Information Rates for Phase Noise Channels (including fiber optic channels)

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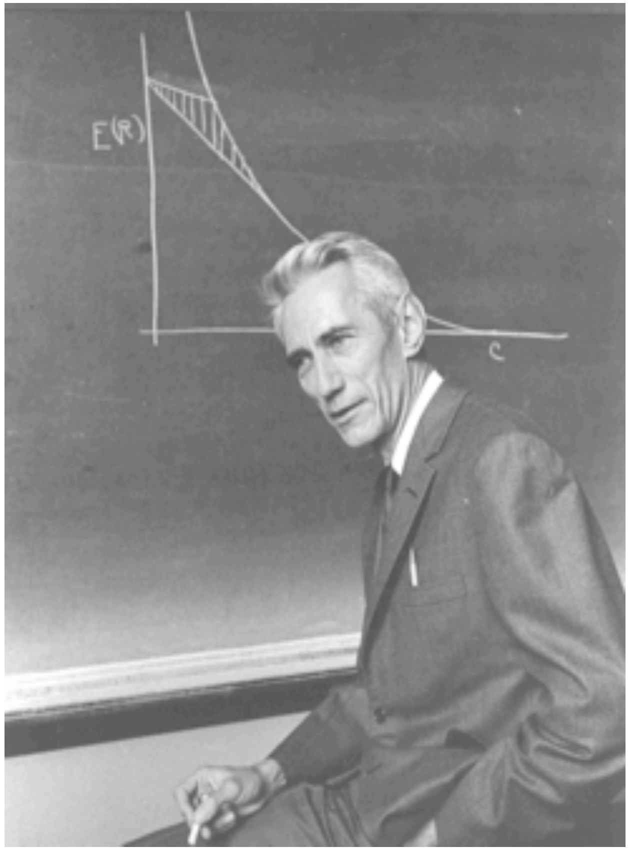
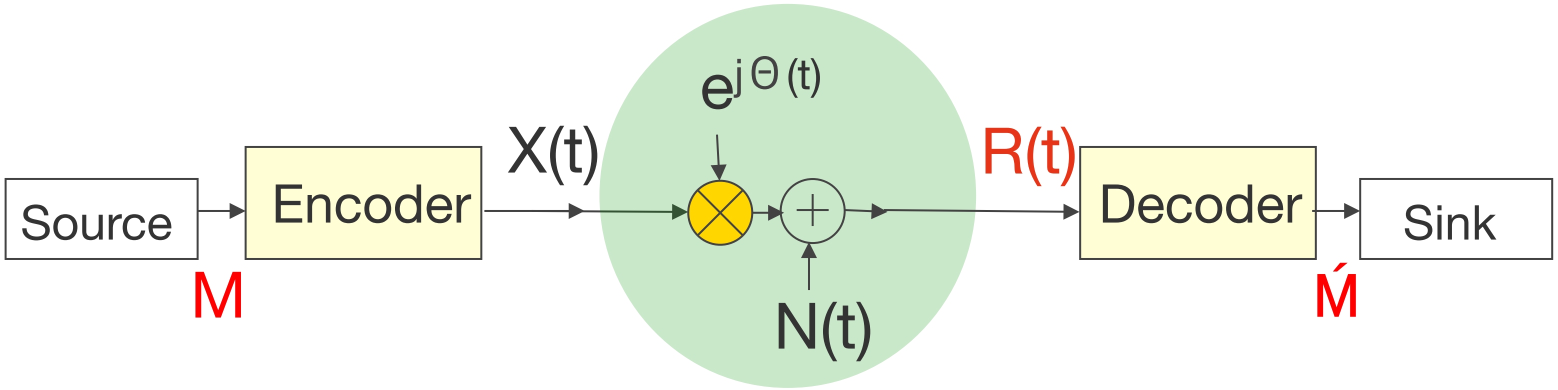


Claude Elwood Shannon

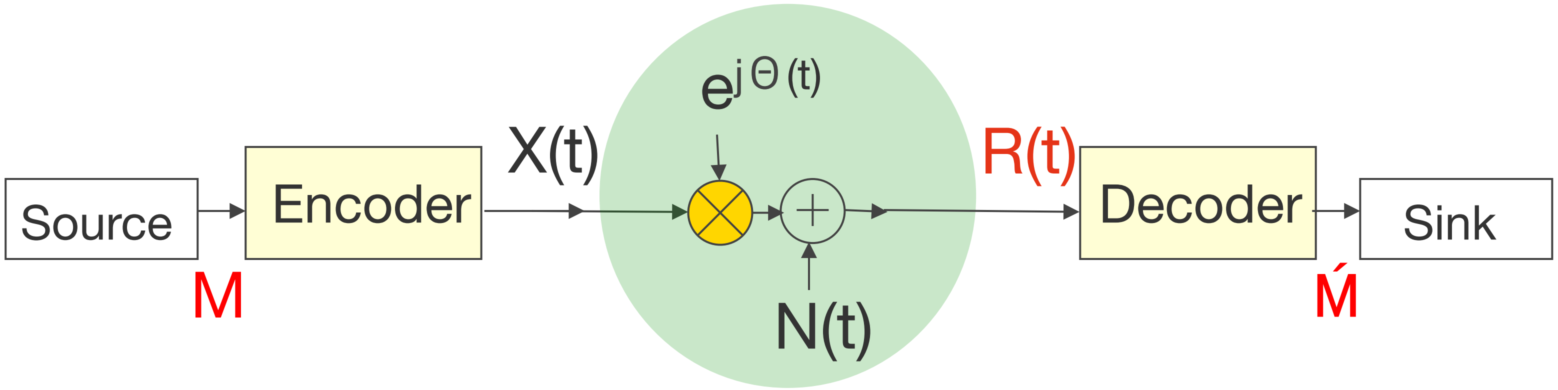
Apr 30, 1916 – Feb 24, 2001

1) Phase Noise Models



- Phase noise due to (1) oscillator instability; (2) fiber non-linearities
- Phase noise statistics:
  - phase-locked loops (PLLs) residual noise: von Mises/Tikhonov distribution
  - satellite (DVB-S2): white Gaussian process filtered by IIR filters
  - fiber-optic lasers: Wiener process
  - Raman amplification: large bandwidth Gaussian process

White Phase Noise



- Simplified model (Barletta-Kramer, 2014)

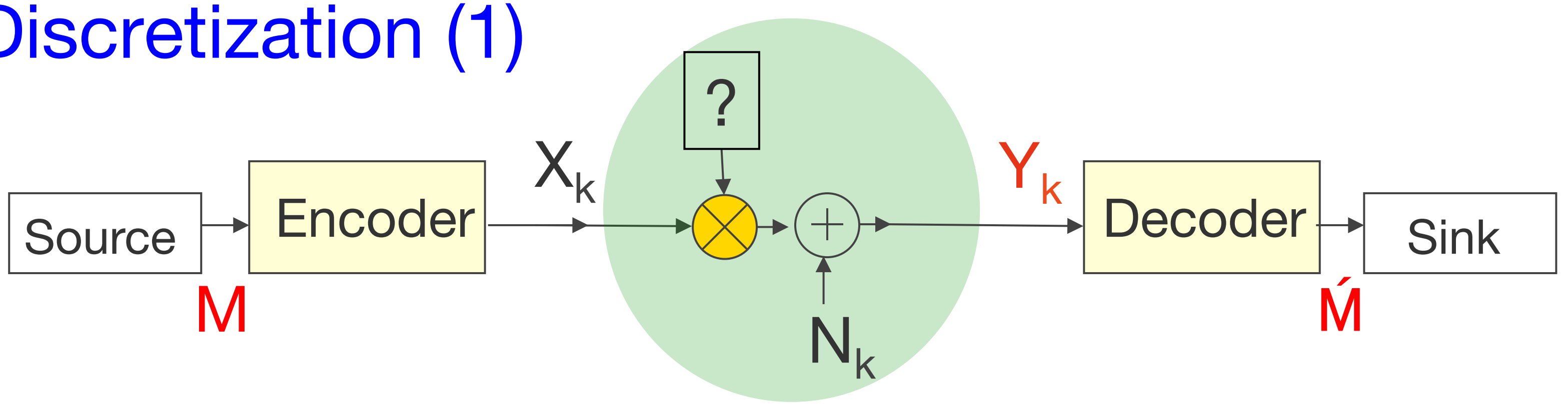
$$R(t) = X(t) \cdot e^{j\Theta(t)} + N(t)$$

  $\Theta(t)$ is white* and N(t) is white Gaussian* (both are idealizations)

- Motivation: phase noise bandwidth much larger than receiver bandwidth
- Mathematically: let $\{\phi_m(t)\}$ be an orthonormal basis of $L^2[0,T]$ and project X(t), N(t), and R(t) onto the $\phi_m(t)$

* We use $E[\Theta(t)\Theta(t+\tau)] = \sigma^2\delta_\tau$ and $E[N(t)N(t+\tau)] = \sigma^2\delta(\tau)$

Discretization (1)



- X(t) and N(t):

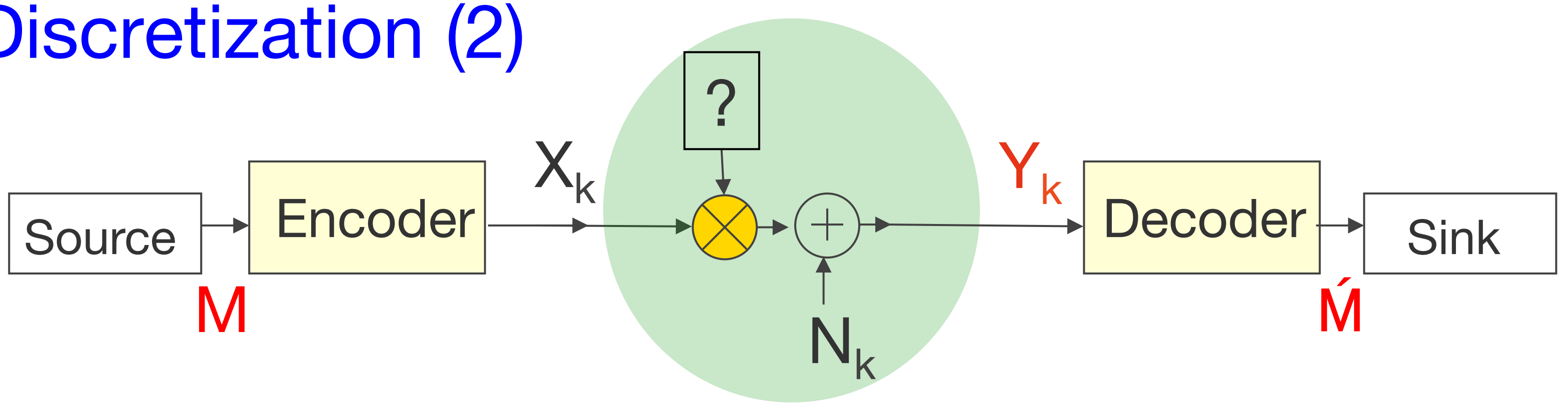
$$X(t) = \sum_{m=1}^{M} X_m \phi_m(t) \qquad N(t) = \sum_{m=1}^{\infty} N_m \phi_m(t)$$

- Receiver:

$$Y_k = \left\langle X(t) e^{j\Theta(t)} + N(t), \phi_k(t) \right\rangle$$

$$= \sum_{m=1}^{M} X_m \underbrace{\left\langle \phi_m(t) e^{j\Theta(t)}, \phi_k(t) \right\rangle}_{\Phi_{m,k}} + N_k$$

Discretization (2)



- Samples:

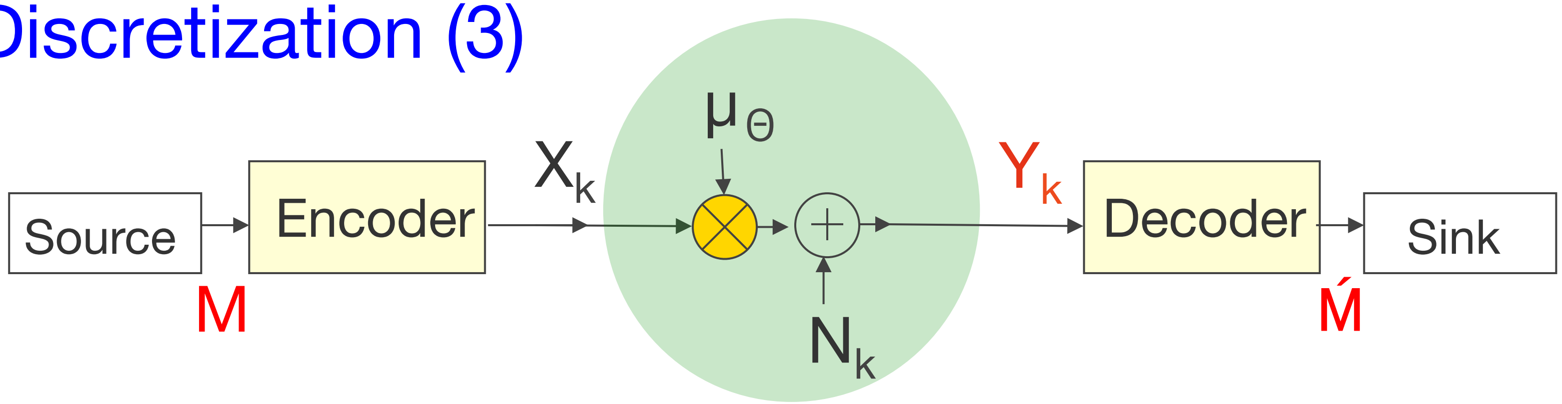
$$\Phi_{m,k} = \int_0^T \phi_m(t) e^{j\Theta(t)} \phi_k(t)^* \, dt$$

$$= \lim_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L} \phi_m\left(t_i^{(L)}\right) e^{j\Theta\left(t_i^{(L)}\right)} \phi_k\left(t_i^{(L)}\right)^*, \quad t_i^{(L)} = \frac{(i-1)T}{L}$$

$$\overset{!}{=} \begin{cases} E\left[e^{j\Theta(t)}\right], & m = k \\ 0, & \text{else} \end{cases}$$

Barletta-Kramer, 2014:
Almost sure convergence for white phase noise with uncorrelated samples of process $\{e^{j\Theta(t)}\}$

Discretization (3)



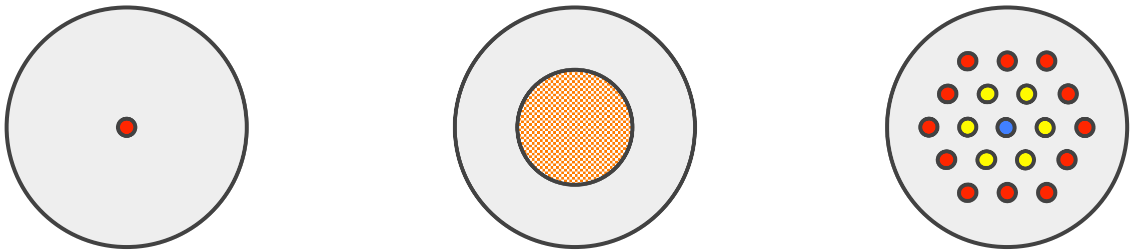
- Model*: $Y_k = X_k \cdot \underbrace{E\left[e^{j\Theta(t)}\right]}_{\mu_\Theta} + N_k$

  An AWGN channel **(!)** but with SNR penalty $|\mu_\Theta|^2$
- Penalty called spectral loss**: "lost" power is spread across all frequencies as white noise
- Proof: use Borel-Cantelli lemma with a classic trick and a simplified step via assumed boundedness of $\int |\phi_m(t)\ \phi_k(t)^*|\ dt$
- I expect this insight to be useful for fiber channels

* Barletta-Kramer 2014, ** Goebel et al. 2011

2) Fiber Channel(s)

- Single-Mode Fiber (SMF): a small core that carries one mode of light
- Here one mode has 2 complex dimensions: two polarizations
- Theory papers often consider one complex dimension; the general case is interesting too of course (see below)
- In fact, a **hot** topic in the fiber community is MIMO fiber



Single-mode fiber (SMF) | Multi-mode fiber (MMF) | Multi-core fiber (MCF)

SMF Pulse Propagation Equation

- Maxwell's equations and low-order approximations* result in a generalized nonlinear Schrödinger equation (GNSE):

Linear | Nonlinear

$$\frac{\partial E}{\partial z} = -\frac{\alpha}{2}E - \frac{i}{2}\beta_2\frac{\partial^2 E}{\partial T^2} + \frac{1}{6}\beta_3\frac{\partial^3 E}{\partial T^3} + i\gamma\,|E|^2\,E + n$$

Distance Evolution

Fiber Loss/Gain

Dispersion

Dispersion Slope

Kerr Nonlinearity

Noise (Gaussian, Bandlimited)

E : Electromagnetic field, function of *z* and *T*
z: Distance
T : Retarded time t-$\beta_1 z$
α : Fiber loss coefficient (~ 3 dB/15 km)
β_1 : Inverse of group velocity
β_2 : Fiber group velocity dispersion
β_3 : Fiber dispersion slope (include if β_2 small)
γ : Fiber nonlinear parameter $(n_2\ \omega)/(c\ A_{eff})$

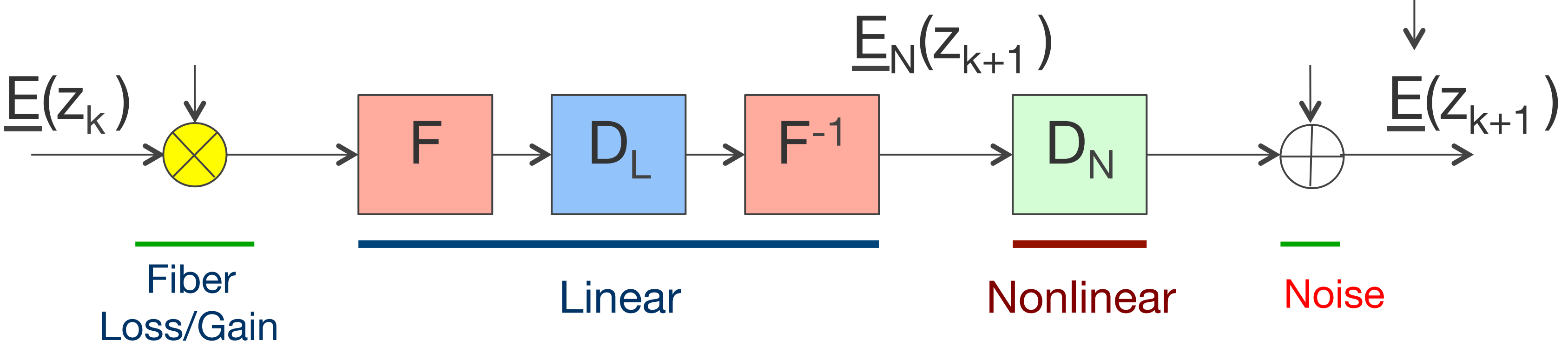
n_2 : Fiber nonlinear coefficient
ω : Angular frequency
c : Speed of light
A_{eff} : Fiber effective area

Figure courtesy of R.-J. Essiambre

*See Ch. 2 in G.P. Agrawal, "Nonlinear Fiber Optics", 3rd ed., 2001

- To simulate, split the fiber length z* into K small steps (Δz) and the time T into **L** small steps (Δt)
- Split-step Fourier method at distance z_k, k=0,1,...,K

time signal:
vector with **L** entries



- Ideal Raman amplification: removes the loss and adds noise
- F = Fourier transform
- D_L = diagonal matrix with fixed entries of unit amplitude (all-pass filter)
- D_N = diagonal matrix with unit amplitude entries; the (ℓ,ℓ)-entry phase shift is proportional to the magnitude-squared of the ℓ^{th} entry of $\underline{E}_N(z_{k+1})$

3) Upper Bound

But First More IT Preliminaries

- Consider a complex column vector $\underline{X} = \underline{X}_c + j\, \underline{X}_s$ with covariance and pseudo-covariance matrices

$$\mathbf{Q}_{\underline{X}} = E\left[\left(\underline{X} - E[\underline{X}]\right)\left(\underline{X} - E[\underline{X}]\right)^{\dagger}\right]$$

$$\tilde{\mathbf{Q}}_{\underline{X}} = E\left[\left(\underline{X} - E[\underline{X}]\right)\left(\underline{X} - E[\underline{X}]\right)^{T}\right]$$

- For interest: $\underline{X}$ is called proper if its pseudo-covariance matrix is **0**

- Example: Consider a complex, zero-mean, scalar $X = X_c + j\, X_s$.

  X is proper if $E[X_c^2]=E[X_s^2]$ and $E[X_cX_s]=0$.

  Note: circularly symmetric X are proper, but proper X are not

  necessarily circularly symmetric (e.g. QAM signal sets)

Maximum Entropy

- Maximum Entropy: consider the correlation matrix $\mathbf{R}_{\underline{X}}=E[\underline{X}\,\underline{X}^\dagger]$ where $\underline{X}$ has L entries. Then

$$h(\underline{X}) \le \log\left[(\pi e)^L \det \mathbf{R}_{\underline{X}}\right]$$

with equality if and only if $\underline{X}$ is Gaussian and proper (or circularly symmetric)

- For a complex square matrix **M** we have

$$h(\mathbf{M}\,\underline{X}) = h(\underline{X}) + 2\log\left|\det(\mathbf{M})\right|$$

In particular, if **M** is unitary then $h(\mathbf{M}\,\underline{X}) = h(\underline{X})$

Entropy Power Inequality

- Entropy Power:

$$V(\underline{X}) = e^{h(\underline{X})/L} / (\pi e)$$

- Entropy Power Inequality: for independent $\underline{X}$ and $\underline{Y}$ we have

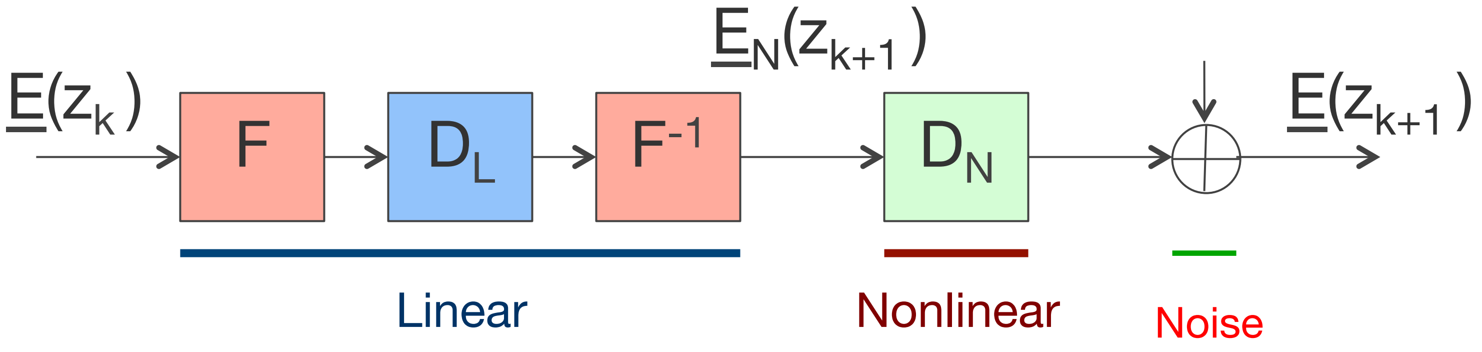
$$\boxed{V(\underline{X}+\underline{Y}) \geq V(\underline{X}) + V(\underline{Y})}$$

- Conditional version: for conditionally independent $\underline{X}$ and $\underline{Y}$ we have

$$V(\underline{X}|\underline{U}) = e^{h(\underline{X}|\underline{U})/L} / (\pi e)$$

$$V(\underline{X}+\underline{Y}|\underline{U}) \geq V(\underline{X}|\underline{U}) + V(\underline{Y}|\underline{U})$$

Energy and Entropy Conservation



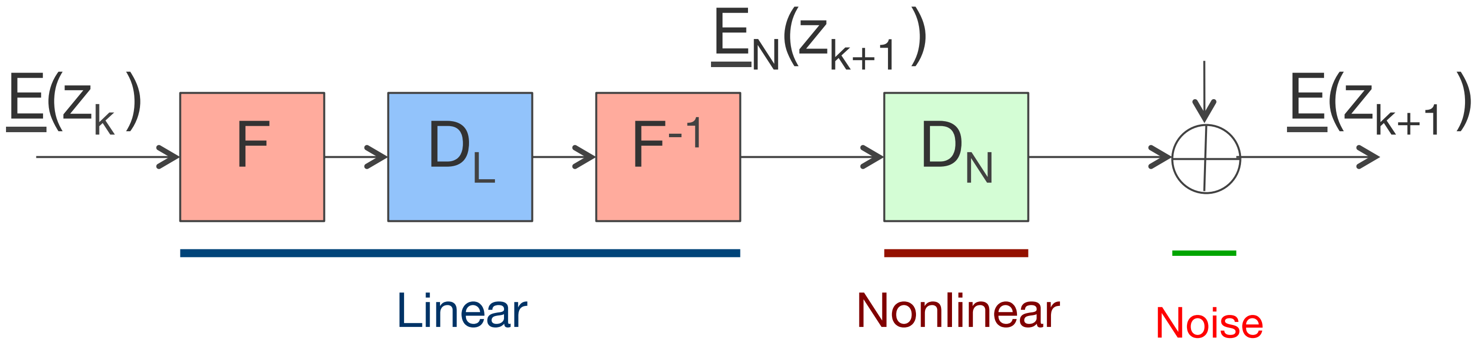
Main Observations

- The linear step conserves energy and entropy
- The non-linear step also conserves energy and entropy **(the key result)**

$$h\left(|a| e^{j\arg(a) + jf(|a|)}\right) = h\left(|a|, \arg(a) + f(|a|)\right) + \mathrm{E}\left[\log|a|\right]$$

$$= \underbrace{h\left(|a|\right) + h\left(\arg(a) + f(|a|) \,\middle|\, |a|\right)}_{h\left(|a|, \arg(a)\right)} + \mathrm{E}\left[\log|a|\right] = h(a)$$

Energy Recursion



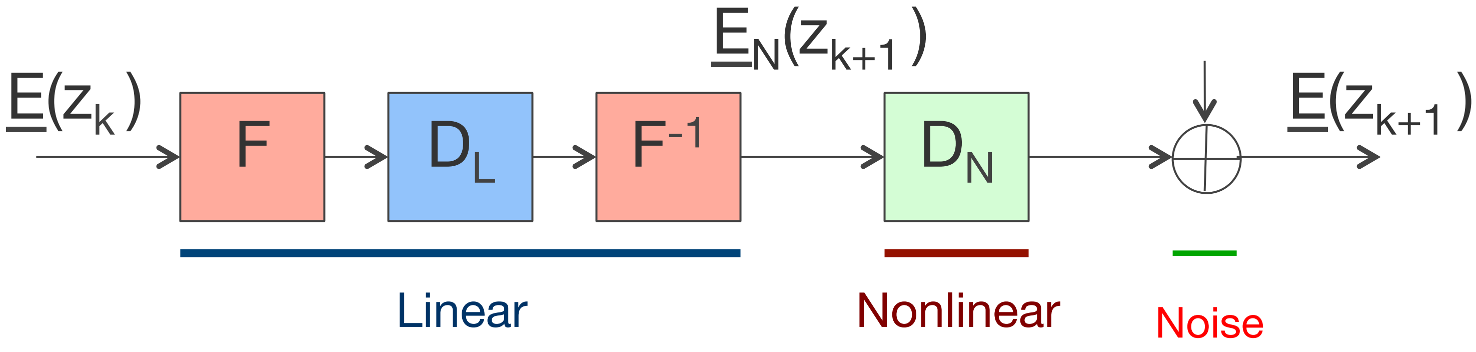
- Energy after K steps: $\text{Energy}_{\text{Launch}} + KN$. We thus have:

$$h\left(\underline{E}(z_K)\right) \le \log\left[(\pi e)^L \det\left(\mathbf{R}\left(\underline{E}(z_K)\right)\right)\right] \quad \ldots \text{ maximum entropy}$$

$$\le \sum_{i=1}^{L} \log\left[\pi e\, R_{i,i}\left(\underline{E}(z_K)\right)\right] \quad \ldots \text{ Hadamard's inequality}$$

$$\le L \cdot \log\left[\pi e\left(Energy_{\text{Launch}} + KN\right)/L\right] \quad \ldots \text{ Jensen's inequality}$$

Entropy Recursion



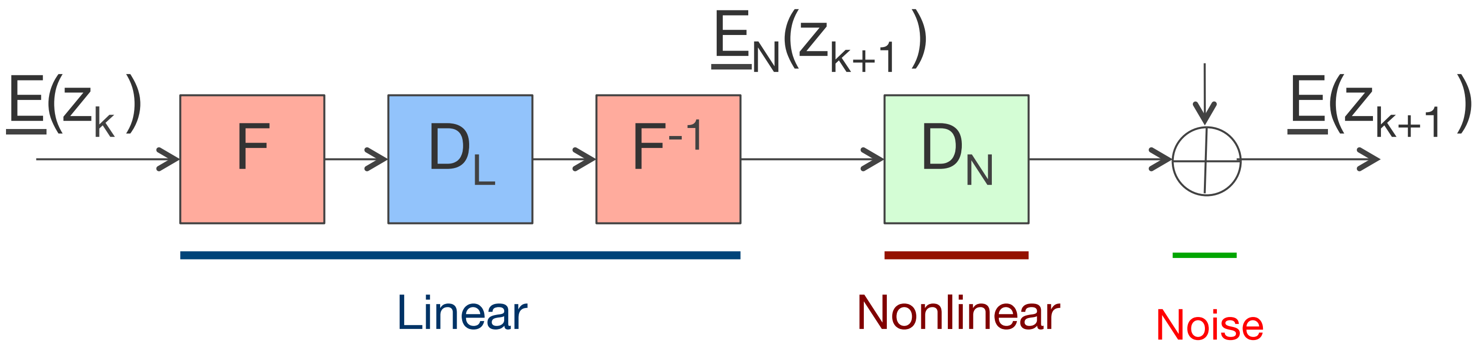
- Entropy recursion:

$$V\left(\underline{E}(z_{k+1})\middle|\underline{E}(z_0)\right) \geq V\left(\underline{E}(z_k)\middle|\underline{E}(z_0)\right) + N/L$$

- We thus have:

$$V\left(\underline{E}(z_K)\middle|\underline{E}(z_0)\right) \geq KN/L$$

or $h\left(\underline{E}(z_K)\middle|\underline{E}(z_0)\right) \geq L\log(\pi e\, KN/L)$



So for every step we have:

- Signal energy grows by the noise variance: can upper bound h($\underline{E}(z_K)$)
- Entropy power grows by at least the noise variance: can lower bound h($\underline{E}(z_K)$ | $\underline{E}(z_0)$)
- Result*:

$$I\left(\underline{E}(z_0);\underline{E}(z_K)\right) = h\left(\underline{E}(z_K)\right) - h\left(\underline{E}(z_K)\middle|\underline{E}(z_0)\right)$$

$$\leq L \cdot \log\left(1 + SNR\right)$$

*SNR = receiver signal-to-noise ratio

$$\Rightarrow \quad \frac{1}{L} I\left(\underline{E}(z_0);\underline{E}(z_K)\right) \le \log\left(1+SNR\right)$$

- Let B = 1/Δt be the “bandwidth” of the simulation
- So L = T/Δt = TB is the time-bandwidth product
- The spectral efficiency is thus bounded by

$$\eta \le \log\left(1+SNR\right) \quad \text{[bits/sec/Hz]}$$

5) Conclusions

1) Spectral efficiency of (an idealized model of) SMF with linear polarization is ≤ log(1+SNR)
2) Many extensions are possible:
   - lumped amplification, 3rd-order dispersion, delayed Kerr effect
   - uniform loss, linear filters (for capacity results)
   - MIMO fiber (MMF or MCF)
3) More difficult:
   - better bounds and understanding at high SNR
   - frequency-dependent loss, dispersion, non-linearity