DEPENDENCE BALANCE BOUNDS

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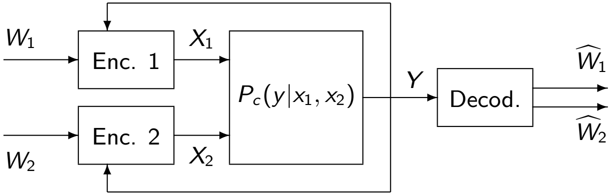
Multiple-Access Channel with Feedback



Message index W_1 is uniform over $\{1, 2, \cdots, M_1\}$ and index W_2 is uniform over $\{1, 2, \cdots, M_2\}$.
Encoders:

$$\begin{aligned} X_{1n} &= E_{1n}(W_1, Y_1, Y_2, \cdots, Y_{n-1}), \\ X_{2n} &= E_{2n}(W_2, Y_1, Y_2, \cdots, Y_{n-1}), n = 1, 2, \cdots, N. \end{aligned}$$

Channel $\{\mathcal{X}_1 \times \mathcal{X}_2, P_c(y|x_1, x_2), \mathcal{Y}\}$ is discrete and memoryless.
Decoder:

$$(\widehat{W}_1, \widehat{W}_2) = D(Y_1, Y_2, \cdots, Y_N).$$

Achievability, Capacity Region

Error probability:

$$P_e = \Pr\{(\widehat{W}_1, \widehat{W}_2) \neq (W_1, W_2)\}.$$

A rate pair (R_1, R_2) with nonnegative R_1 and R_2 is **achievable** for a MAC with FB if for any $\epsilon > 0$ there exist for all N large enough encoders and a decoder with

$$\begin{aligned} \log_2 M_1 &\geq N(R_1 - \epsilon), \\ \log_2 M_2 &\geq N(R_2 - \epsilon), \\ P_e &\leq \epsilon. \end{aligned}$$

The set of achievable rate pairs (R_1, R_2) is the **capacity region** $\mathcal{R}_f$.

An Inner and an Outer Bound for Capacity Region $\mathcal{R}_f$

Theorem (Cover-Leung (1981) Inner Bound)

$$\mathcal{R}_f \supseteq \{(R_1, R_2) \quad : \quad \begin{array}{l} 0 \leq R_1 \leq I(X_1; Y|X_2, U), \\ 0 \leq R_2 \leq I(X_2; Y|X_1, U), \\ R_1 + R_2 \leq I(X_1, X_2; Y), \\ \textit{for } P(u, x_1, x_2, y) = P(u)P(x_1|u)P(x_2|u)P_c(y|x_1, x_2)\} \end{array}$$

Theorem (Cut-Set Outer Bound)

$$\mathcal{R}_f \subseteq \{(R_1, R_2) \quad : \quad \begin{array}{l} 0 \leq R_1 \leq I(X_1; Y|X_2), \\ 0 \leq R_2 \leq I(X_2; Y|X_1), \\ R_1 + R_2 \leq I(X_1, X_2; Y), \\ \textit{for } P(x_1, x_2, y) = P(x_1, x_2)P_c(y|x_1, x_2)\} \end{array}$$

QUESTION: Can arbitrary joint distributions $\{P(x_1, x_2), x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\}$ be realized?

A Result for the Binary Adder MAC

X_1	X_2	Y
0	0	0
0	1	1
1	0	1
1	1	2

Note that $X_1 \equiv Y - X_2$ for the binary adder MAC.

Theorem (W. (1982))

For MACs for which there exist a mapping ϕ such that $X_1 \equiv \phi(Y, X_2)$ the Cover-Leung region is the feedback capacity region, hence

$$\begin{aligned}\mathcal{R}_f = \{(R_1, R_2) \quad : \quad & 0 \leq R_1 \leq I(X_1; Y|X_2, U) = H(X_1|U),\\ & 0 \leq R_2 \leq I(X_2; Y|X_1, U),\\ & R_1 + R_2 \leq I(X_1, X_2; Y),\\ & \textit{for } P(u, x_1, x_2, y) = P(u)P(x_1|u)P(x_2|u)P_c(y|x_1, x_2)\}\end{aligned}$$

THEREFORE the Cover-Leung region is the capacity region for the binary adder MAC.

Gaarder & Wolf Scheme

X_1	X_2	Y
0	0	0
0	1	1
1	0	1
1	1	2

- Assume that first the encoders independently transmit k binary, uniformly distributed, digits over the adder channel.
- If a 0 or 2 is received, the decoder understands.
- However if the output of the channel is 1 (this happens with probability 1/2), the encoders have to resolve 1 bit of uncertainty extra to the decoder. They can however do this later in full cooperation, transmitting $\log_2 3$ bits/transmission.
- Rates (Gaarder & Wolf (1975)):

$$R_1 = R_2 = \frac{k}{k + \frac{k/2}{\log_2 3}} = 0.7602 \text{ bits/transm.}$$

The system can operate in an **independent** mode but also in a **full-cooperation, dependent** mode.

QUESTION: How much dependence can be created?

Preliminaries

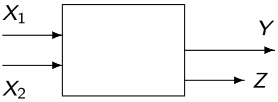
A. Some identities:

$$\begin{aligned} I(A;B|C) - I(A;B) &= H(B|C) - H(B|A,C) - H(B) + H(B|A) \\ &= I(B;C|A) - I(B;C) \\ &\quad \dots \\ &= I(C;A|B) - I(C;A). \end{aligned}$$

B: Let Z be an extra output of the MAC. Hence

$$P(y,z|x_1,x_2) = P_c(y|x_1,x_2)P_e(z|x_1,x_2,y) \quad (1)$$

where P_e is a channel with output Z having X_1, X_2, and Y as inputs.



Definition

A MAC with an extra output Z is said to be in class $\mathcal{K}$ if $I(X_1;X_2) = 0$ implies that $I(X_1;X_2|YZ) = 0$.

Local Dependence Balance Bound

$$
\begin{aligned}
&I(W_1; W_2|(YZ)_n, (YZ)^{n-1}) - I(W_1; W_2|(YZ)^{n-1}) \\
&\quad = I((YZ)_n; W_1|W_2, (YZ)^{n-1}) - I((YZ)_n; W_1|(YZ)^{n-1}) \\
&\quad = H((YZ)_n|W_2, (YZ)^{n-1}) - H((YZ)_n|W_1, W_2, (YZ)^{n-1}) \\
&\qquad -H((YZ)_n|(YZ)^{n-1}) + H((YZ)_n|W_1, (YZ)^{n-1}) \\
&\quad \overset{(a)}{\leq} H((YZ)_n|X_{2n}, (YZ)^{n-1}) - H((YZ)_n|X_{1n}, X_{2n}, (YZ)^{n-1}) \\
&\qquad -H((YZ)_n|(YZ)^{n-1}) + H((YZ)_n|X_{1n}, (YZ)^{n-1}) \\
&\quad = I((YZ)_n; X_{1n}|X_{2n}, (YZ)^{n-1}) - I((YZ)_n; X_{1n}|(YZ)^{n-1}) \\
&\quad = I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) - I(X_{1n}; X_{2n}|(YZ)^{n-1}).
\end{aligned}
$$

In (a) we use that

$$
\begin{aligned}
H((YZ)_n|W_2, (YZ)^{n-1}) &= H((YZ)_n|W_2, X_{2n}, (YZ)^{n-1}) \\
&\leq H((YZ)_n|X_{2n}, (YZ)^{n-1}),
\end{aligned}
$$

etc., and

$$
\begin{aligned}
H((YZ)_n|W_1, W_2, (YZ)^{n-1}) &= H((YZ)_n|W_1, W_2, X_{1n}, X_{2n}, (YZ)^{n-1}) \\
&= H((YZ)_n|X_{1n}, X_{2n}, (YZ)^{n-1}).
\end{aligned}
$$

Lemma (Local dependence balance bound)

$$
\begin{aligned}
& I(W_1; W_2|(YZ)_n, (YZ)^{n-1}) - I(W_1; W_2|(YZ)^{n-1}) \\
& \quad \leq \quad I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) - I(X_{1n}; X_{2n}|(YZ)^{n-1}).
\end{aligned}
$$

Increase in dependence between W_1 and W_2 by observing $(YZ)_n$ is upper bounded by increase in dependence between X_{1n} and X_{2n} by observing $(YZ)_n$, all given $(YZ)^{n-1}$.

Lemma

For a MAC with extra output Z in class K we have that $I(W_1; W_2|(YZ)^{n-1}) = 0$ for all $n = 1, 2, \cdots, N$.

PROOF: By induction.

1. First note that $I(W_1; W_2) = 0$.
2. Let $I(W_1; W_2|(YZ)^{n-1}) = 0$ for some $n = 1, 2, \cdots, N$. Then also $I(X_{1n}; X_{2n}|(YZ)^{n-1}) = 0$ and for a MAC with extra output Z in $\mathcal{K}$ this implies that $I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) = 0$. Now from the local DBB it follows that $I(W_1; W_2|(YZ)_n, (YZ)^{n-1}) = 0$.

Consequences of the Local DBB

Using standard techniques we can now prove:

Theorem

An outer bound for a MAC with FB is

$$\begin{aligned}\mathcal{R}_f \subseteq \{(R_1, R_2) \quad : \quad & 0 \leq R_1 \leq I(X_1; YZ|X_2, U), \\ & 0 \leq R_2 \leq I(X_2; YZ|X_1, U), \\ & R_1 + R_2 \leq I(X_1, X_2; Y), \ \textit{for} \\ & P(u, x_1, x_2, y, z) = P(u)P(x_1|u)P(x_2|u)P(y, z|x_1, x_2)\}\end{aligned}$$

if the MAC with an extra output Z is in class $\mathcal{K}$.

CONSEQUENCE: A MAC for which there is a function ϕ such that $X_1 = \phi(Y, X_2)$ with extra output $Z = X_1$ is in class $\mathcal{K}$, since $I(X_1; X_2|Y, Z) = I(X_1; X_2|Y, X_1) = 0$.
By the above theorem again the Cover-Leung region is the capacity region for such channels, since $I(X_1; YZ|X_2, U) = I(X_1; Y|X_2, U)$ and $I(X_2; YZ|X_1, U) = I(X_2; Y|X_1, U)$.

Global Dependence Balance Bound

$$\begin{aligned}
0 &\leq I(W_1; W_2|(YZ)^N) - I(W_1; W_2) \\
&= \sum_{n=1}^{N} [I(W_1; W_2|(YZ)^{n-1}), (YZ)_n) - I(W_1; W_2|(YZ)^{n-1})] \\
&\overset{(b)}{=} \sum_{n=1}^{N} [I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) - I(X_{1n}; X_{2n}|(YZ)^{n-1})].
\end{aligned}$$

where (b) follows from the local DBB.

Lemma (Global dependence balance bound)

$$\sum_{n=1}^{N} [I(X_{1n}; X_{2n}|(YZ)_n, (YZ)^{n-1}) - I(X_{1n}; X_{2n}|(YZ)^{n-1})] \geq 0.$$

A Consequence of the Global DBB

Using standard techniques we can now prove:

Theorem

An outer bound for a MAC with FB is

$$\begin{aligned}\mathcal{R}_f \subseteq \{(R_1, R_2) \quad : \quad & 0 \leq R_1 \leq I(X_1; YZ|X_2, U),\\ & 0 \leq R_2 \leq I(X_2; YZ|X_1, U),\\ & R_1 + R_2 \leq I(X_1, X_2; Y), \text{ for}\\ & P(u, x_1, x_2, y, z) = P(u)P(x_1, x_2|u)P(y, z|x_1, x_2)\\ & \text{such that } I(X_1; X_2|YZ, U) \geq I(X_1; X_2|U)\}\end{aligned}$$

if the MAC has an extra output Z.

DEPENDENCE BALANCE: $I(X_1; X_2|YZ, U) \geq I(X_1; X_2|U)$, i.e. produced dependence cannot be smaller than consumed dependence.

Concluding Remarks

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- Bounds and capacity regions for MACs with FB (Hekstra-W. (1985, 1989)).
- Similar results for Two-Way Channels, especially for the binary multiplying channel (Hekstra-W. (1985, 1989)).
- Gaussian MAC and Interference Channels with FB (Gastpar-Kramer (2006, 2007))
- Recent results on MACs with FB, MACs with noisy FB or with user cooperation, Interference Channels with user cooperation, Gaussian cases (Tandon-Ulukus (2007 ...)
- Relay Channel?
- Does something similar exist for the broadcast channel?