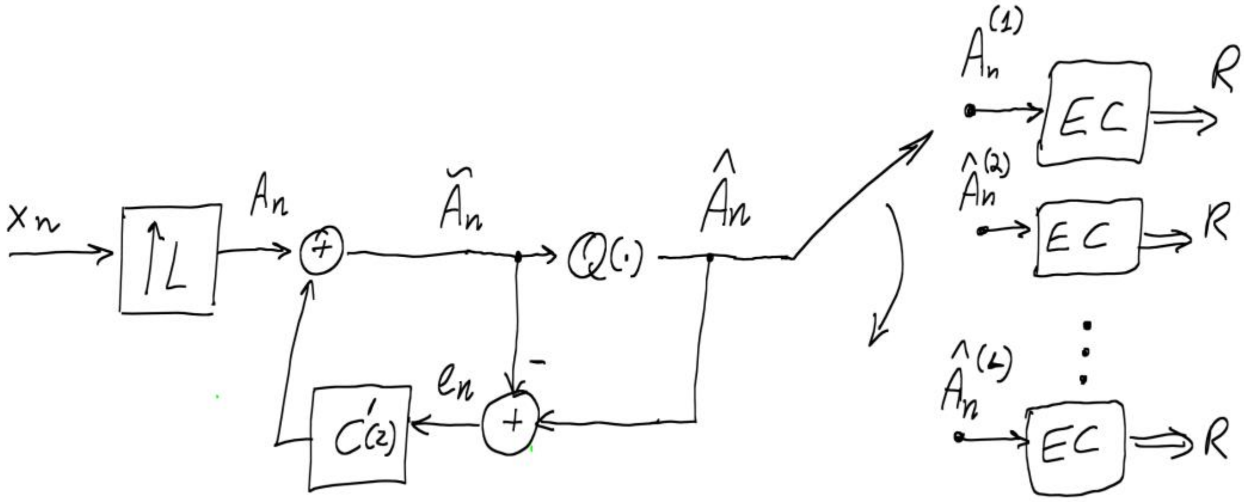
$$d_c = a/2$$

$$d_s = \frac{a+b}{2}$$

arithmetic average

$d_s/d_c = \frac{a+b}{a} \Rightarrow b/a$ controls operation point

L-Description DSQ



* Optimal for the "1-or-L problem"
[Mashiach - Ostergaard - Zamir 2010]

* With random binning, optimal for the "K-or-L problem" [M-O-Z 2013]

Creating asymmetric descriptions by grouping of many symmetric descriptions

Adam Mashiach – TAU
Yuval Kochman – HUJI
Jan Østergäard – Aalborg U., Denmark
Ram Zamir – TAU

How MD-DSQ can be extended to Asymmetric case

?

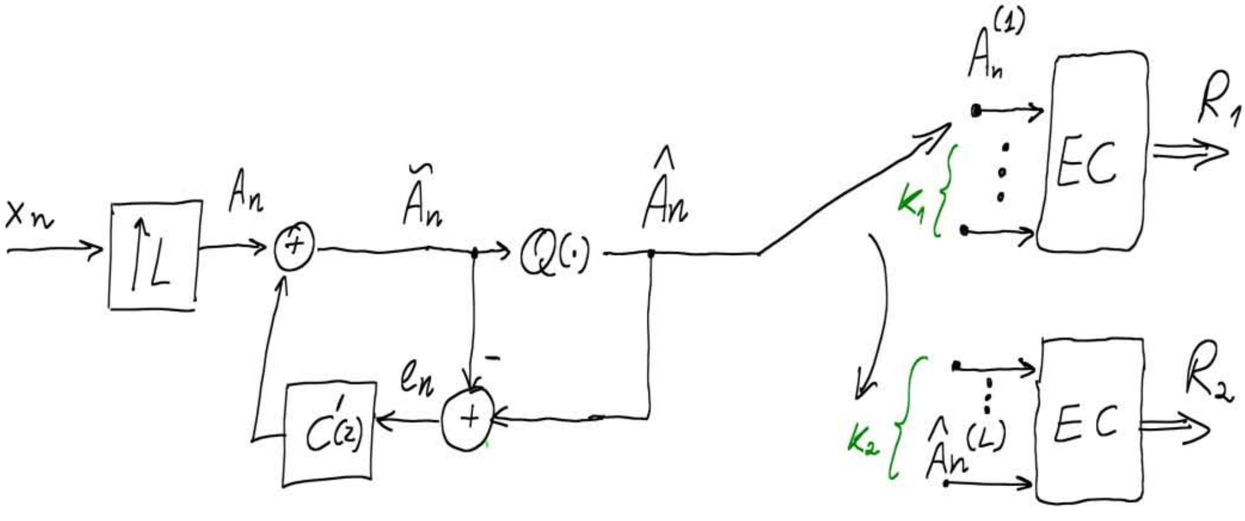
How MD-DSQ can be extended to Asymmetric case

?

Asymmetric Sample allocation ?

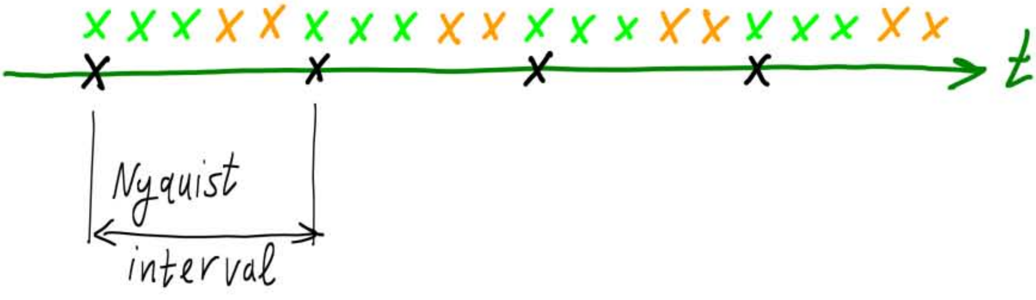
Asymmetric 2-descriptions via grouping

$$L = k_1 + k_2$$

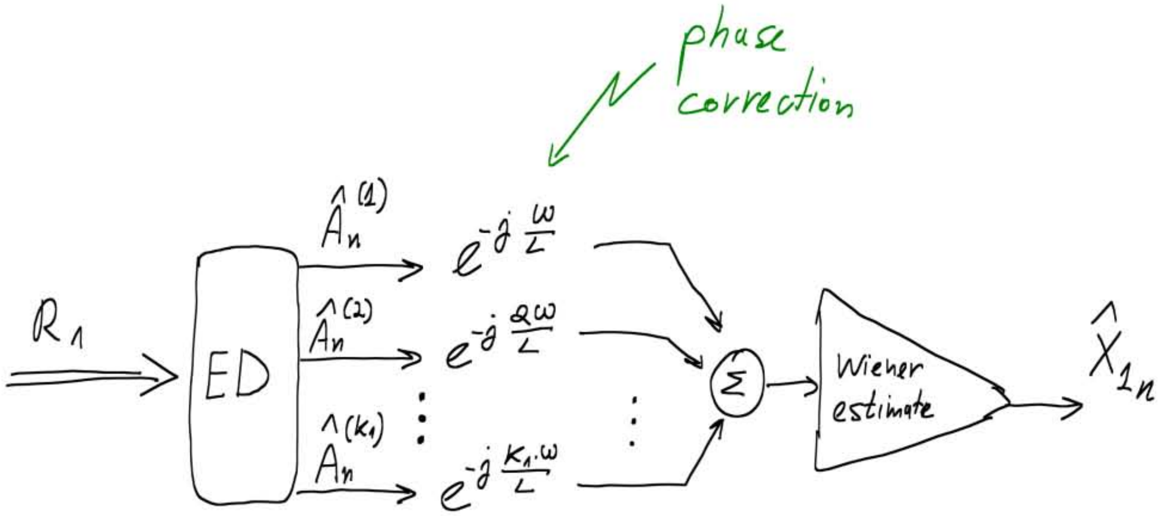


Example: $L = 5$, $k_1 = 3$, $k_2 = 2$

description 1 = x description 2 = x x = Nyquist (source)



Decoding



* Similar for 2nd and common receivers.

Asymmetric MD-DSQ: Performance

- Distortions:

$$d_0 = \frac{\sigma_x^2 \cdot a/L}{\sigma_x^2 + a/L} \quad , \quad d_i = \frac{\sigma_x^2 \cdot \left(\frac{a}{L} + \frac{L-k_i}{k_i} \cdot \frac{b}{L}\right)}{\sigma_x^2 + \frac{a}{L} + \frac{L-k_i}{k_i} \cdot \frac{b}{L}}$$

$i = 1, 2$

- Rate:

$$R_i = \frac{1}{2} \log \left(\frac{k_i \cdot \sigma_x^2 + k_i \cdot \frac{a}{L} + (L-k_i) \cdot \frac{b}{L}}{\sqrt[L]{a^{k_i} \cdot b^{L-k_i}}} \right)$$

$i = 1, 2$

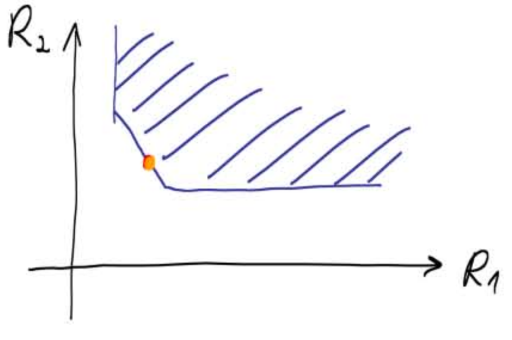
- Both rates & distortions depend only on the sample subset <u>sizes</u> k_1 & k_2, not on the specific patterns:

Main Result

Theorem: The asymmetric MD-DSQ can arbitrarily approach any distortion triplet (d_0, d_1, d_2) (by sufficiently large L, K_1, K_2).
The resulting rate-sum is optimal:

$$R_1 + R_2 = R^*(d_0) + \varphi(d_0, d_1, d_2)$$

(for Gaussian dither and optimal noise shaping)



Summary

- General idea: Asymmetric descriptions by grouping several symmetric descriptions
- Asymmetric 2-description MD-DSQ scheme is optimal.
- Successive refinement case.
- Scalar (finite dim) quantization: Rate loss $\propto L$.
- Extention to many (more than 2) descriptions

Thank you