Spatially Coupled LDPC Codes – Theory and Applications

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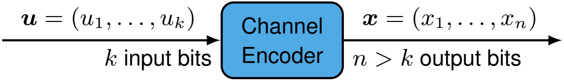
Overview

1. Introduction: Channel Coding and LDPC Codes
2. Spatially Coupled LDPC Codes
   - Motivation and Definition
   - Performance of SC-LDPC Codes
   - Practical Implementation of SC-LDPC Codes
   - Improvement of SC-LDPC Codes by Non-Uniform Coupling
3. Burst Correction Capabilities of Spatially Coupled LDPC Codes
   - Error Probability after Burst Erasures
   - Application Example
4. Conclusions

Chanel Coding



- Channel coding is indispensible in any communication and storage device
- The channel encoder encodes k information bits into n output bits, adding $n - k$ parity bits

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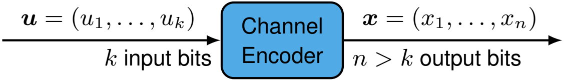

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Channel code

A block code $\mathcal{C}$ of length n and cardinality $M = 2^k$ over a field $\mathbb{F}$ is a collection of M elements $\boldsymbol{x}^{[i]}$ from $\mathbb{F}^n$

$$\mathcal{C}(n, M) := \left\{\boldsymbol{x}^{[1]}, \boldsymbol{x}^{[2]}, \ldots, \boldsymbol{x}^{[M]}\right\}, \boldsymbol{x}^{[m]} \in \mathbb{F}^n, 1 \leq m \leq M$$

The elements $\boldsymbol{x}^{[i]}$ are called *codewords*. Here we only consider *binary codes*, where $\mathbb{F} = \{0, 1\}$.

Linear Block Codes

- We consider only the sub-class of **linear block codes**

Linear block code

A linear block code of length n and dimension k is a k-dimensional linear subspace of the vector space $\mathbb{F}^n$. As a linear subspace, the code can be represented also as the *span* of k codewords forming a basis. These basis codewords are the *rows* of a **generator matrix** $\boldsymbol{G} \in \mathbb{F}^{k \times n}$. The code is generated by multiplying all possible information vectors by the generator matrix

$$\mathcal{C}(n,k) = \{\boldsymbol{x}^{[i]} = \boldsymbol{u}^{[i]}\boldsymbol{G} : \forall \boldsymbol{u}^{[i]} \in \mathbb{F}^k\}$$

Alternative, the linear code can be seen as the *null space* of a **parity-check matrix** $\boldsymbol{H}$ with $\boldsymbol{G}\boldsymbol{H}^T = 0$

$$\mathcal{C}(n,k) = \{\boldsymbol{x} \in \mathbb{F}^n : \boldsymbol{H}\boldsymbol{x}^T = \boldsymbol{0}\}$$

Linear Block Code – Example

- Consider the following parity-check matrix

$$\boldsymbol{H} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

- For example: $\boldsymbol{x} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$ is a codeword

Low-Density Parity-Check (LDPC) Codes

Definition: Low-Density Parity-Check (LDPC) Codes

A low-density parity-check (LDPC) code is a ***linear block code*** with a ***sparse*** parity-check matrix $\boldsymbol{H}$, i.e., the number of “1”s in $\boldsymbol{H}$ is very small, compared to the number of “0”s

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- We consider the sub-class of ***regular*** LDPC codes

Definition: Regular $[d_\mathsf{v}, d_\mathsf{c}]$ LDPC Codes

A regular $[d_\mathsf{v}, d_\mathsf{c}]$ LDPC code is an LDPC code where each ***column of H contains exactly d_v "1"s*** and each ***row of H contains exactly d_c "1"s***. We assume $d_\mathsf{v} \geq 2$ and $d_\mathsf{c} \geq 2$.

- As the number of "1"s in $\boldsymbol{H}$ is small, the memory complexity is only $O(n)$ as the number of "1"s per column is constant (and independent of n)

Example: $[3,6]$ LDPC Code

- Rate $r_d = \frac{1}{2}$, $n = 48$, $m = 24$, $d_{\mathsf{v}} = 3$, $d_{\mathsf{c}} = 6$

$$
\boldsymbol{H} = \left(\begin{array}{*{48}{c}}
& 1 &&&&&&&&&&&& 1 &&&&&&&& 1 &&&&&&&&&& 1 &&&&&& 1 &&&&& 1 &&&&& \\
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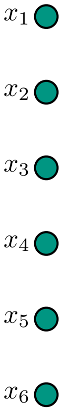
&&&&& 1 &&&&&&&&& 1 &&&&&&& 1 &&&&& 1 &&&&&&&& 1 &&&&&&& 1 &&&&&&
\end{array}\right)
$$

- Density of parity-check matrix: 12.5%

Graph Representation of LDPC Codes

- **Example:** $[2, 4]$ LDPC code with $n = 6$



$$\boldsymbol{H} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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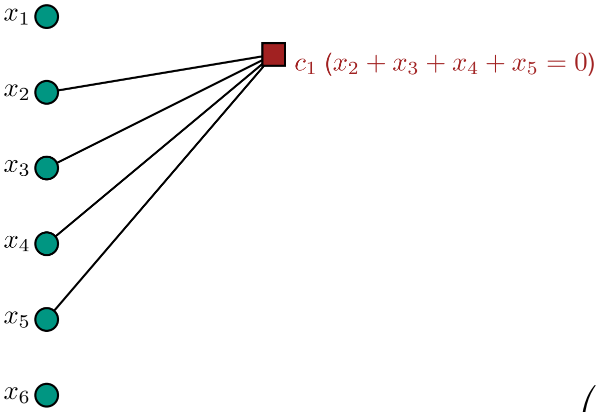

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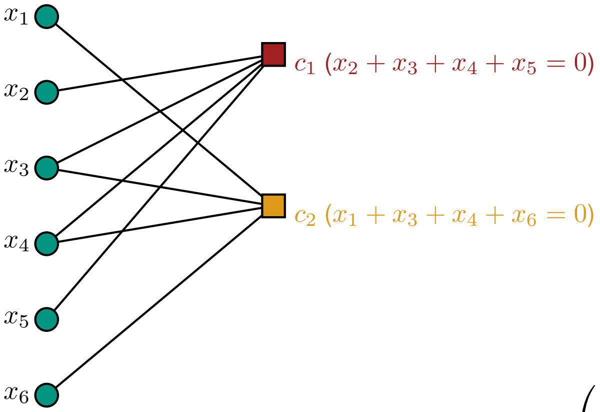

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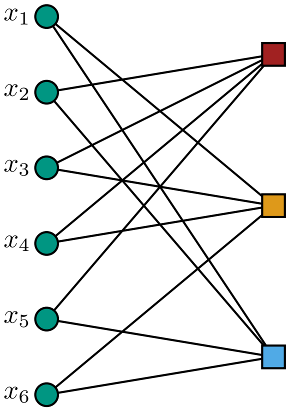


$c_1\ (x_2+x_3+x_4+x_5=0)$

$c_2\ (x_1+x_3+x_4+x_6=0)$

$c_3\ (x_1+x_2+x_5+x_6=0)$

$$\boldsymbol{H}=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Example: $[3, 6]$ Regular Code

Consider the following parity-check matrix of the $d_\mathsf{v} = 3$, $d_\mathsf{c} = 6$ code

$$\boldsymbol{H} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}$$

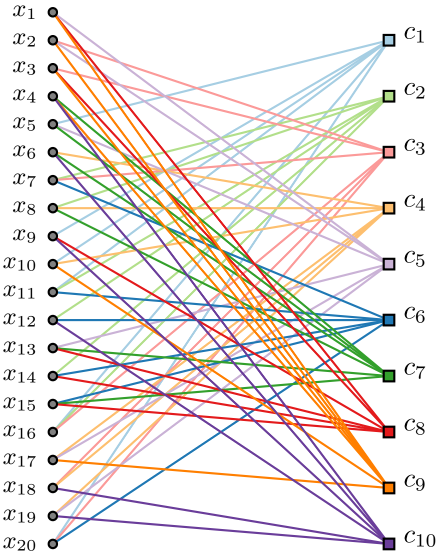
Example: $[3,6]$ Regular Code (2)

Consider the following parity-check matrix of the $d_\mathsf{v} = 3$, $d_\mathsf{c} = 6$ code

$$\boldsymbol{H} = \begin{pmatrix}
 & & & & 1 & & & & 1 & 1 & 1 & & & & & 1 & & & & 1 \\
 & & & & & & 1 & 1 & & & 1 & 1 & & 1 & & 1 & & & & \\
 & 1 & 1 & & & & 1 & & & & & & & & & 1 & & 1 & & 1 \\
 & & & & & 1 & & 1 & & 1 & & & & & & & 1 & 1 & 1 & \\
1 & 1 & & & 1 & & & & & & & & 1 & & & & 1 & & 1 & \\
 & & & & & & 1 & & & & 1 & 1 & & 1 & 1 & & & & & 1 \\
 & & & 1 & 1 & 1 & & 1 & & & & & 1 & & 1 & & & & & \\
1 & & 1 & & & & & & 1 & & & & 1 & 1 & 1 & & & & & \\
1 & 1 & 1 & 1 & & & & & & 1 & & & & & & & 1 & & & \\
 & & & 1 & & 1 & & & 1 & & & 1 & & & & & & 1 & 1 &
\end{pmatrix}$$

- For the sake of readability, we only display “1”s and do not show the “0”s

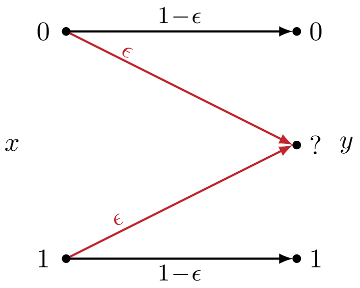
Graph of $[3, 6]$ Regular Code



- Every constraint $c_1, \dots, c_{10}$ corresponds to a row of the parity-check matrix of previous slide
- Every code bit (variable) has $d_\mathsf{v} = 3$ outgoing edges
- Every constraint node c_i has $d_\mathsf{c} = 6$ connected edges

Binary Erasure Channel (BEC)

- Simple channel model that enables easy analysis but is close to practice
- Channel with 3 output symbols, where "?" denotes an ***erasure***, i.e., no information about transmitted symbol
- BEC can be used to model, e.g., packet losses or high SNR channels



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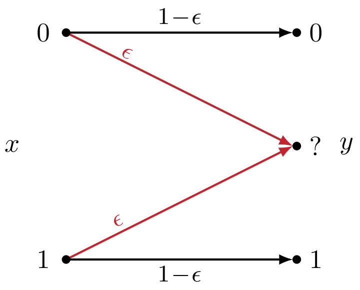
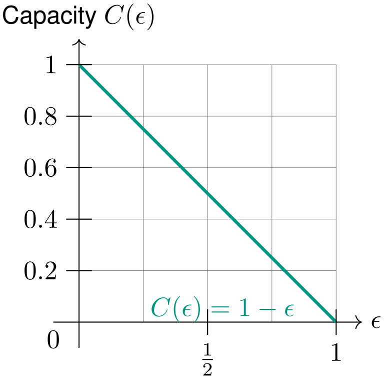




Density Evolution – Variable Nodes

- ***What is the necessary ϵ for iterative decoding to be successful?***

Density Evolution – Variable Nodes

- ***What is the necessary ϵ for iterative decoding to be successful?***
- Denote the channel erasure probability by ϵ
- Iterative message passing decoder exchanging ***beliefs*** or ***messages*** along the edges of the Tanner graph
- Analyse decoding behavior of the code using a simple update equation

Update equation for regular $[d_\mathsf{v}, d_\mathsf{c}]$ LDPC codes on the BEC

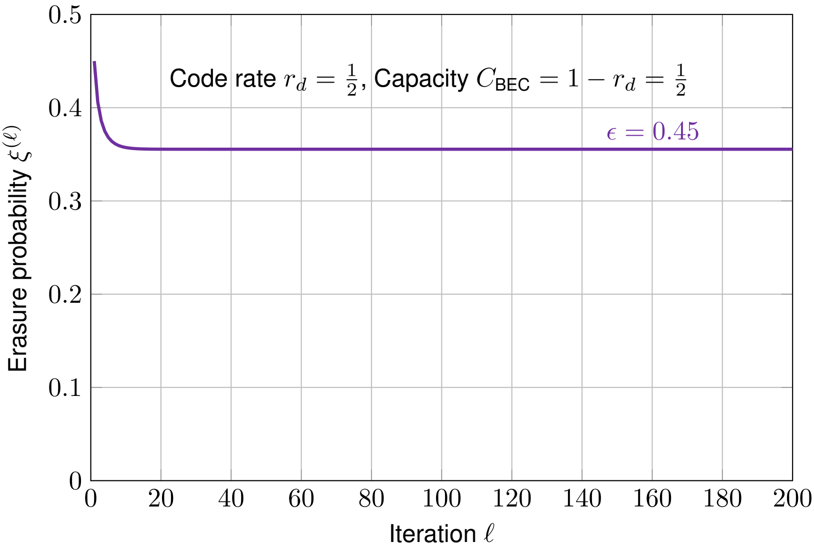
The average message erasure probability ξ_ℓ after ℓ decoding iterations is given by

$$\xi^{(\ell)} = \epsilon\left(1-(1-\xi^{(\ell-1)})^{d_\mathsf{c}-1}\right)^{d_\mathsf{v}-1}$$

with $\xi_0 = 1$.

- We can decode successfully (if $n \to \infty$) if $\xi^{(\ell)} \to 0$
- The **threshold** $\epsilon^\star$ is the largest ϵ for which $\xi^{(\ell)} \to 0$

Density Evolution for the Regular [3,6] LDPC Code



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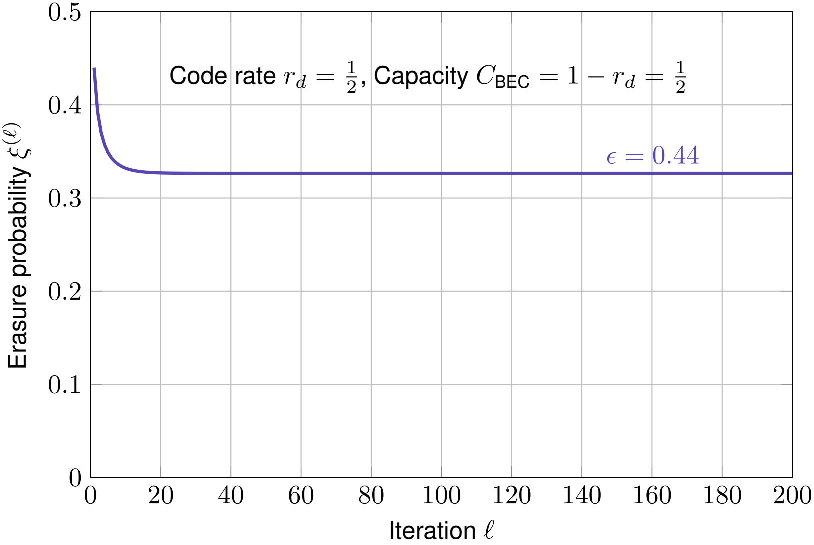

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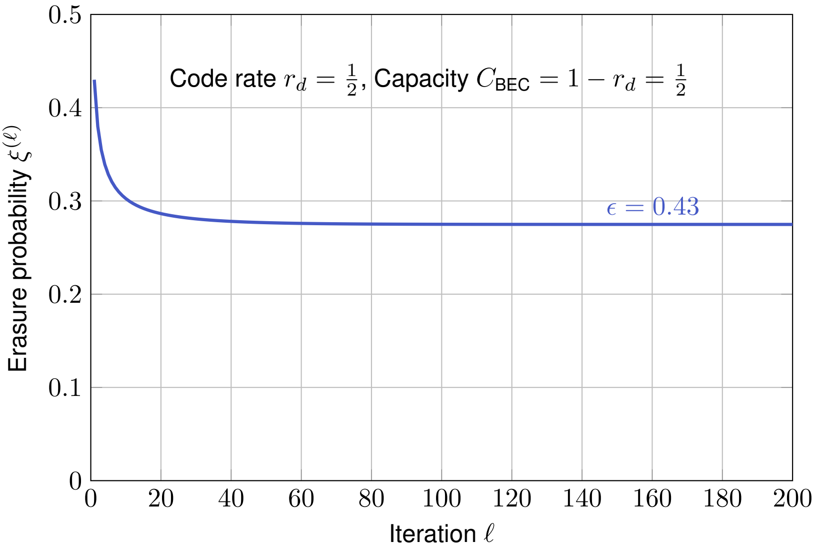

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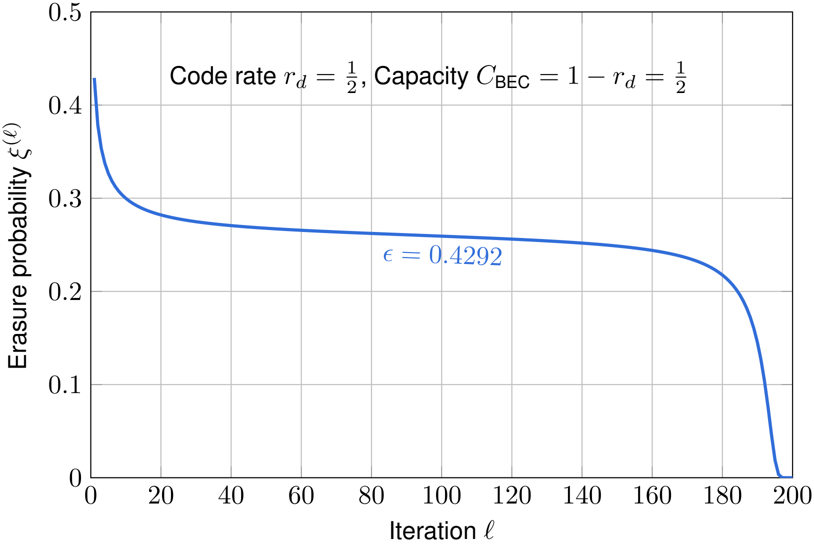

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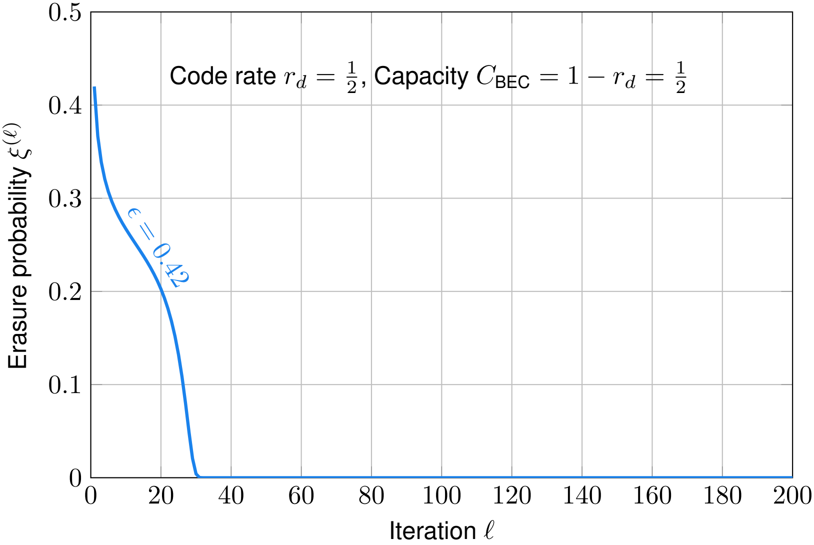

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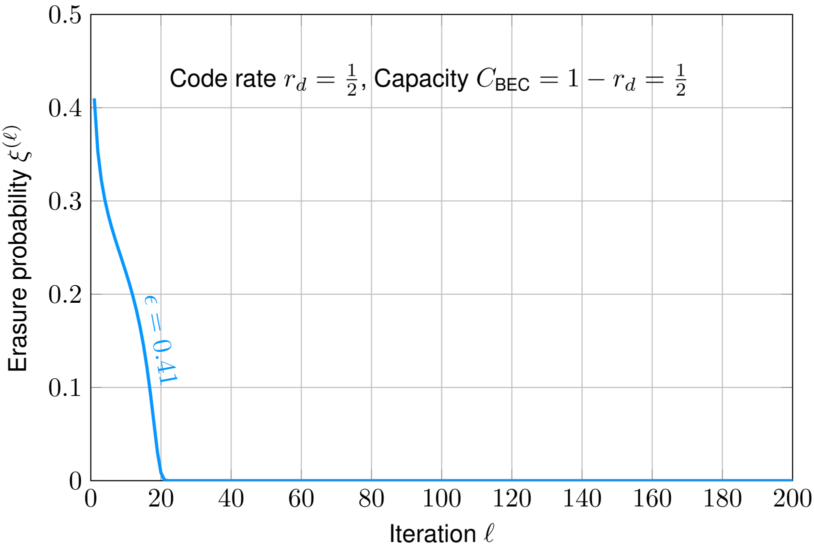

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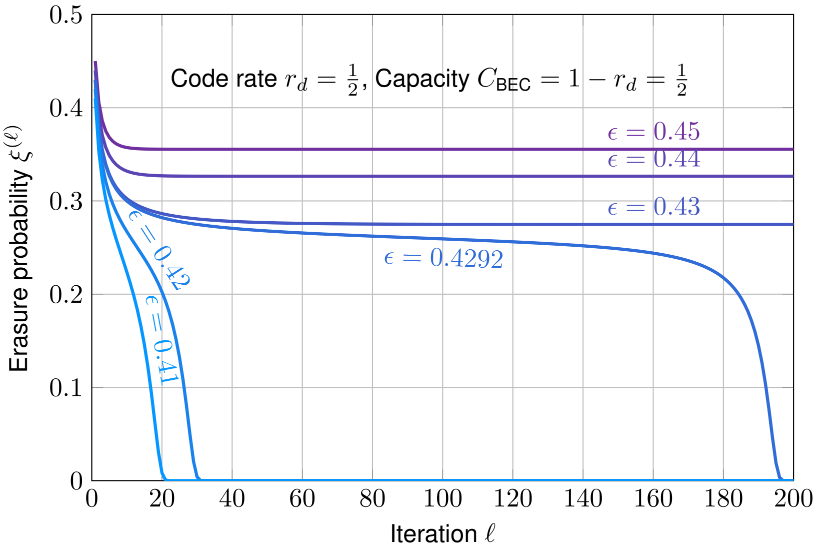
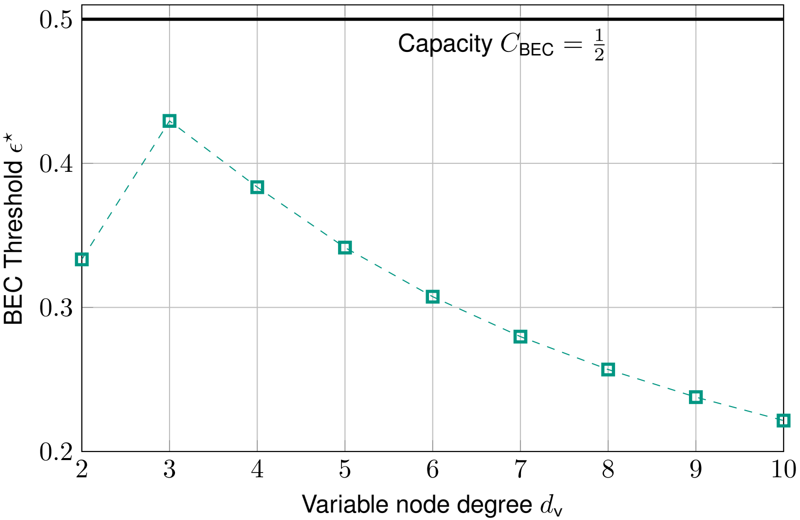


Thresholds of Rate-$\frac{1}{2}$ $[d_\mathsf{v}, d_\mathsf{c} = 2d_\mathsf{v}]$ Codes



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Regular LDPC Codes

- **Gap to capacity** with regular LDPC codes

- Remedy A: **Irregular LDPC Codes** **[RSU01]**
  - Can close the gap to capacity
  - Drawback: Increased error floor, not suitable for high-reliability applications

- Remedy B: **Protograph LDPC Codes** **[Tho03]**
  - Generalizes the construction of LDPC Codes
  - Optimization of very good codes with low error floors still difficult

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- New approach needed that extends protograph LDPC codes
- **Spatially coupled LDPC codes**

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Spatially Coupled LDPC Codes

- Block codes:
  - encode a group of k input bits $\boldsymbol{u}$ into a codeword $\boldsymbol{x}$
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  - decode $\boldsymbol{y}$ and proceed to next block
- Block processing independent of previous and next block

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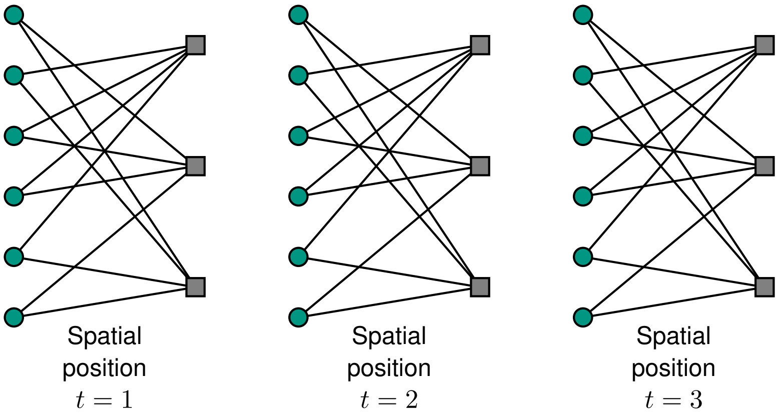
- ***Spatial coupling:*** introduce dependencies between the neighboring blocks so that they can help each other during decoding.
- We introduce a "spatial" dimension t to describe blocks. It is called "spatial" dimension as the temporal dimension is reserved for the bits ***inside*** a codeword.
- Let w denote the coupling width.

Example: $L = 3$ Codes Along Spatial Dimension



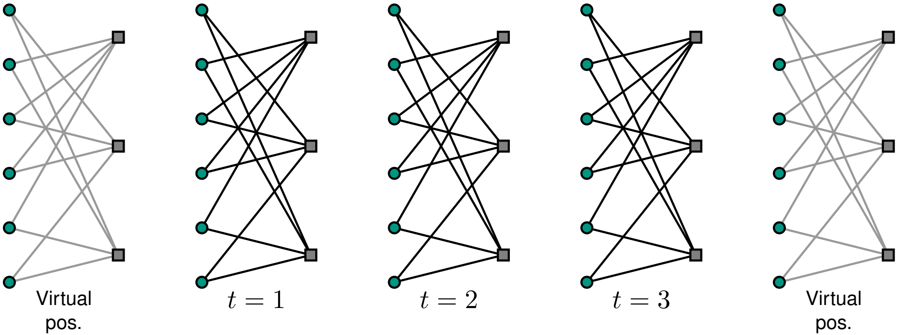
- No coupling, independent codes ($w = 1$)

Spatial Coupling of LDPC Codes

Spatial Coupling

- Consider a spatial chain of L codes ($t = 1, \dots L$).
- Consider each edge of the Tanner graph at spatial position t and connect it ***randomly*** to a check node of spatial positions $t, t+1, \dots, t+w-1$ with probability $1/w$ such that the ***local degree distributions*** are ***preserved***
- At each boundary insert $w-1$ ***virtual*** spatial positions at $-w+2, \dots, 0$ (left boundary) and $L+1, \dots, L+w-1$ (right boundary) to fulfill the degree distributions at the boundaries.
- The code bits of these virtual positions are fixed to "0" and need not be transmitted. As the code bits are "0", the edges can be removed in a later step

Spatial Coupling of $L = 3$ Codes ($w = 2$)



- Starting position, no coupling yet
- Two ($2 \cdot (w-1)$) virtual positions at boundaries
- Next remove all edges and start coupling for $t = 1$

Spatial Coupling of $L = 3$ Codes ($w = 2$) (2)

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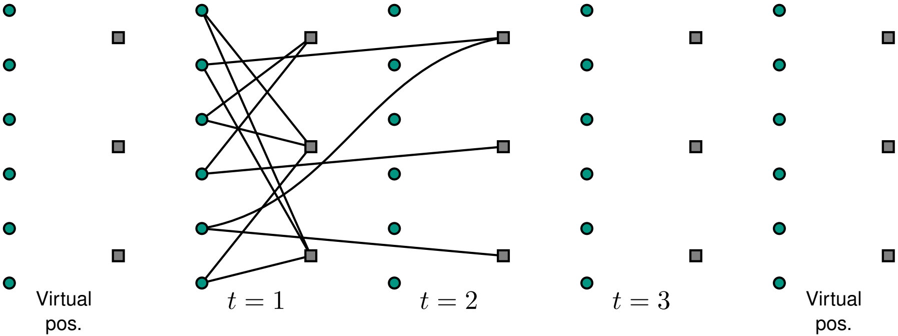
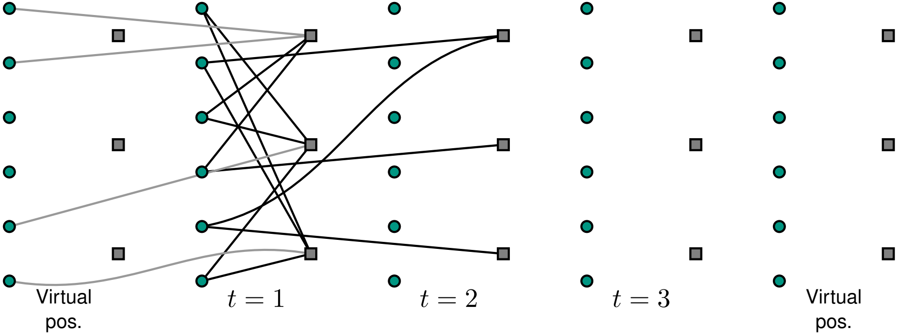


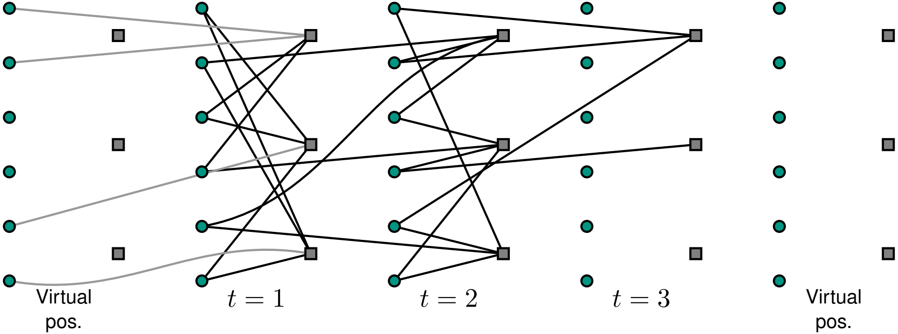
- As check nodes in $t = 1$ cannot achieve their degree distribution anymore from variable nodes at $t = 1$, we need to fill them using edges from the virtual position

Spatial Coupling of $L = 3$ Codes ($w = 2$) (3)



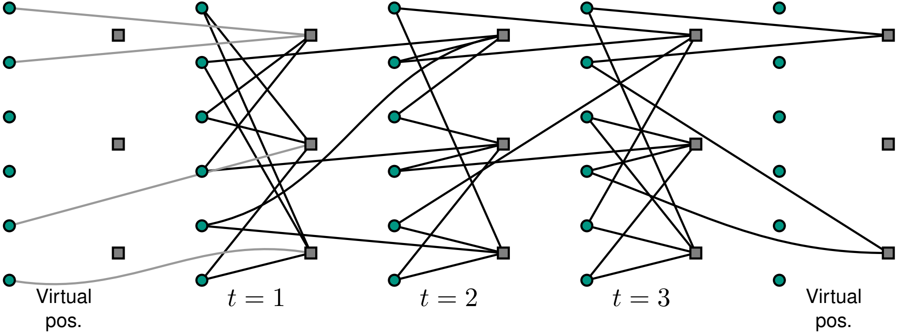
- We can now proceed to couple position $t = 2$.

Spatial Coupling of $L = 3$ Codes ($w = 2$) (4)



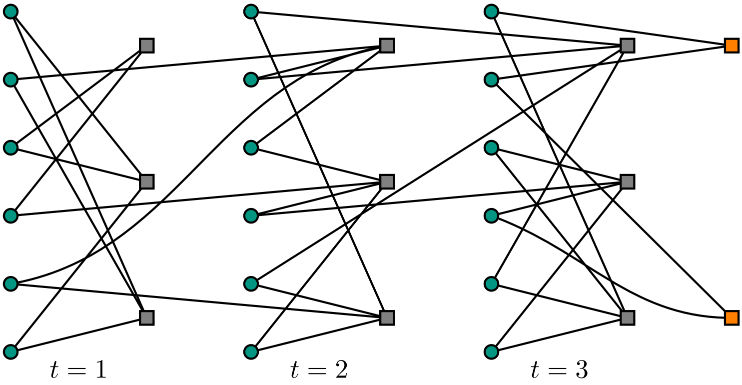
- Finally, we couple the edges of position $t = 3$.

Spatial Coupling of $L = 3$ Codes ($w = 2$) (5)



- We next remove the unconnected nodes and set the variable nodes at the virtual positions to be “0”, so we can remove these edges

Spatial Coupling of $L = 3$ Codes ($w = 2$) (6)



- We have created a new code of $n' = Ln = 21$ code bits
- Due to coupling we have created *two extra check nodes*
- These extra check nodes decrease the rate of the code

The Regular SC-LDPC $[d_{\mathsf{v}}, d_{\mathsf{c}}, w, L]$ Code Ensemble

- For simplicity, we consider again regular $[d_{\mathsf{v}}, d_{\mathsf{c}}]$ LDPC codes and couple them
- We denote the spatially coupled code by the 4-tuple $[d_{\mathsf{v}}, d_{\mathsf{c}}, w, L]$ where
  - d_{v} denotes the variable node degree
  - d_{c} denotes the check node degree
  - w denotes the coupling width
  - L denotes the ***replication factor***, i.e., the number of spatial positions

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  - d_v denotes the variable node degree
  - d_c denotes the check node degree
  - w denotes the coupling width
  - L denotes the ***replication factor***, i.e., the number of spatial positions
- Note that the check nodes have only regular degree d_c at spatial positions $w, \ldots, L$
  - check nodes at positions $1, \ldots, w-1$ have ***lower degree*** $\leq d_\mathsf{c}$
  - extra check nodes due to right virtual positions also have ***lower degree*** $\leq d_\mathsf{c}$
- The extra check nodes induce a rate loss

Rate of Regular $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ Code Ensemble

KIT Karlsruhe Institute of Technology

Theorem (Rate of regular $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ SC-LDPC codes)

Consider the regular $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ SC-LDPC code ensemble and assume $w < L$. The design rate of this code ensemble is given by

$$r_{d,\mathsf{SC}} = 1 - \frac{d_\mathsf{v}}{d_\mathsf{c}} - \frac{d_\mathsf{v}}{d_\mathsf{c}} \frac{w + 1 - 2\sum_{i=0}^{w}\left(\frac{i}{w}\right)^{d_\mathsf{c}}}{L}$$

$$= r_d - \frac{d_\mathsf{v}}{d_\mathsf{c}} \frac{w + 1 - 2\sum_{i=0}^{w}\left(\frac{i}{w}\right)^{d_\mathsf{c}}}{L}$$

where $r_d := 1 - \frac{d_\mathsf{v}}{d_\mathsf{c}}$ is the design rate of the regular LDPC code ensemble.

Proof:

- We obtain the result by counting the number of variable nodes V and the average number of check nodes C. The rate is obtained as $r = 1 - \frac{C}{V}$

[KRU11] S. Kudekar, T. Richardson and R. Urbanke, "Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform So Well over the BEC," *IEEE Trans. Inform. Theory*, vol. 57, no. 2, Feb. 2011

Overview

Karlsruhe Institute of Technology

1 Introduction: Channel Coding and LDPC Codes

2 Spatially Coupled LDPC Codes
- Motivation and Definition
- Performance of SC-LDPC Codes
- Practical Implementation of SC-LDPC Codes
- Improvement of SC-LDPC Codes by Non-Uniform Coupling

3 Burst Correction Capabilities of Spatially Coupled LDPC Codes
- Error Probability after Burst Erasures
- Application Example

4 Conclusions

Density Evolution Analysis of $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ SC-LDPC Codes

- We can track the asymptotic behavior using a similar technique as for LDPC codes
- For simplicity, we restrict ourselves to the BEC case
- Contrary to LDPC codes, we need to track L distinct erasure probabilities $\xi_t^{(\ell)}$, $t \in \{1, \ldots, L\}$, as each spatial position must be treated differently

- We need to take care of the boundary conditions
- As the code bits at virtual positions are fixed to be “0” and not transmitted, we can fix

$$\xi_t^{(\ell)} = 0 \quad \forall t \in \{-w+2, -w+3, \ldots, -1, 0, L+1, L+2, \ldots, L+w-1\}$$

BEC Density Evolution of $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ SC-LDPC Codes

BEC Density Evolution of $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ SC-LDPC Codes

1. Initialize

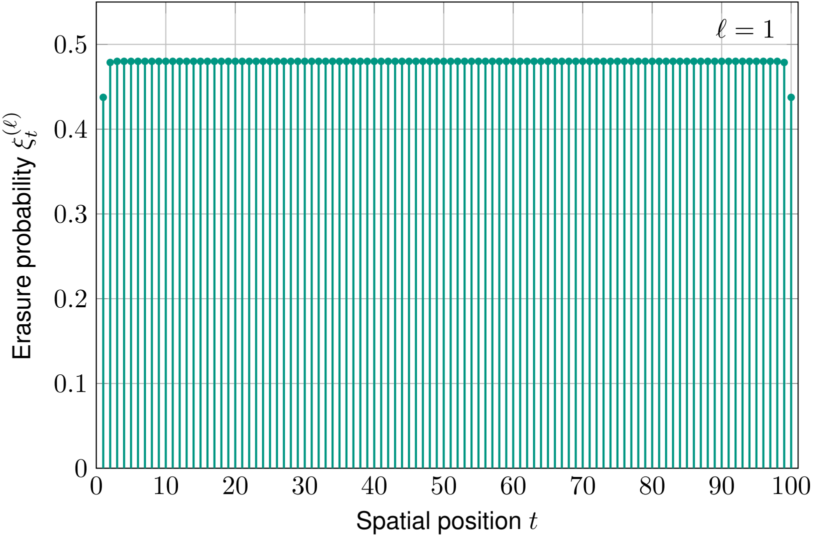
$$\xi_t^{(0)} = \begin{cases} 1 & \text{if } t \in \{1, 2, \ldots, L\} \\ 0 & \text{otherwise} \end{cases}$$

2. Update

$$\xi_t^{(\ell)} = \epsilon \left(1 - \frac{1}{w} \sum_{i=0}^{w-1} \left(1 - \frac{1}{w} \sum_{j=0}^{w-1} \xi_{t+i-j}^{(\ell-1)}\right)^{d_\mathsf{c}-1}\right)^{d_\mathsf{v}-1} \qquad \forall t \in \{1, \ldots, L\}$$

3. If all $\xi_t^{(\ell)} = 0$, $\forall t$, declare ***success*** and abort
4. If all $\xi_t^{(\ell)} = \xi_t^{(\ell-1)} > 0$, $\forall t \in \{1, \ldots, L\}$, we have reached a fixed point. Declare ***failure*** and abort
5. Otherwise, go to step 2)

Example: $[d_\mathsf{v} = 3, d_\mathsf{c} = 6, w = 3, L = 100]$, $\epsilon = 0.48$



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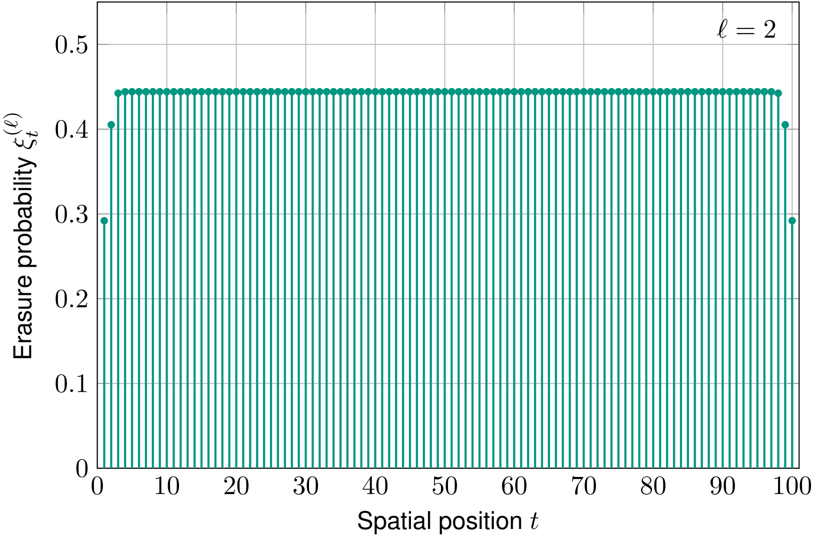

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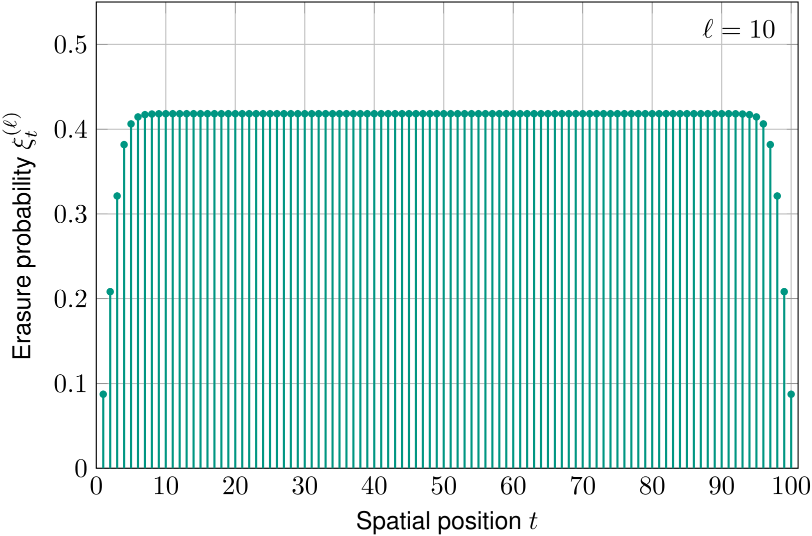

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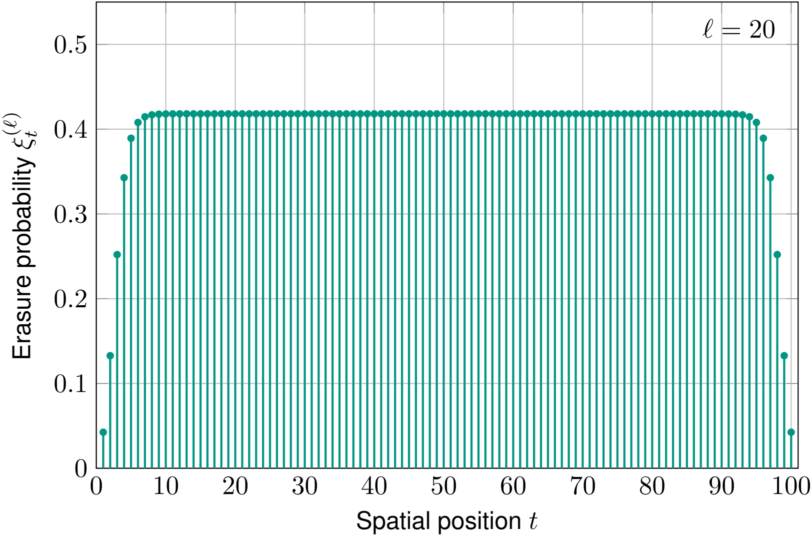

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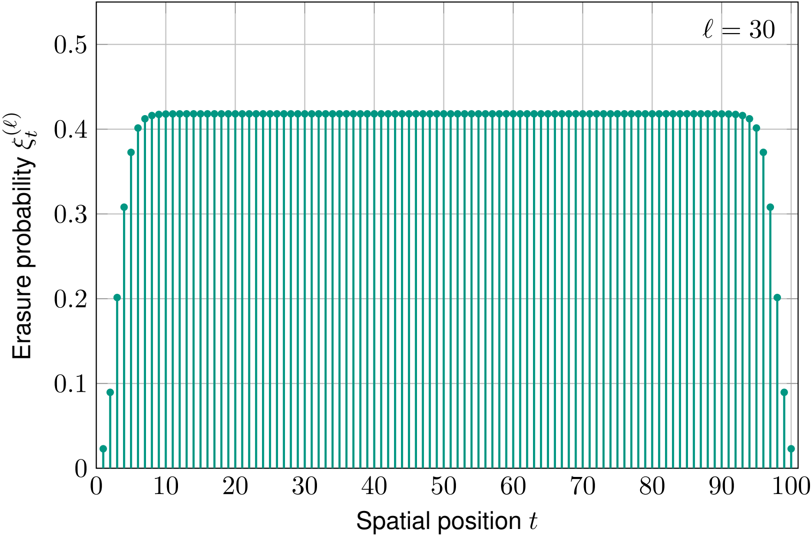

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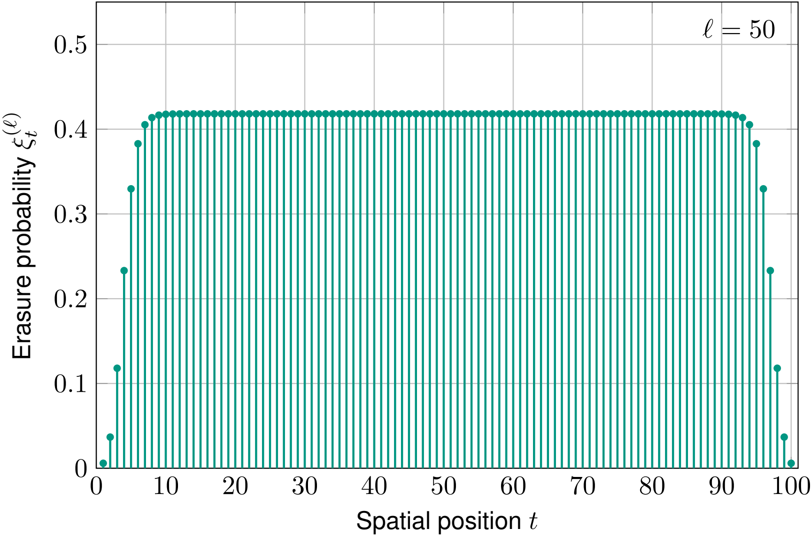

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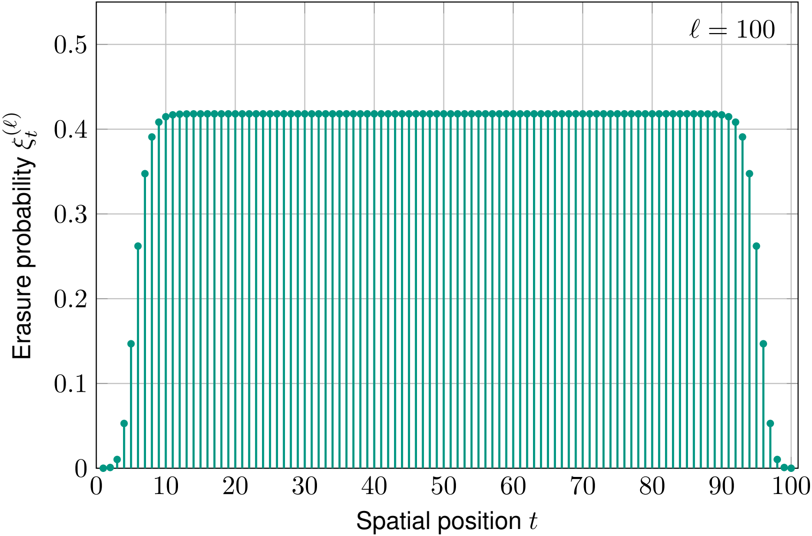

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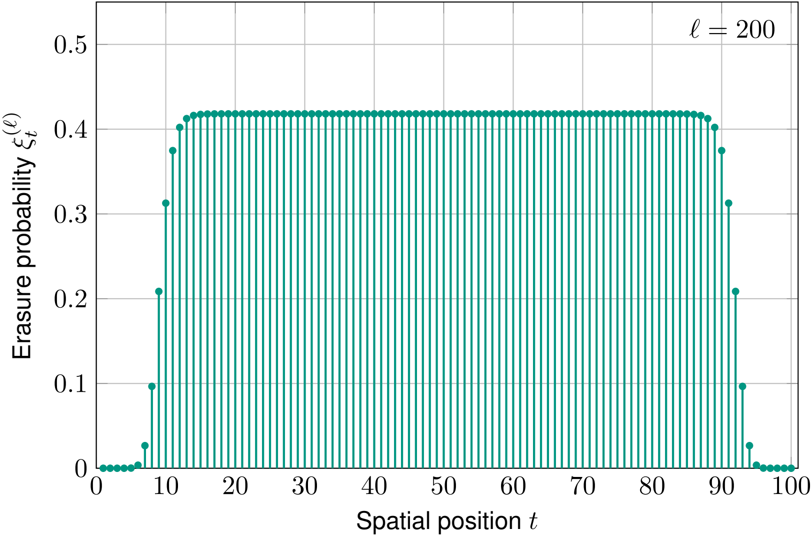

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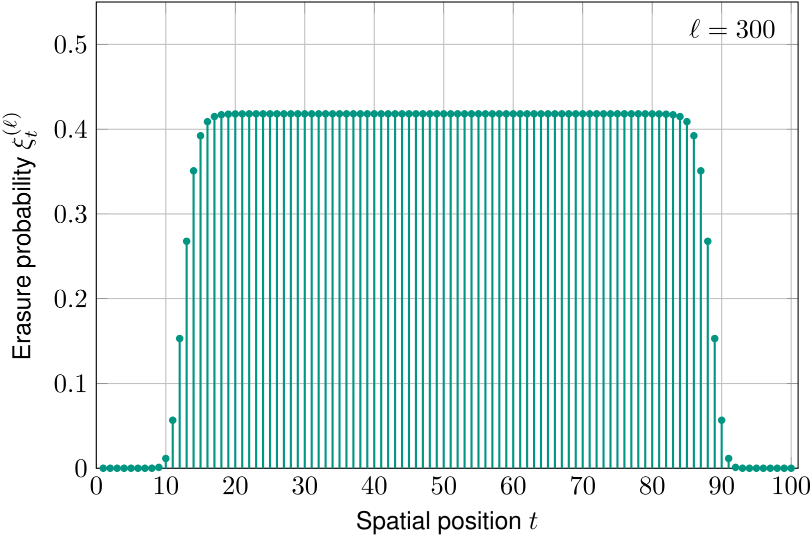

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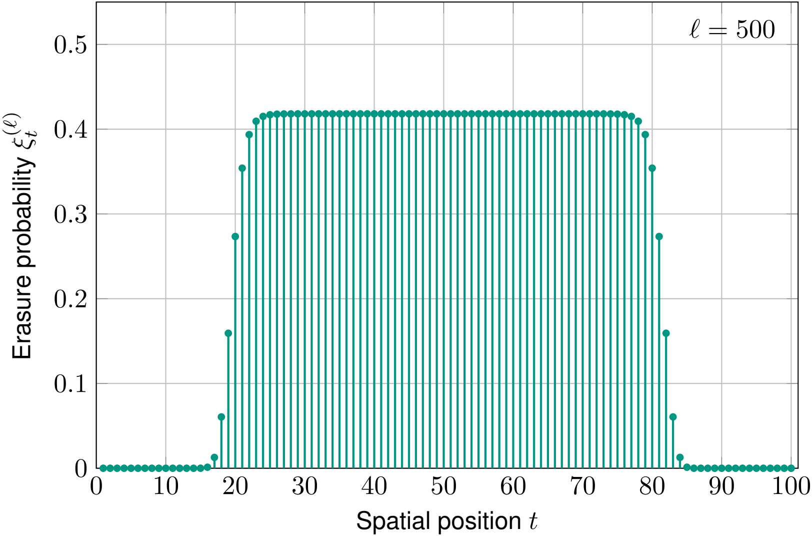

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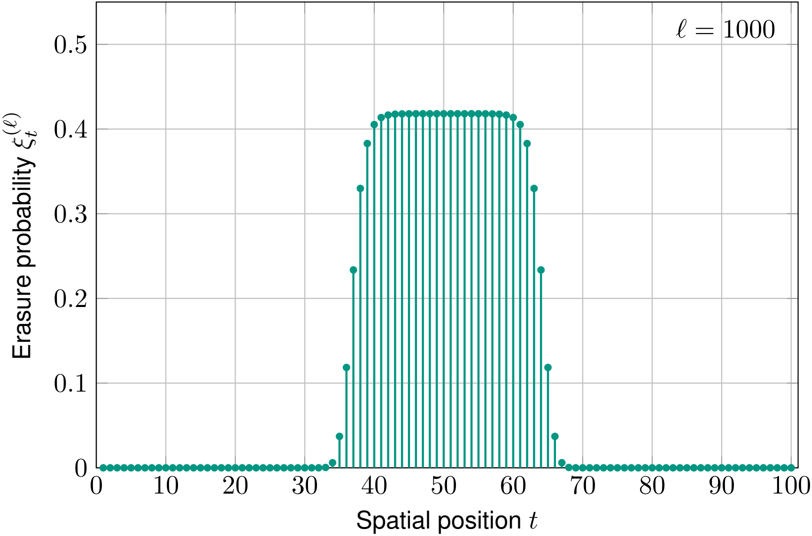

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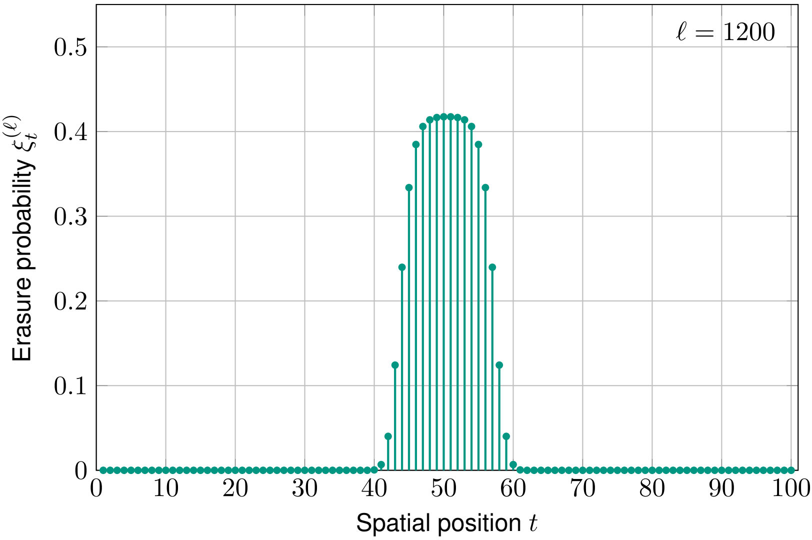

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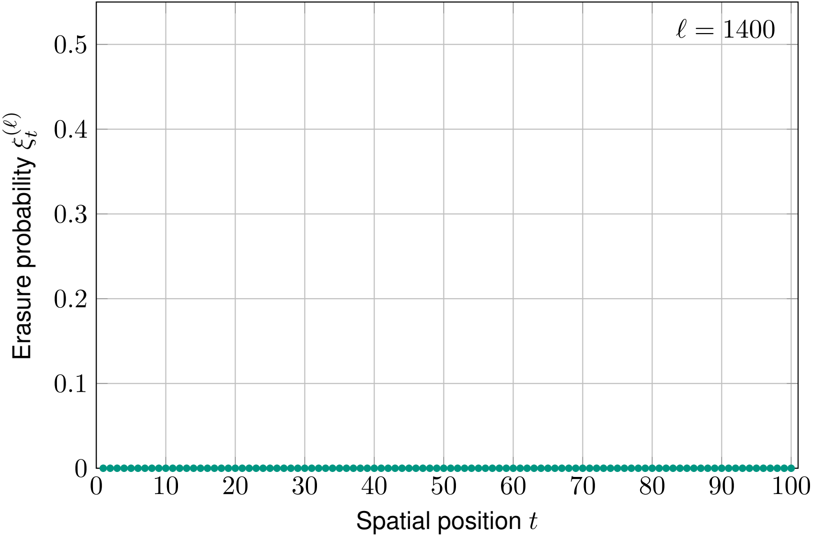

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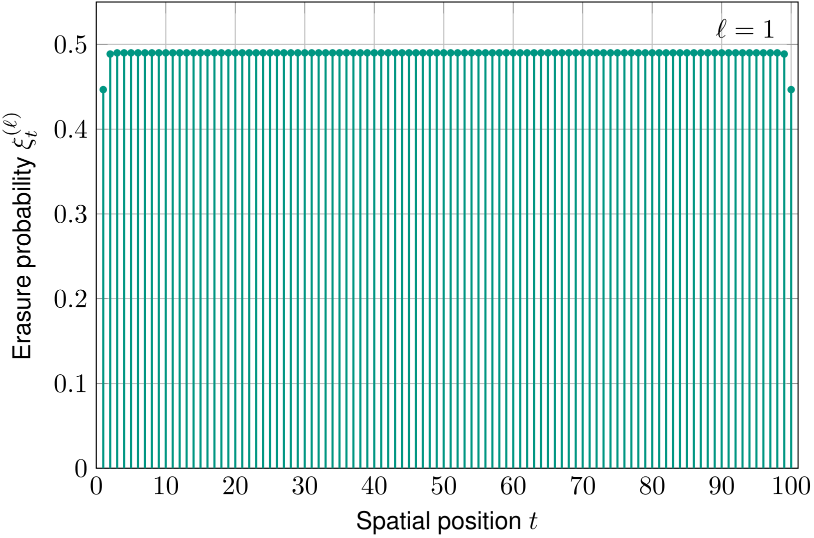

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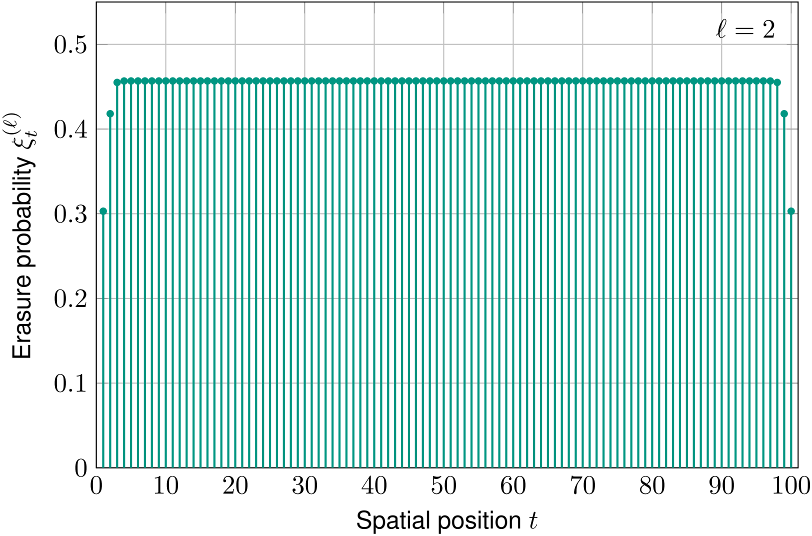

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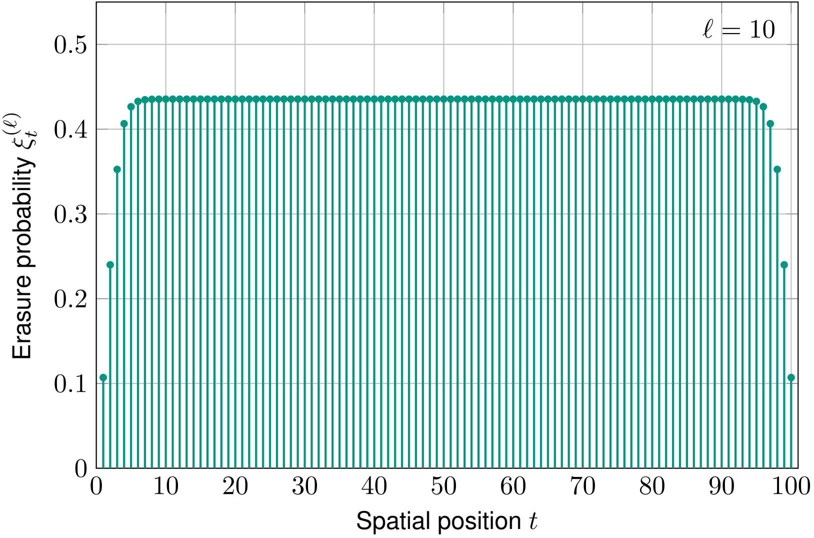

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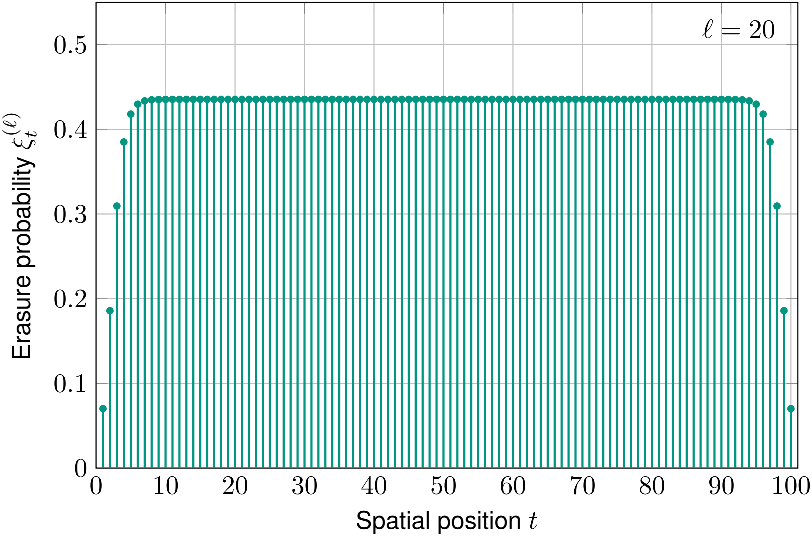

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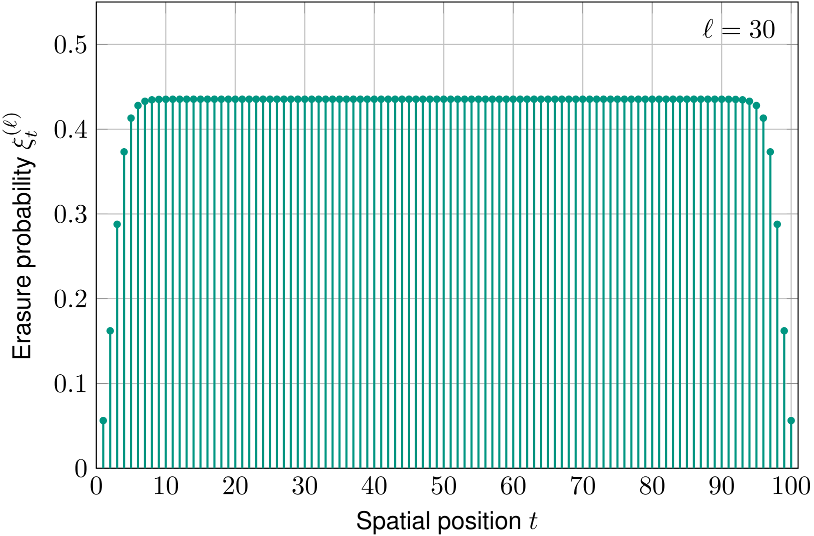

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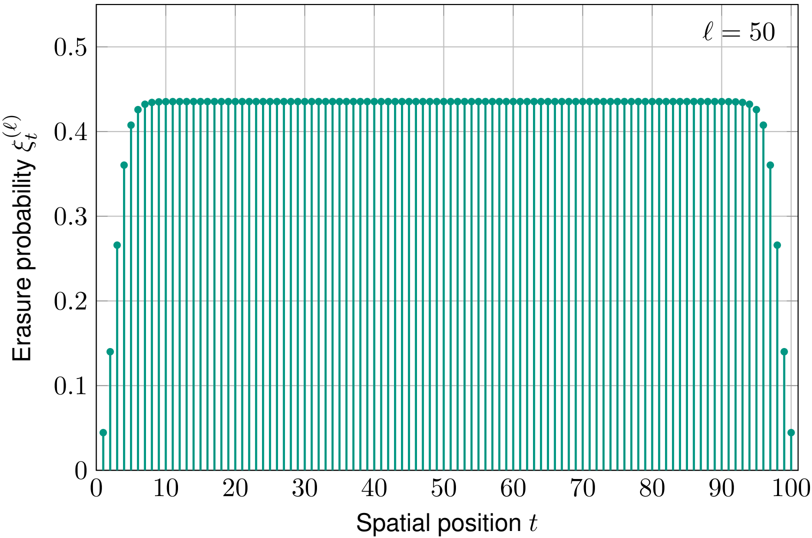

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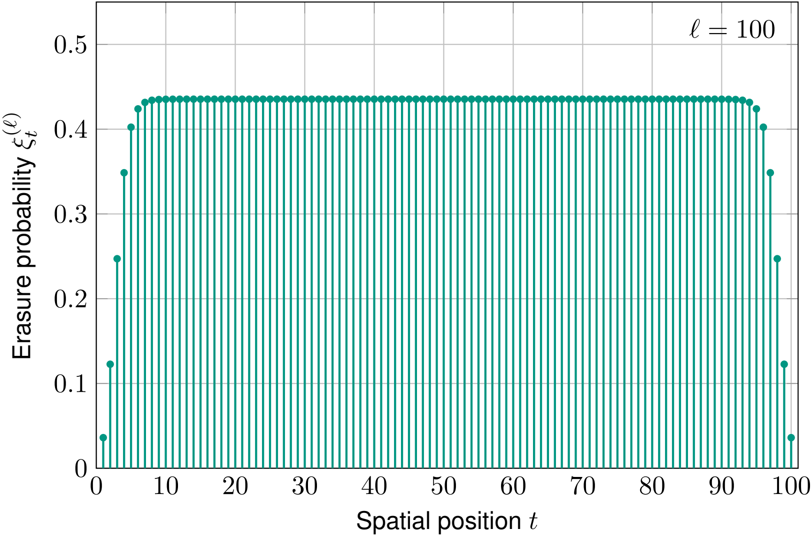

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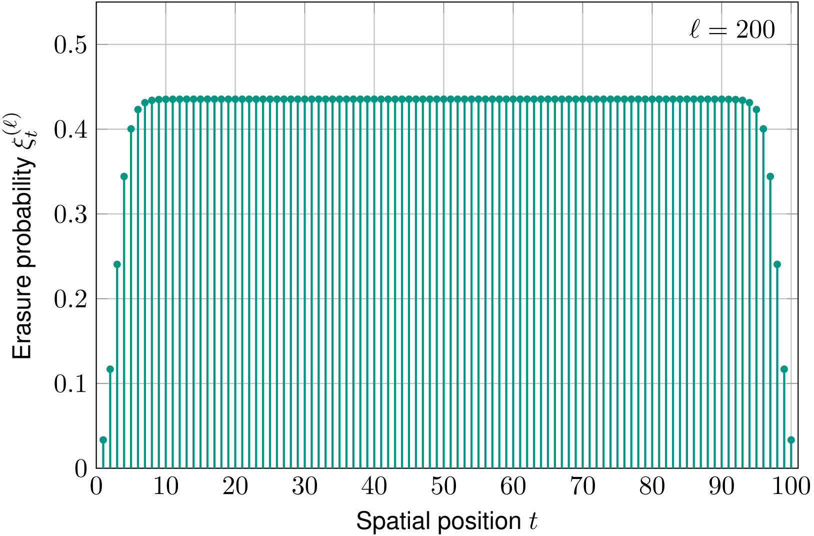

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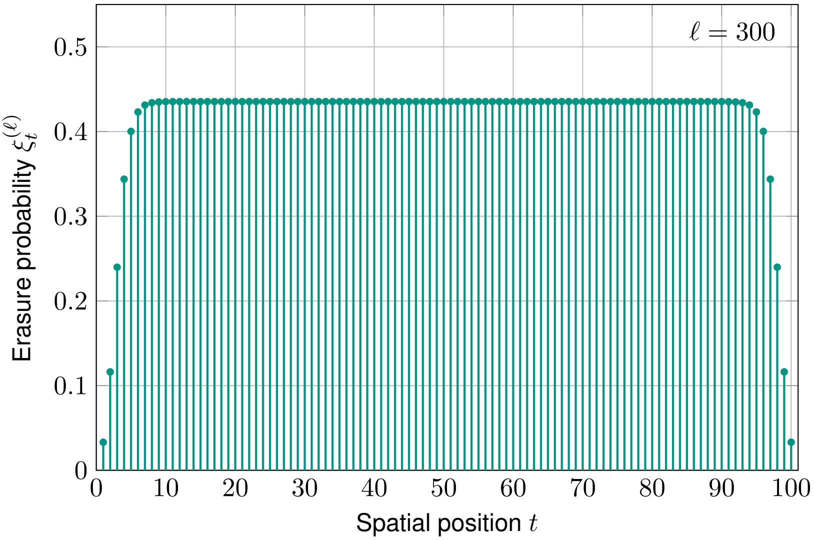

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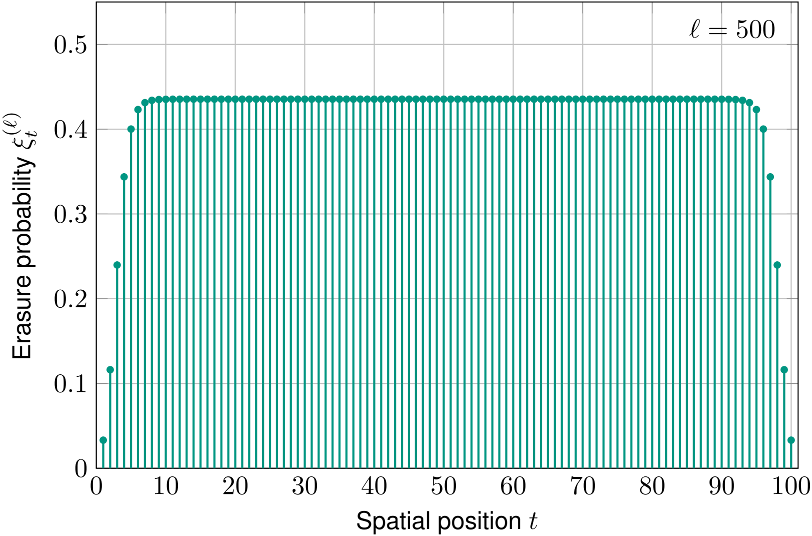


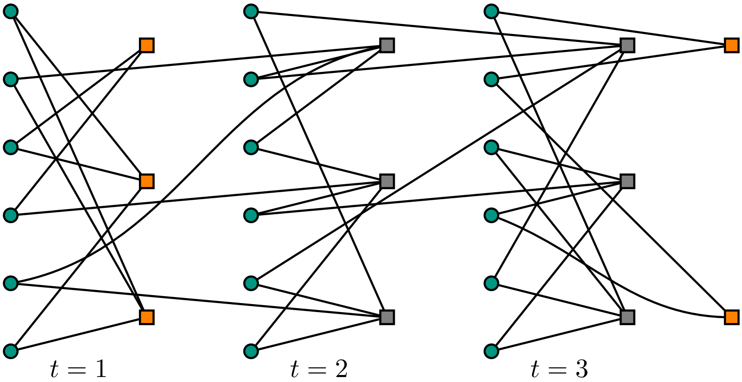
Threshold of SC-LDPC Codes

- Similarly to LDPC codes, we define the decoding threshold $\epsilon^\star_{\mathsf{SC}}$ as

$$\epsilon^\star_{\mathsf{SC}} = \sup\Big\{\epsilon \in [0,1] : \xi_t^{(\ell)} \xrightarrow{\ell\to\infty} 0, \forall t \in \{1,\ldots,L\},$$
$$\xi_t^{(0)} = 1\ \forall t \in \{1,\ldots,L\}, \text{and } \xi_t^{(0)} = 0\ \forall t \notin \{1,\ldots,L\}\Big\}$$

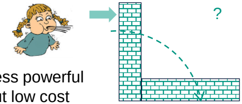
- Recall that the threshold of the $[3,6]$ regular LDPC code ensemble was 0.4292
- Spatial coupling dramatically improves the threshold
- The key are slightly different check node degrees at the boundaries
- Nucleation effect, also observed in physics (e.g., supercooled water)
- Video ...

Boundary Effects of Spatial Coupling: Example

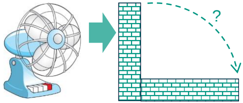


- Check nodes at boundaries have lower degrees then other check nodes
- Lower degree check nodes decrease the possibility of an erasure
- Better protection of bits at boundaries

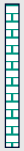
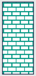
- A Toy Example



Less powerful
but low cost



more powerful
but higher cost

Wall width			
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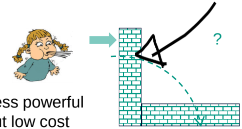
Triggering the Decoding Wave

- A Toy Example

LDPC code



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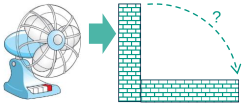

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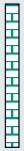
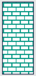
Wall width			
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Triggering the Decoding Wave

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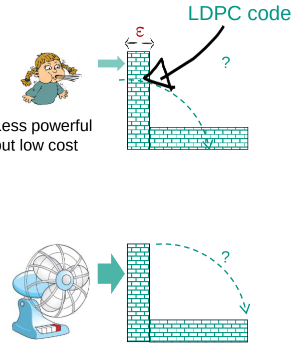

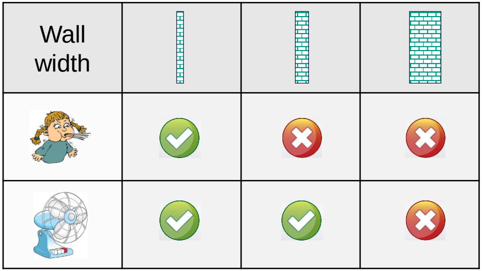

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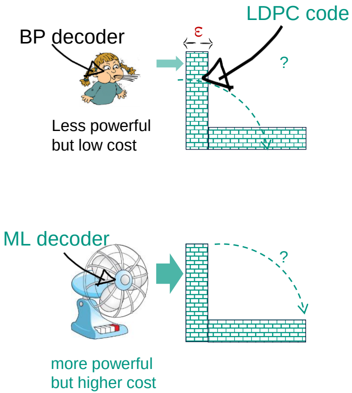

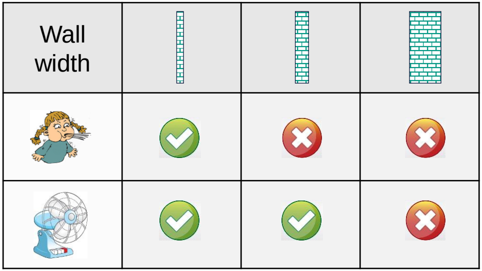

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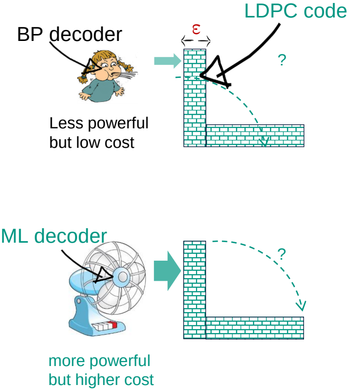

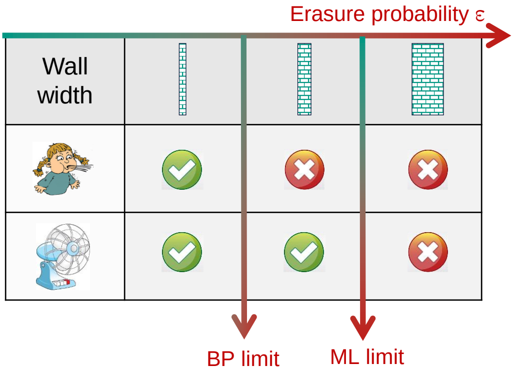

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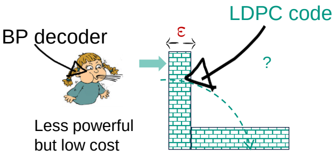
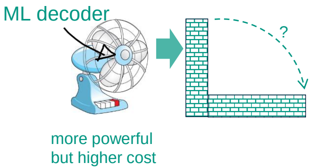
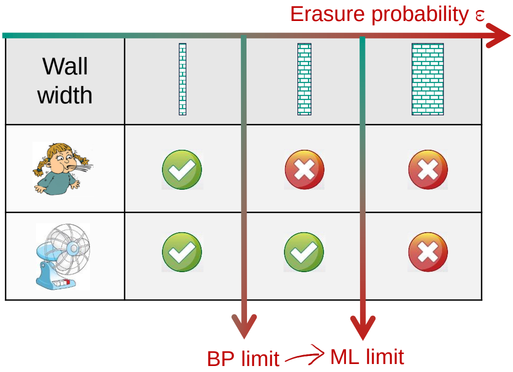
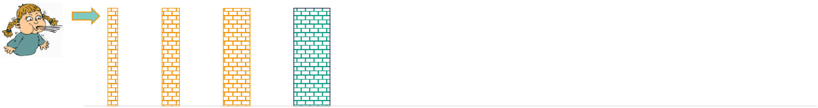






Triggering the Decoding Wave

- Triggering similar to dominoes



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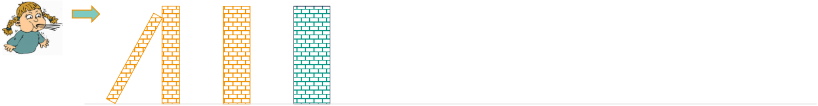

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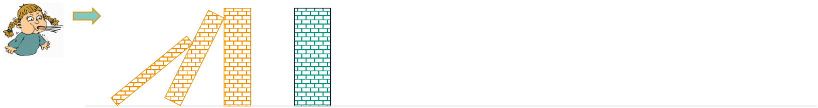

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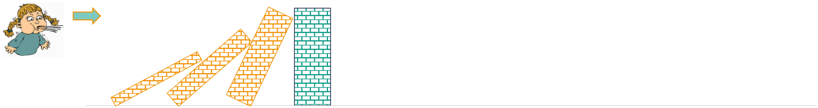

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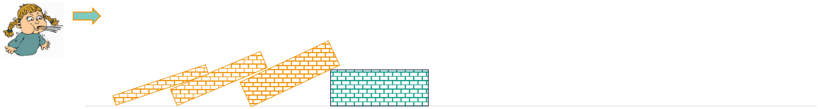

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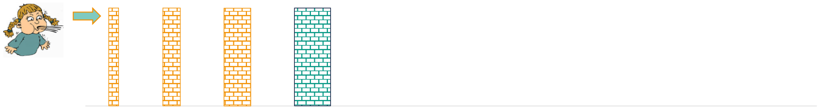

Cost of 3 extra bricks!

Triggering the Decoding Wave

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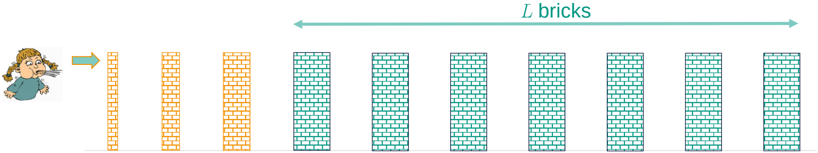
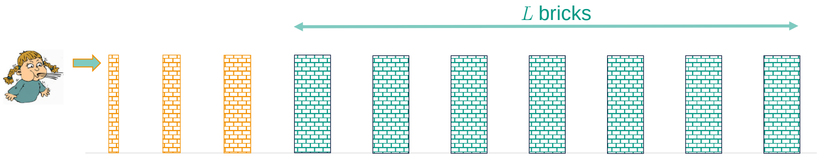


Cost of 3 extra bricks! ➡ The average extra cost per brick is $3/L$

Triggering the Decoding Wave

- Triggering similar to dominoes



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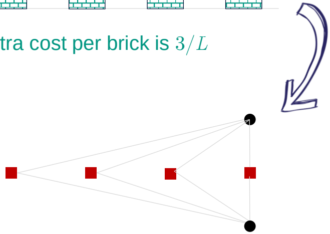
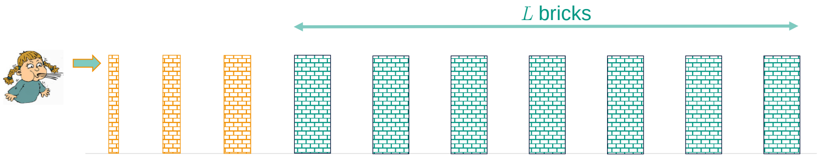


LDPC code

Triggering the Decoding Wave

- Triggering similar to dominoes



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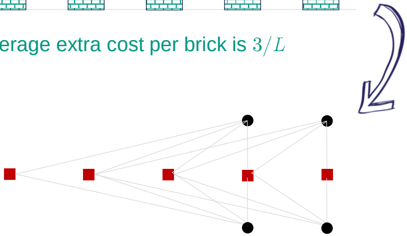

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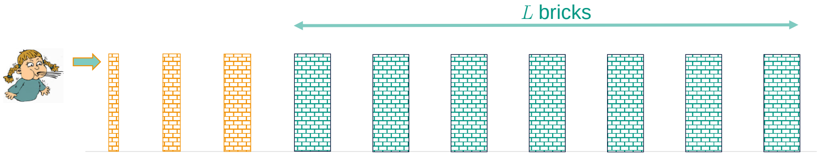

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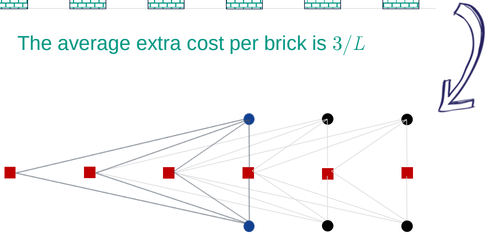

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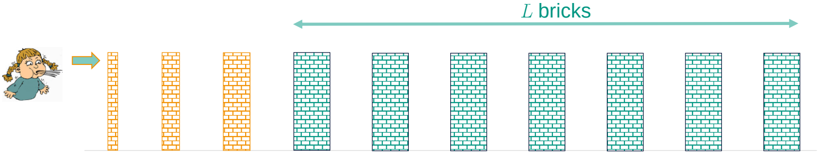

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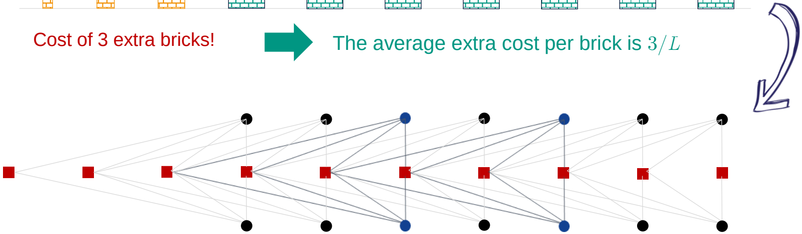

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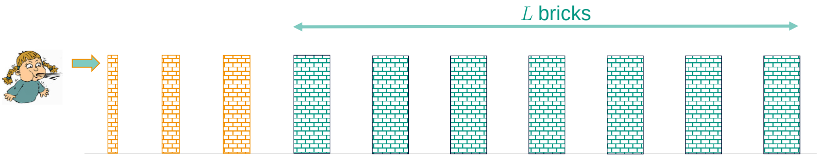

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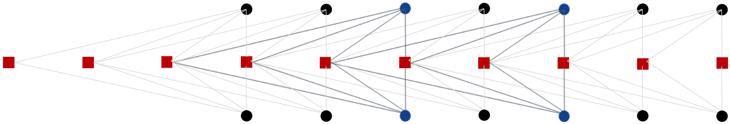
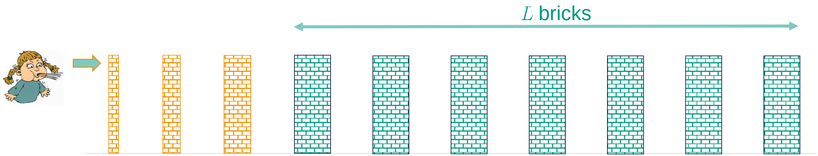


Spatially Coupled LDPC code

Triggering the Decoding Wave

- Triggering similar to dominoes



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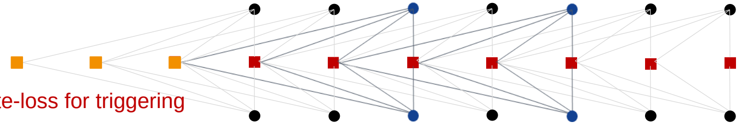

Rate-loss for triggering

Spatially Coupled LDPC code

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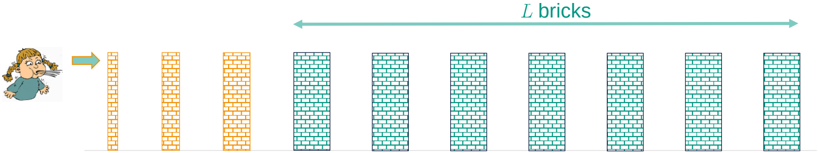

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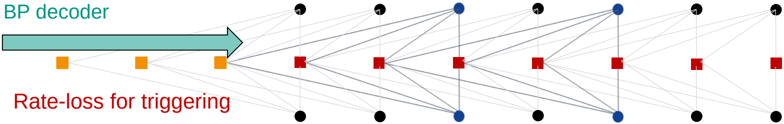
BP decoder



Rate-loss for triggering

Spatially Coupled LDPC code

Spatial Coupling Improves the Threshold

- By running density evolution, we can compute the thresholds

d_v	d_c	w	L	$r_{d,\mathsf{SC}}$	r_d	$\epsilon^\star_{\mathsf{SC}}$	$\epsilon^\star$
3	6	2	100	0.49516	0.5	0.48837	0.42944
3	6	3	100	0.49089	0.5	0.48833	0.42944
4	8	2	100	0.49504	0.5	0.49446	0.38344
4	8	3	100	0.49039	0.5	0.49788	0.38344
5	10	2	100	0.49501	0.5	0.48268	0.34154
5	10	3	100	0.49017	0.5	0.49895	0.34154
5	10	4	100	0.48557	0.5	0.49971	0.34154

- Thresholds of SC-LDPC codes are very close to capacity limit $1 - r_{d,\mathsf{SC}}$
- Note that $r_{d,\mathsf{SC}}$ can be made arbitrarily close to r_d by increasing L
- Note that $\epsilon^\star$ decreases with increasing d_v while $\epsilon^\star_{\mathsf{SC}}$ increases!

Threshold Saturation of Spatially Coupled LDPC Codes

- It has been recently shown [KRU11]

$$\lim_{w\to\infty} \lim_{L\to\infty} r_{d,\mathsf{SC}} = 1 - \frac{d_{\mathsf{v}}}{d_{\mathsf{c}}}$$
$$\lim_{w\to\infty} \lim_{L\to\infty} \epsilon^{\star}_{\mathsf{SC}} = \lim_{w\to\infty} \lim_{L\to\infty} \epsilon^{\mathsf{ML}}_{\mathsf{SC}} = \epsilon^{\mathsf{ML}}$$

- This means that in the limit of L and w (in that order!) the threshold ***saturates*** to the ML performance of the spatially coupled LDPC code
- The ML performance of the spatially coupled code equals (in the limit of L and w) the ML performance of the uncoupled LDPC code!

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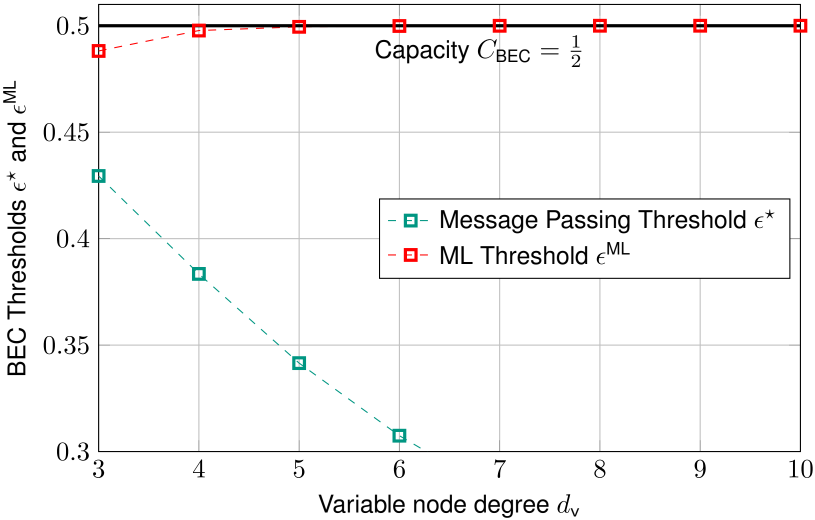
- The ML performance of the spatially coupled code equals (in the limit of L and w) the ML performance of the uncoupled LDPC code!
- This ***threshold saturation*** allows us to
  - Take a block LDPC code which we know has good ML performance
  - Apply spatial coupling with large enough w and L
  - With simple message passing decoding, ML performance can be approached

[KRU11] S. Kudekar, T. Richardson and R. Urbanke, “Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform So Well over the BEC,” IEEE Trans. Inform. Theory, vol. 57, no. 2, Feb. 2011

ML Performance of Regular $[d_{\mathsf{v}}, d_{\mathsf{c}}]$ LDPC Codes



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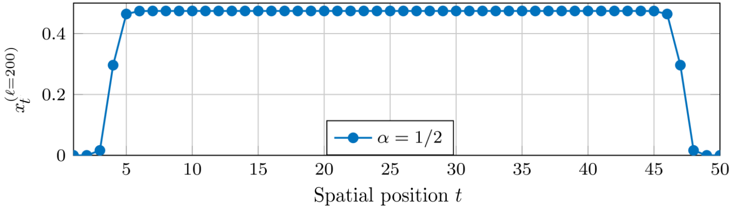
Density Evolution ($\varepsilon = 0.48,\ L = 50$)

- Spatially coupled LDPC Code $[d_\mathrm{v} = 5,\ d_\mathrm{c} = 10,\ w = 2,\ L = 50]$
- Performance after decoding iterations



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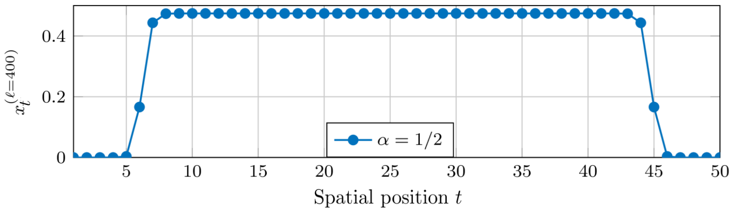

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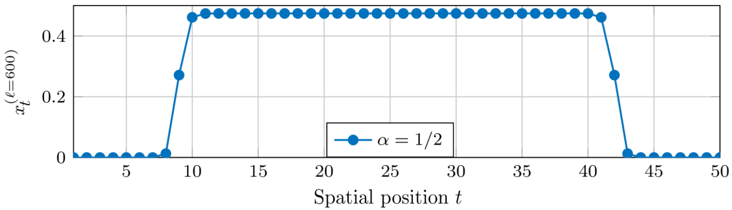

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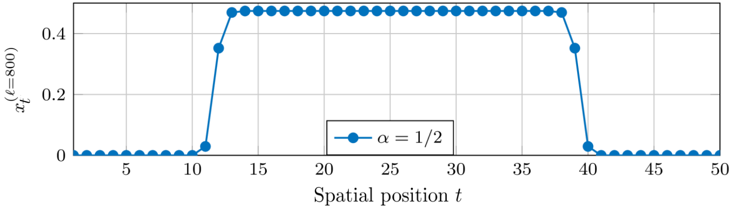

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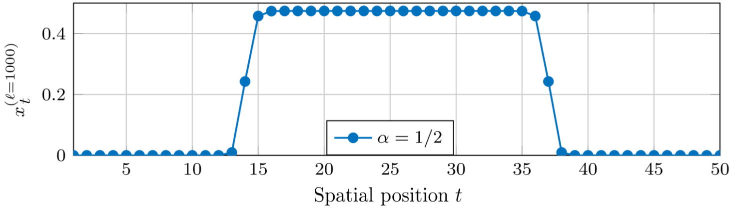

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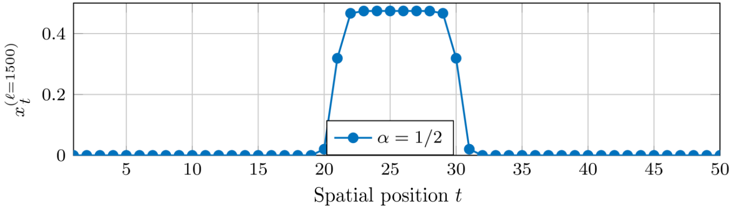

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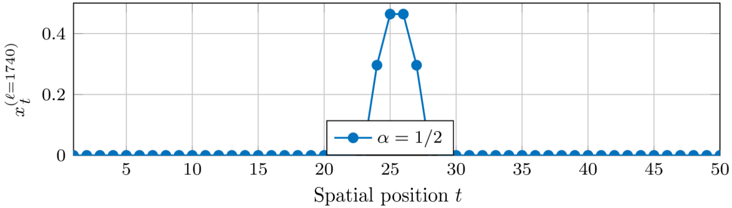

Windowed Decoding

KIT Karlsruhe Institute of Technology

- Spatially coupled LDPC Code $[d_\mathrm{v} = 5,\ d_\mathrm{c} = 10,\ w = 2,\ L = 50]$
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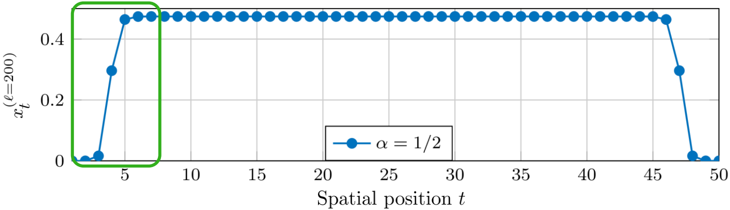

- Windowed decoding sufficient to achieve capacity [ISU+13]
- To save latency, we are only interested in left-most portion of wave and use windowed decoder of size W_D for this part (decode while receive)
- Window latency of order $W_D + w$ ($W_D + w - 1$ SPs in window)
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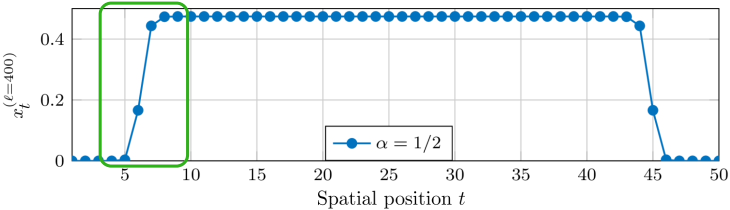

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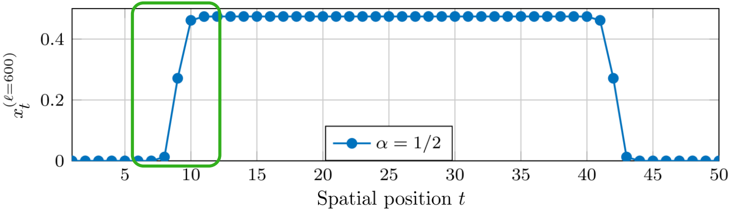

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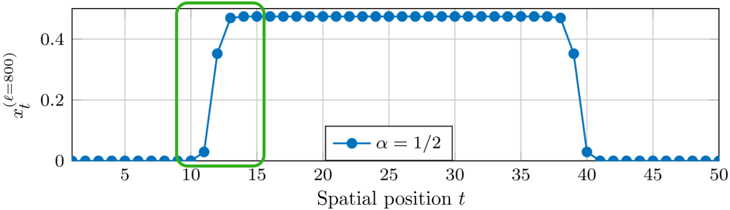

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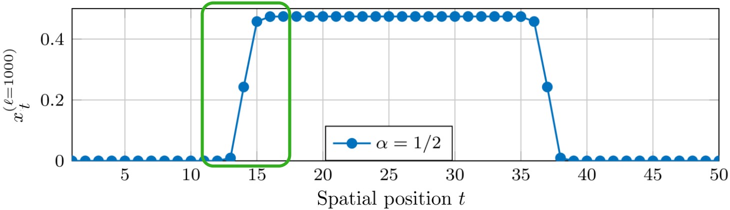

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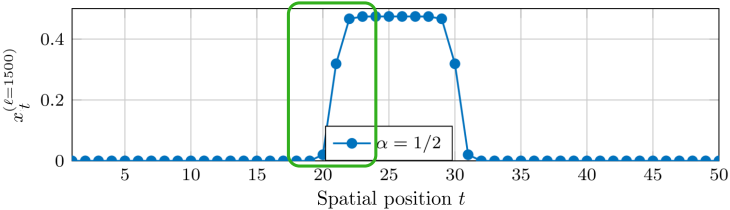

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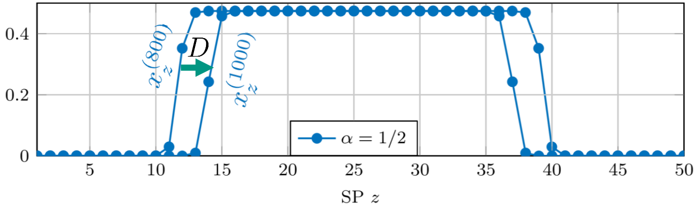
Decoding Velocity and Windowed Decoding



- Decoding velocity as displacement of erasure profile per decoding iteration [AStB13], [EM16]
- Decoding velocity v defined as D/I, where I is the number of iterations required to advance the profile by D, i.e., here $v = D/200$

[AStB13] V. Aref, L. Schmalen, S. ten Brink, "On the convergence speed of spatially coupled LDPC ensembles," *Proc. Allerton Conf.*, 2013
[EM16] R. E-Khatib, N. Macris, "The velocity of the decoding wave for spatially coupled codes on BMS channels," *Proc. ISIT*, 2016
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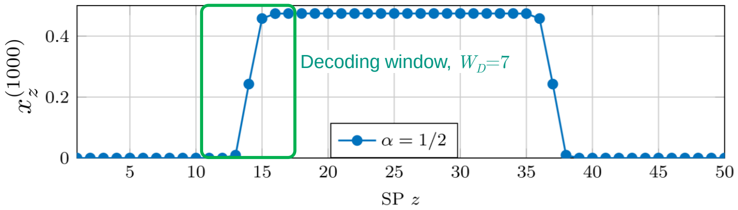

- Windowed decoding only carries out decoding operations on W_D spatial positions that benefit from decoding [ISU+13]
- Complexity of windowed decoding directly linked to the velocity of the profile

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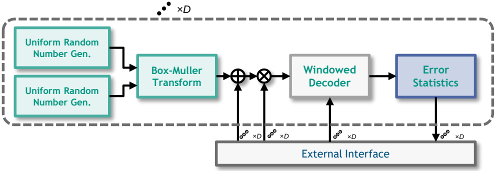
Windowed Decoding

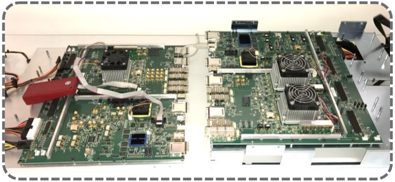
- Use your favorite LDPC decoding algorithm
- Only operate on **parts of the Tanner graph** corresponding to $W_D + w - 1$ spatial positions (the **window**
- After carrying out I iterations on the window, shift the window by 1 spatial position
  - **Important:** Keep all the edge messages, do not re-initialize messages
- Virtual positions assumed to be received without error (shortened)

FPGA-Based Code Evaluation

- High throughputs & large coding gains necessary in optical communication networks & submarine cables
- Required BER: less than 0.0000000000001% (10^{-15})
- Less than 10 bit errors per day at line rate of 100 Gbit/s
- Requirements might become more strict in the future (now 1 Tbit/s)

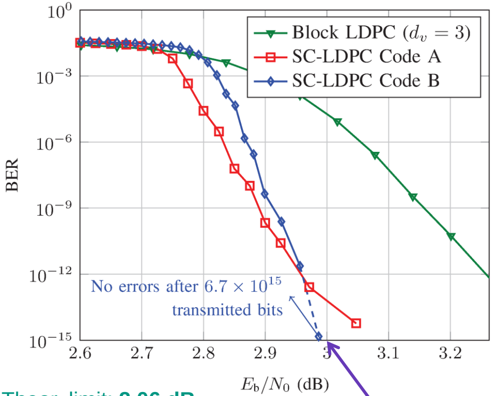
- Simulation of codes using FPGA-platform with windowed decoder





Result of FPGA Decoding Platform

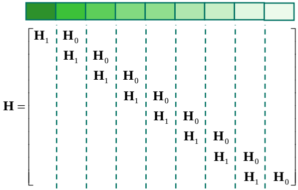
- Codes of rate $r = 0.8$



Theor. limit: **2.06 dB** (asymptotic)

Net coding gain 12.01 dB

- Comparison of two different codes
  - Code A: optimized velocity
  - Code B: optimized error floor
- Dual engine decoder, extra iteration without extra latency [SSR+15]



[SSR+15] L. Schmalen, D. Suikat, D. Rösener, V. Aref, A. Leven, S. ten Brink "Spatially coupled codes and optical fiber communications: An ideal match?," *Proc. Workshop on Signal Processing Advances in Wireless Communications (SPAWC)*, 2015

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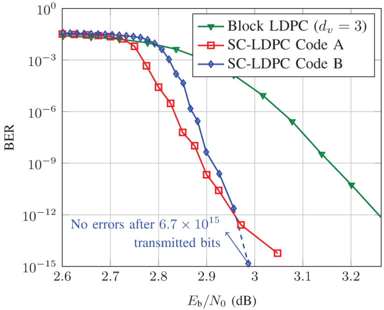

- **0.4dB** gain correspond to **900km reach increase** in trans-pacific cables
- Optical fiber communication systems **age** (material, lasers, photodiodes) and the **SNR will decay over time**
- In this case, additional gains increase lifetime/reduce margins of a system
- **More gains** are possible **with higher decoding complexity**
- However, **we want even more gains**!

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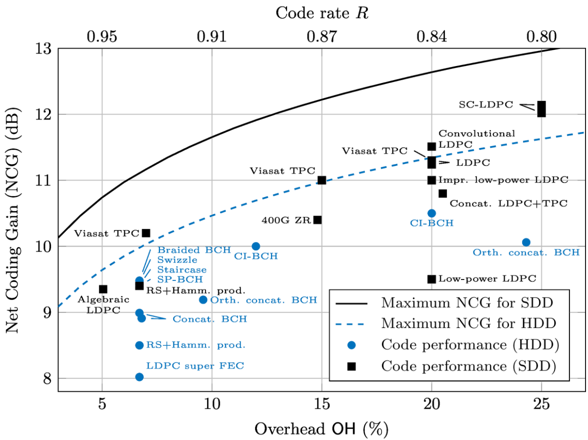
Comparison of Coding Schemes used in Optical Communications



- State-of-the-art coding schemes proposed for practical implementation
- Performance verified or reasonably estimated at 10^{-15} BER

[GS20] A. Graell i Amat and L. Schmalen, “Forward Error Correction for Optical Transponders”, *Springer Handbook of Optical Networks*, B. Mukherjee, I. Tomkos, M. Tornatore, P. Winzer, and Y. Zhao (editors), Springer, October 2020

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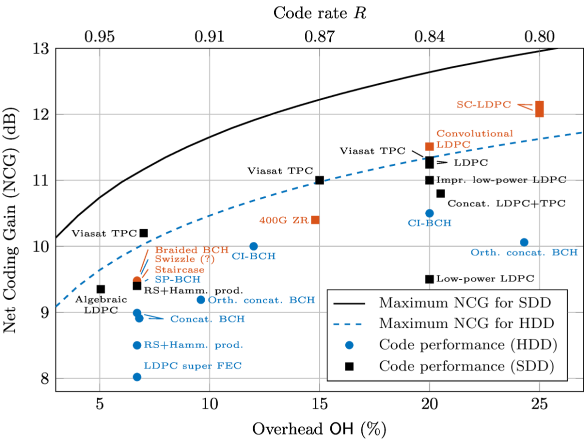

- State-of-the-art coding schemes proposed for practical implementation
- Performance verified or reasonably estimated at 10^{-15} BER
- The best performing schemes are spatially coupled codes

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Improve SC-LDPC Codes

- By running density evolution, we can compute the thresholds

d_v	d_c	w	L	$r_{d,SC}$	r_d	$\epsilon^\star_{SC}$	$\epsilon^\star$
3	6	2	100	0.49516	0.5	0.48837	0.42944
3	6	3	100	0.49089	0.5	0.48833	0.42944
4	8	2	100	0.49504	0.5	0.49446	0.38344
4	8	3	100	0.49039	0.5	0.49788	0.38344
5	10	2	100	0.49501	0.5	0.48268	0.34154
5	10	3	100	0.49017	0.5	0.49895	0.34154
5	10	4	100	0.48557	0.5	0.49971	0.34154

- We see that there is **still a gap to capacity**

If you do not get what you expect...

Find ways to generalize the construction!

New: Non-Uniformly Coupled LDPC Codes

- Start with SC-LDPC Code ensemble with $w = 2$
- We fixed: probability of coupling an edge with neighboring position ($1/w$)
- **New: generalization of coupling**

[SAJ17] L. Schmalen, V. Aref, F. Jardel, "Non-Uniformly Coupled LDPC Codes: Better Thresholds, Smaller Rate-loss, and Less Complexity," *Proc. ISIT*, 2017
[SA19] L. Schmalen and V. Aref, "Spatially Coupled LDPC Codes with Non-uniform Coupling for Improved Decoding Speed," *Proc. ITW*, 2019, available online at https://arxiv.org/abs/1904.07026

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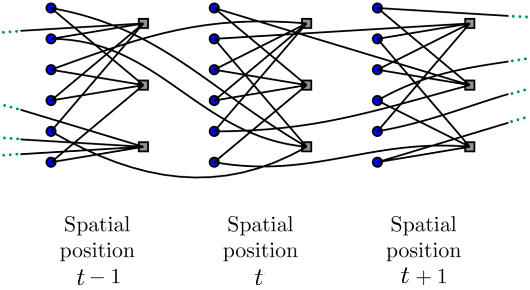

Definition

Connect each edge from variable node at SP z to

- check node at position z with **probability α** and to
- Check node at position $z+1$ with **probability $1-\alpha$**

[SAJ17] L. Schmalen, V. Aref, F. Jardel, "Non-Uniformly Coupled LDPC Codes: Better Thresholds, Smaller Rate-loss, and Less Complexity," *Proc. ISIT*, 2017
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Density Evolution

- Let $\boldsymbol{\nu} = (\nu_0, \ldots, \nu_{w-1})$ denote the coupling vector with $\sum_{i=0}^{w-1} \nu_i = 1$
- In uniform coupling, we have $\boldsymbol{\nu} = \left(\frac{1}{w}, \ldots, \frac{1}{w}\right)$
- For $w = 2$, we have $\boldsymbol{\nu} = (\alpha, 1-\alpha)$

Density Evolution of Non-Uniformy Coupled Codes

The DE update equation for a $[d_{\mathsf{v}}, d_{\mathsf{c}}, \boldsymbol{\nu}, L]$ SC-LDPC code ensemble becomes

$$\xi_t^{(\ell)} = \epsilon \left(1 - \sum_{i=0}^{w-1} \nu_i \left(1 - \sum_{j=0}^{w-1} \nu_j \xi_{t+i-j}^{(\ell-1)} \right)^{d_{\mathsf{c}}-1} \right)^{d_{\mathsf{v}}-1}$$

- Free parameters ν_i can be optimized to yield good thresholds

Rate of Non-Uniformy Coupled Codes

- Non-uniform coupling has an impact on the rate

Rate of Non-Uniformly Coupled Codes

The rate of a non-uniformy coupled $[d_{\mathsf{v}}, d_{\mathsf{c}}, \boldsymbol{\nu}, L]$ SC-LDPC code ensemble amounts $r_{d,\mathsf{SC}} = \left(1 - \frac{d_{\mathsf{v}}}{d_{\mathsf{c}}}\right) - \frac{1}{L}\Delta$, where

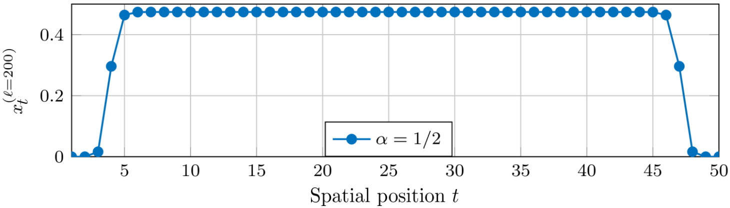
$$\Delta = \frac{d_{\mathsf{v}}}{d_{\mathsf{c}}}\left(w - 1 - \sum_{k=0}^{w-2}\left[\left(\sum_{i=0}^{k}\nu_i\right)^{d_{\mathsf{c}}} + \left(\sum_{i=k+1}^{w-1}\nu_i\right)^{d_{\mathsf{c}}}\right]\right)$$

- Interestingly, for $w = 2$, Δ becomes maximal for $\alpha = \nu_0 = \nu_1 = \frac{1}{2}$, i.e., for $w = 2$, non-uniform coupling reduces the rate loss!
- For general $w > 3$, non-uniform coupling may also lead to larger rate losses

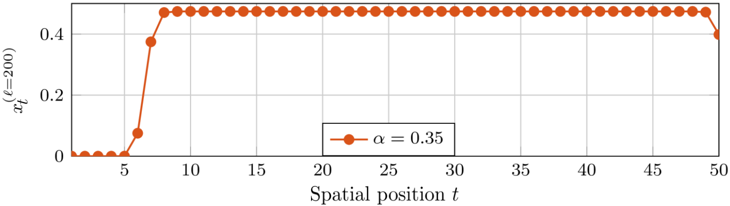
Density Evolution ($\varepsilon = 0.48,\ L = 50$)

Conventional Spatially Coupled LDPC Code
$d_v = 5,\ d_c = 10$

$I = 200$ iter.



New: Non-uniformly coupled code with
$d_v = 5,\ d_c = 10$
Single-side convergence

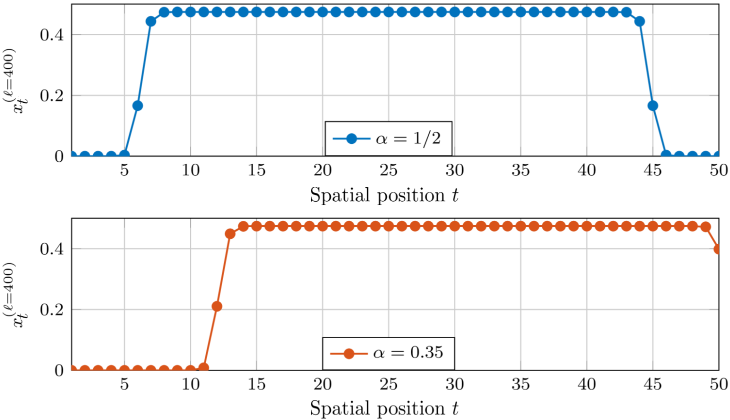


Density Evolution ($\varepsilon = 0.48,\ L = 50$)

Conventional Spatially Coupled LDPC Code
$d_v = 5,\ d_c = 10$

$I = 400$ iter.

New: Non-uniformly coupled code with
$d_v = 5,\ d_c = 10$
Single-side convergence



Density Evolution ($\varepsilon = 0.48,\ L = 50$)

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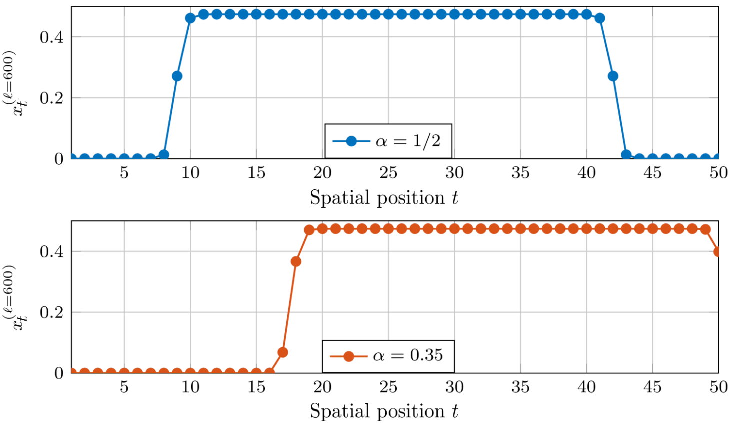
Conventional Spatially Coupled LDPC Code
$d_v = 5,\ d_c = 10$

$I = 600$ iter.

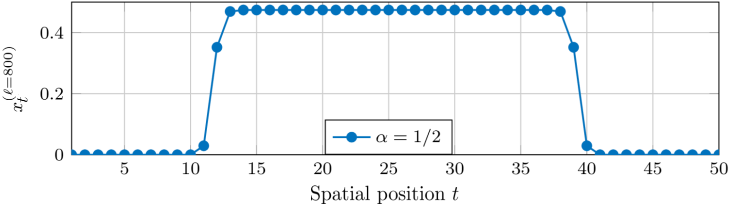
New: Non-uniformly coupled code with
$d_v = 5,\ d_c = 10$
Single-side convergence



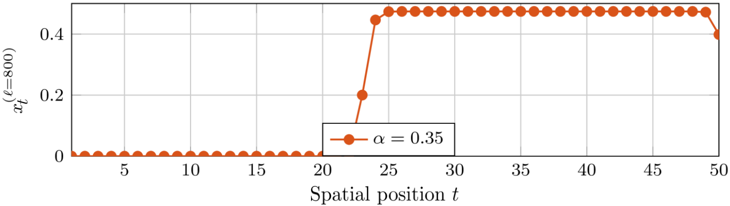
Density Evolution ($\varepsilon = 0.48,\ L = 50$)

Conventional Spatially Coupled LDPC Code
$d_v = 5,\ d_c = 10$

$I = 800$ iter.



New: Non-uniformly coupled code with $d_v = 5,\ d_c = 10$
Single-side convergence

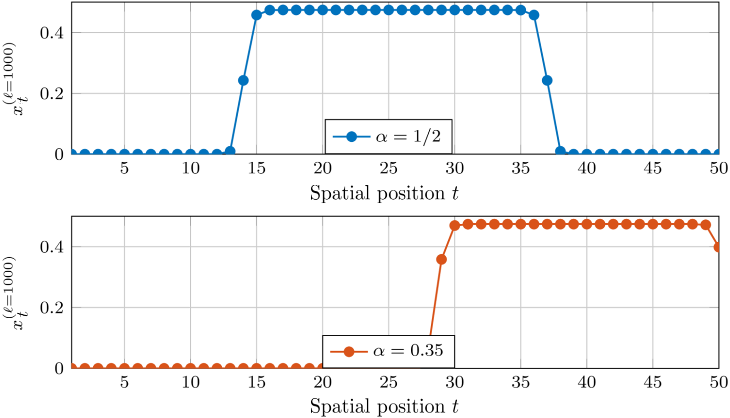


Density Evolution ($\varepsilon = 0.48,\ L = 50$)

Conventional Spatially Coupled LDPC Code
$d_v = 5,\ d_c = 10$

$I = 1000$ iter.

New: Non-uniformly coupled code with
$d_v = 5,\ d_c = 10$
Single-side convergence

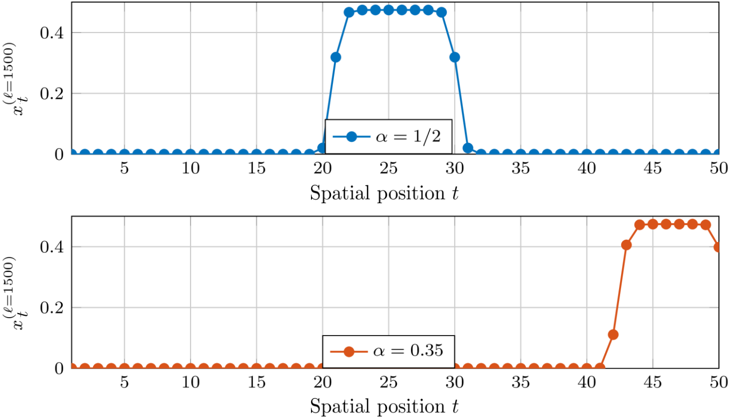


Density Evolution ($\varepsilon = 0.48,\ L = 50$)

Conventional Spatially Coupled LDPC Code
$d_v = 5,\ d_c = 10$

$I = 1500$ iter.

New: Non-uniformly coupled code with
$d_v = 5,\ d_c = 10$
Single-side convergence



Density Evolution ($\varepsilon = 0.48,\ L = 50$)

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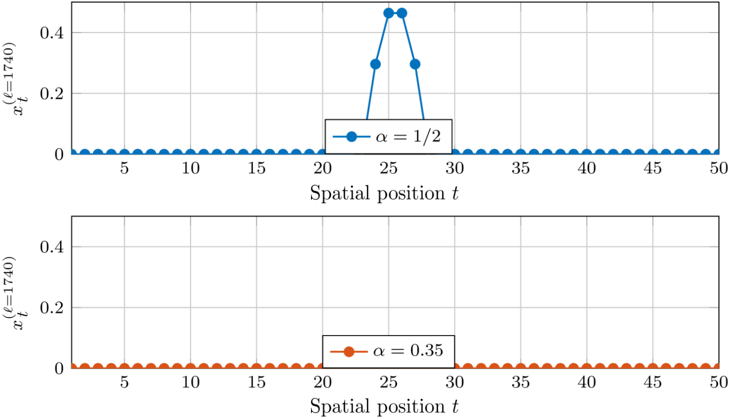
Conventional Spatially Coupled LDPC Code
$d_v = 5,\ d_c = 10$

$I = 1740$ iter.

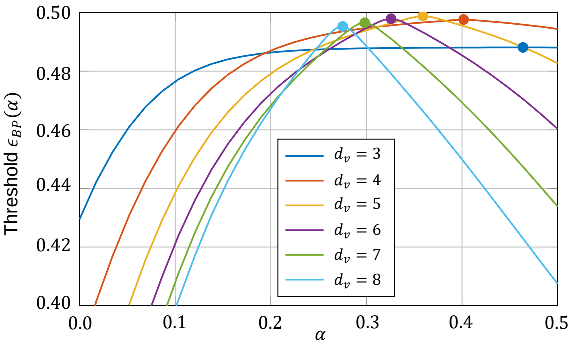


New: Non-uniformly coupled code with $d_v = 5,\ d_c = 10$
Single-side convergence

Less iterations required for full convergence!

Non-Uniform Coupling vs. Conventional Uniform Coupling (1)

- Decoding thresholds: SC-LDPC $[d_v,\ 2d_v,\ w=2,\ \alpha,\ L=100]$ over the BEC



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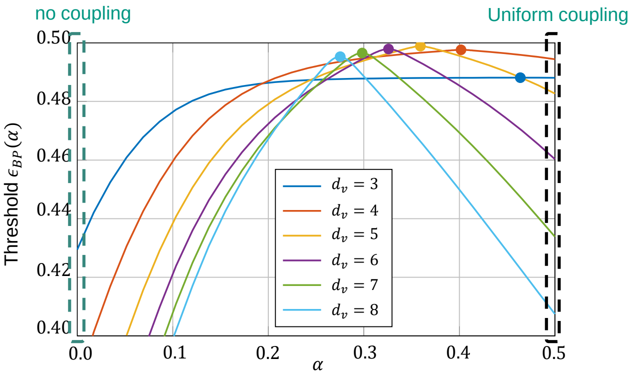

Non-Uniform Coupling with $w > 2$

- Thresholds already sufficiently close to capacity with $w > 2$
- Improvement of decoding speed (speed of wave)

- Optimizing of coupling vectors $\boldsymbol{\nu}$ using numerical methods
- Using tool of **differential evolution**
- Cost function is speed of decoding wave (estimated by curve fitting from profile)
- Details omitted here but can be found in **[SA19]**

[SA19] L. Schmalen and V. Aref, “Spatially Coupled LDPC Codes with Non-uniform Coupling for Improved Decoding Speed,” *Proc. ITW*, 2019, available online at `https://arxiv.org/abs/1904.07026`

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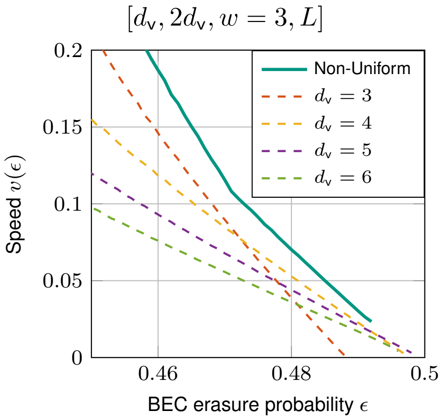

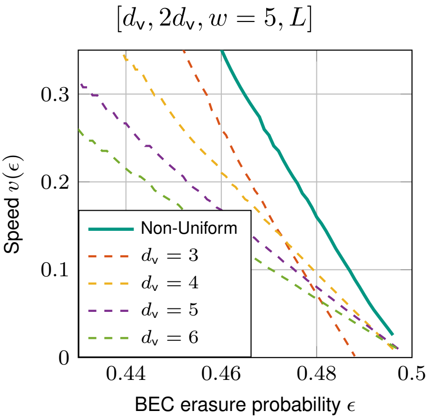

Demonstration

- Demonstration of SC-LDPC Code Decoding

Overview

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   - Motivation and Definition
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   - Improvement of SC-LDPC Codes by Non-Uniform Coupling
3. Burst Correction Capabilities of Spatially Coupled LDPC Codes
   - Error Probability after Burst Erasures
   - Application Example
4. Conclusions

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4 Conclusions

Burst Erasure of a Complete Spatial Position

- SC-LDPC codes have some other unique properties, we will highlight just one of them
- Consider the following scenario:
  - All spatial positions have been correctly received with the exception of position t_b, which is ***completely erased***
  - This can model different scenarios, for instance packet loss in data networks, or server failure in distributed storage, or impulse noise in xDSL systems
- When can we recover from this burst?

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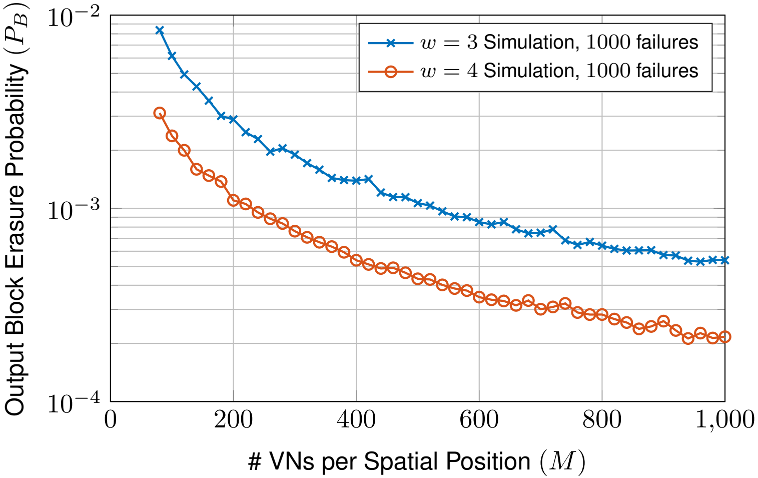
- When can we recover from this burst?

Theorem

Consider an $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ SC-LDPC code where $L-1$ spatial positions have been correctly received and one spatial position is completely erased. A necessary condition for recovering this spatial position is that $w > (\epsilon^\star)^{-1}$, where $\epsilon^\star$ is the BEC decoding threshold of the regular $[d_\mathsf{v}, d_\mathsf{c}]$ LDPC code.

Example: $[d_\mathsf{v} = 3, d_\mathsf{c} = 6, w, L]$ SC-LDPC

- Construct codes with M variable nodes per spatial position
- One position erased, other positions without error



What is going on?

- We should be able to recover the burst
- What happened?

What is going on?

- We should be able to recover the burst
- What happened?

- **Finite-length effect due to the construction**
- A burst on some structures is not recoverable
- We will just give a few hints on what is going on
- Full analysis and many extensions in **[ARS18]**

[ARS18] V. Aref, N. Rengaswamy and L. Schmalen, "Finite-length analysis of spatially-coupled regular LDPC ensembles on burst erasure channels," *IEEE Transactions on Information Theory*, vol. 64, no. 5, pp. 3431–3449, 2018, available online at `https://arxiv.org/abs/1611.08267`

Probability of Stopping Set

Lemma (Probability that two variable nodes form a stopping set [ARS18])

Consider a code sampled randomly from the uniformly coupled $[d_{\mathsf{v}}, d_{\mathsf{c}}, w, L]$ *ensemble with* M *variable nodes per spatial position (SP). The probability that two variable nodes from SP* t *form a stopping set is*

$$P_{\mathcal{R}} = \frac{\left(1 - \frac{1}{d_{\mathsf{c}}}\right)^{d_{\mathsf{v}}}}{\sum_{\ell=0}^{d_{\mathsf{v}}} \binom{d_{\mathsf{v}}}{\ell} \binom{wM\frac{d_{\mathsf{v}}}{d_{\mathsf{c}}} - d_{\mathsf{v}}}{d_{\mathsf{v}} - \ell} \left(1 - \frac{1}{d_{\mathsf{c}}}\right)^{\ell}} \quad (1)$$

- The proof is based on a counting argument and purely combinatoric [ARS18]
- If M grows, the probability $P_{\mathcal{R}}$ decreases

[ARS18] V. Aref, N. Rengaswamy and L. Schmalen, “Finite-length analysis of spatially-coupled regular LDPC ensembles on burst erasure channels,” *IEEE Transactions on Information Theory*, vol. 64, no. 5, pp. 3431–3449, 2018, available online at https://arxiv.org/abs/1611.08267

Error Probability Due to Stopping Sets

Lemma (Second moment method)

Let $X \geq 0$ be a positive random variable with finite variance. Then

$$P(X > 0) \geq \frac{(\mathbb{E}\{X\})^2}{\mathbb{E}\{X^2\}}$$

Lemma (Joint expectation of two stopping sets [ARS18])

Consider one SP of a code sampled from uniformly coupled $[d_{\mathsf{v}}, d_{\mathsf{c}}, w, L]$ ensemble with M variable nodes per spatial position (SP). Recall the indicator function $U_{ij} = 1$ if VNs v_i and v_j in that SP form a size-2 stopping set. Assuming $d_{\mathsf{c}} > 2$ and $wM \geq 2(d_{\mathsf{v}} + 1)d_{\mathsf{c}}$, we have the following bound on the joint expectation

$$\mathbb{E}\{U_{ij}U_{kl}\} \leq \frac{2\mathbb{E}\{U_{ij}\}}{\binom{wM\frac{d_{\mathsf{v}}}{d_{\mathsf{c}}} - 2d_{\mathsf{v}}}{d_{\mathsf{v}}}}$$

Error Probability After Decoding

Theorem

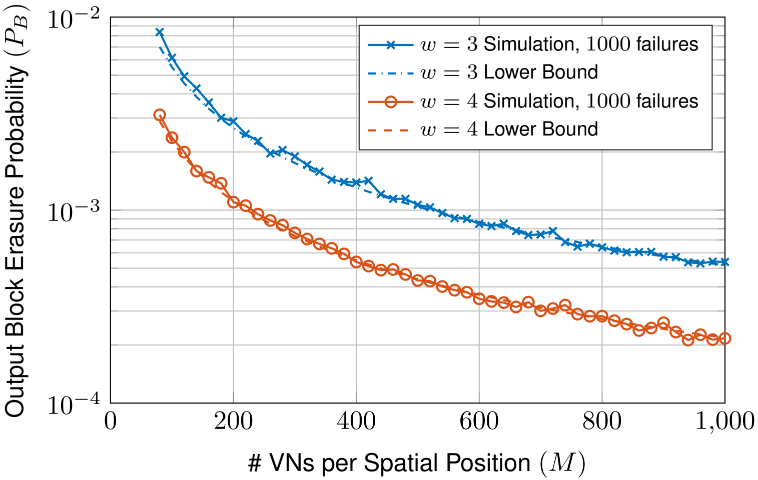
Consider a code sampled uniformly from the $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ *ensemble with* M *variable nodes per spatial position (SP) with w* $d_\mathsf{c} > 2$ *and* $wM \geq 2(d_\mathsf{v}+1)d_\mathsf{c}$. *If all variable nodes of a randomly chosen SP are erased, the (average) probability of BP decoding failure is lower-bounded by*

$$P_\mathrm{B}^\mathrm{SPBC} \geq \binom{M}{2}\left(1 - \frac{M^2}{\left(\frac{w}{d_\mathsf{c}}M - 3\right)^{d_\mathsf{v}}}\right)P_\mathcal{R}$$

where $P_\mathcal{R}$ *is the probability that two variable nodes from an SP of the code form a stopping set, given by* (1).

Example: $[d_\mathsf{v}=3, d_\mathsf{c}=6, w, L]$ SC-LDPC

- Construct codes with M variable nodes per spatial position



Can We Do Better?

- The error rates are pretty depressing, even with large w
- How can we do better?

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- How can we do better?

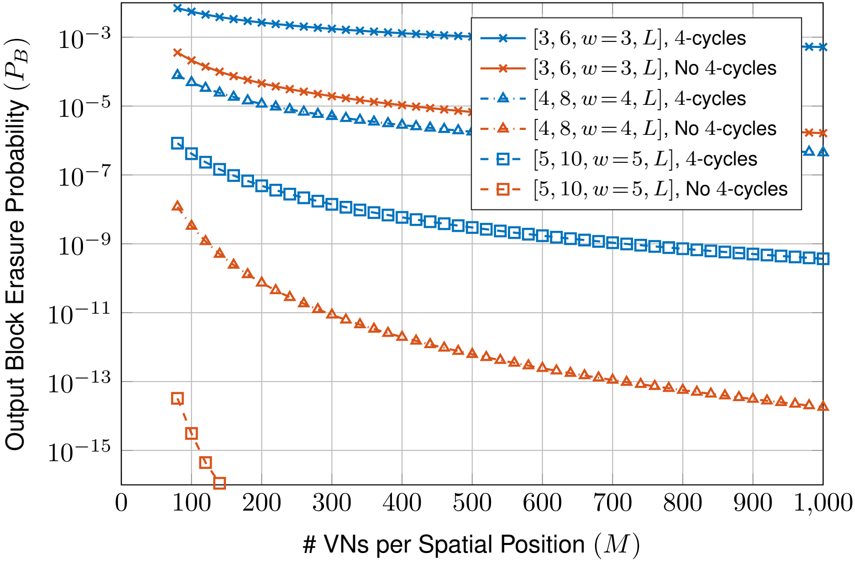
- If size-2 stopping sets cause most of the error floor, **expurgate them!**
- Increasing the girth of the graph to 6 (no length-4 cycles) yields to a minimum stopping set size of

$$s_{\min} = d_{\mathsf{v}} + 1$$

- We can approximate the probability that $d_{\mathsf{v}} + 1$ VNs form a stopping set as

$$P_{\mathcal{R},6} \approx \frac{\left(1 - \frac{1}{d_{\mathsf{c}}}\right)^{\frac{1}{2}d_{\mathsf{v}}(d_{\mathsf{v}}+1)} \frac{(d_{\mathsf{v}}!)^{d_{\mathsf{v}}+1}}{\left(\frac{1}{2}d_{\mathsf{v}}(d_{\mathsf{v}}+1)\right)!}}{\binom{wM\frac{d_{\mathsf{v}}}{d_{\mathsf{c}}} - \frac{1}{2}d_{\mathsf{v}}\left(d_{\mathsf{v}}+1\right)}{\frac{1}{2}d_{\mathsf{v}}\left(d_{\mathsf{v}}+1\right)}}$$

Effect of Expurgation



Additional Erasures in Remaining SPs

- We now consider the case where

$$\epsilon_{t_b} = 1$$
$$\epsilon_{\sim t_b} = \epsilon$$

i.e., one SP is completely erased, all other SPs transmitted over BEC(ϵ)

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Error Probability for Both Errors and Erasures

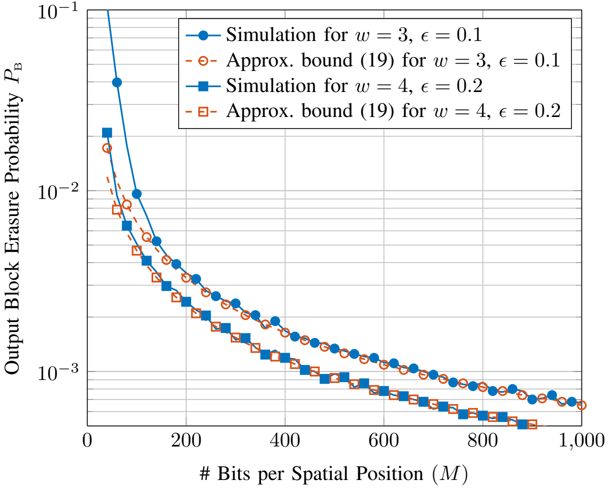
By counting all possibilities for stopping sets (which can now form between all possible positions), we get the approximation

$$P_B \gtrapprox \lambda_0 + \epsilon(2-\epsilon)\sum_{k=1}^{w}\lambda_k + \epsilon^2\sum_{k=0}^{w-1}(L-k-1)\lambda_k$$

with

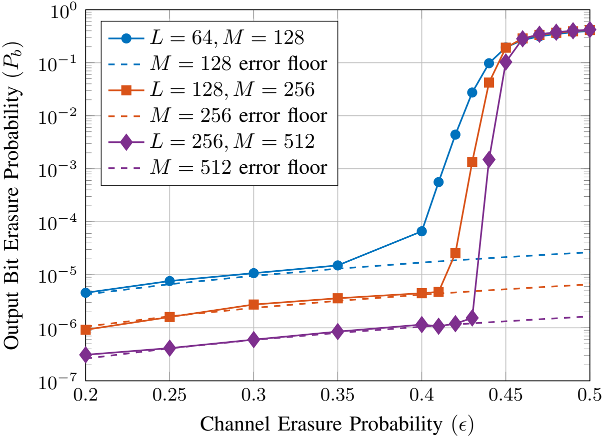
$$\lambda_0 = \binom{M}{2}P_{\mathcal{R}} \qquad \text{and} \qquad \lambda_k = M^2\left(1-\frac{k}{w}\right)^{d_\mathsf{v}} P_{\mathcal{R}} \quad (k \in \{1,\ldots,w-1\})$$

Simulation Results (1)



- $[3, 6, w, L = 10]$ ensemble (design rate $\frac{1}{2}$
- $\epsilon_{t_b} = 1$, $\epsilon_{\sim t_b} = \epsilon$
- Again, we can accurately predict the error floor

Simulation Results (2)



- $[3, 6, w, L]$ ensemble (design rate $\frac{1}{2}$
- $\epsilon_{t_b} = 1$, $\epsilon_{\sim t_b} = \epsilon$
- Again, we can accurately predict the error floor

Random Burst Channel

- Now: Burst of length b starts at variable node S at SP t_b
- Erased bit positions in codeword

$$E = \{(t_b - 1)M + S, \ldots, (t_b - 1)M + S + b - 1\}$$

- We introduce the normalized quantities

$$\beta = \frac{b}{M} \qquad \text{and} \qquad s = \frac{S}{M}$$

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Theorem

Consider the $[d_\mathsf{v}, d_\mathsf{c}, w, L)$ SC-LDPC ensemble and transmission over a channel where a random selection of b consecutive VNs are erased and all other VNs are received without erasure. Let $\beta = \frac{b}{M}$ denote the normalized burst length. Furthermore, let $w \geq 1 + \lceil \beta \rceil$. Asymptotically (in the limit of M), the erased VNs can be recovered when $w \geq \left\lceil (1 + \lceil \beta \rceil)/\epsilon^\star_{\mathsf{LDPC}[d_\mathsf{v}, d_\mathsf{c}]} \right\rceil$, where $\epsilon^\star_{\mathsf{LDPC}[d_\mathsf{v}, d_\mathsf{c}]}$ is the BP threshold of the underlying $[d_\mathsf{v}, d_\mathsf{c}])$ uncoupled LDPC ensemble.

Correctable Burst

- We consider a random starting position β
- A burst of length βM and random starting position is recoverable in the limit of M if

$$\beta < \beta_{\max} = \min_{0 \leq s < 1} \beta(s)$$

- We can determine $\beta_{\max}$ using density evolution **[ARS16]**

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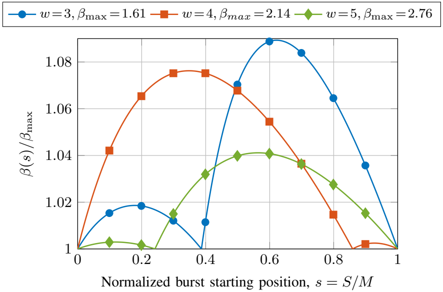


- $[d_{\mathsf{v}} = 3, d_{\mathsf{c}} = 6, w, L]$ ensemble
- Normalized maximum correctable burst length for a given starting position $S = sM$

[ARS16] V. Aref, N. Rengaswamy, and L. S., "Spatially coupled LDPC codes affected by a single random burst of errors," in *Proc. ISTC*, Sep. 2016

Finite-Length Analysis

Theorem

Consider the $[d_{\mathsf{v}}, d_{\mathsf{c}}, w, L]$ *SC-LDPC ensemble affected by a burst of length* $b = \beta M$ *with a random starting bit* $M(t_b - 1) + S$*, where* $1 \leq S \leq M$ *and* $\beta \ll \beta_{\max}$*. The number of erased VNs in SP* z *is given by*

$$m_z = \begin{cases} 0 & z < t_b \\ \min\{b, M - S + 1\} & z = t_b \\ \max\{0, \min\{M, b + S - 1 - (z - t_b)M\}\} & z > t_b. \end{cases}$$

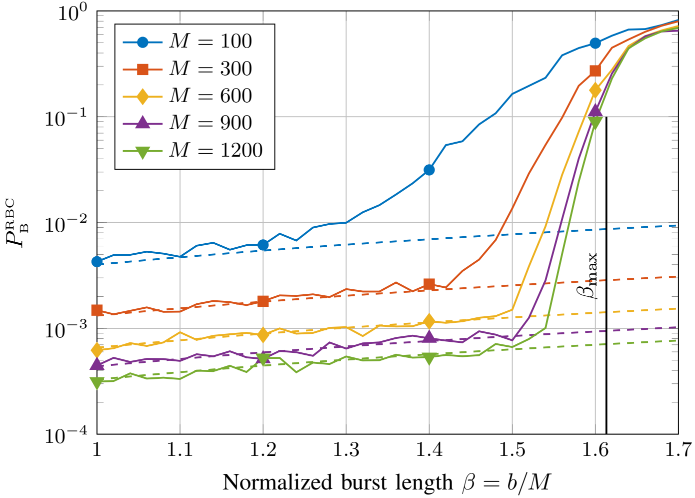
Then the expected number of size-2 *stopping sets formed by VNs erased by the burst is given by*

$$\mathbb{E}[N_2(S, t_b, b)] \gtrapprox \frac{L - \lceil \beta \rceil}{(L - \beta)M + 1} \sum_{S=1}^{M} \sum_{z=1}^{\lceil \beta \rceil + 1} \left(\binom{m_z}{2} q_0 + \sum_{k=1}^{w-1} m_z m_{z+k} q_k \right)$$

with $q_k = P_{\mathcal{R}} \left(1 - \frac{k}{w}\right)^{d_{\mathsf{v}}} \mathbb{1}_{\{k \in \{0, \ldots, w-1\}\}}$.

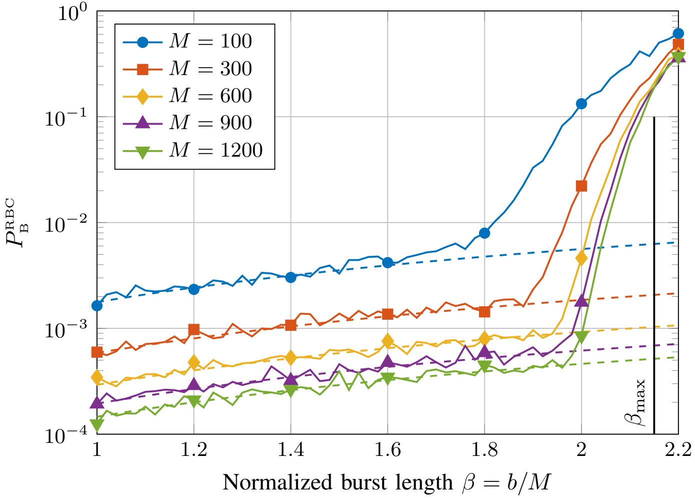
Monte-Carlo Simulations

- The $[d_\mathsf{v} = 3, d_\mathsf{c} = 6, w = 3, L]$ ensemble



Monte-Carlo Simulations

- The $[d_\mathsf{v} = 3, d_\mathsf{c} = 6, w = 4, L]$ ensemble



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3 Burst Correction Capabilities of Spatially Coupled LDPC Codes
- Error Probability after Burst Erasures
- Application Example

4 Conclusions

Applications

- Burst erasures come in many varieties
- Collision in coded random access schemes (e.g., slotted ALOHA) **[LPL+12]**
- Strong fading in block fading channels **[ULA+14]**
- Error due to misestimation of channel
- Impulse noise in wireline communications (e.g., DSL)
- Server failure in distributed storage (cloud storage) **[JBS+15]**

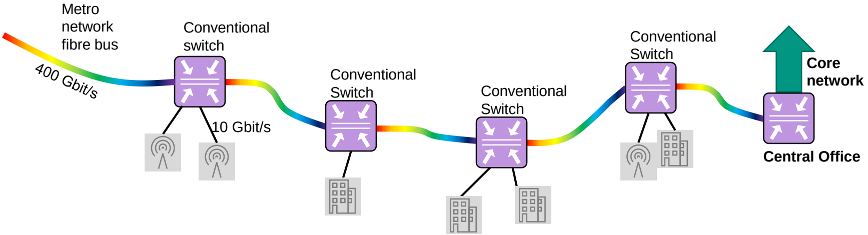
- Application: **Distributed coding**

[LPL+12] G. Liva, E. Paolini, M. Lentmaier, and M. Chiani, “Spatially-coupled random access on graphs,” in Proc. IEEE Int. Symp. Inform. Theory (ISIT), pp. 478–482, 2012.

[ULA+14] N. Ul Hassan, M. Lentmaier, I. Andriyanova, and G. Fettweis, “Improving code diversity on block-fading channels by spatial coupling,” in Proc. IEEE Int. Symp. Inform. Theory (ISIT), pp. 2311–2315, June 2014

[JBS+15] F. Jardel, J. J. Boutros, M. Sarkiss, and G. Rekaya-Ben Othman, “Spatial coupling for distributed storage and diversity applications,” in Proc.IEEE International Conference in Communications and Networking (ComNet), (Hammamet, Tunisia), Nov. 2015

Multi-Point-to-Point Aggregation Bus



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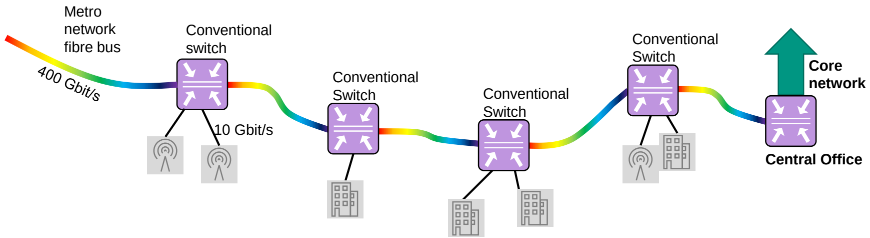

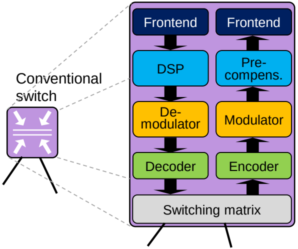

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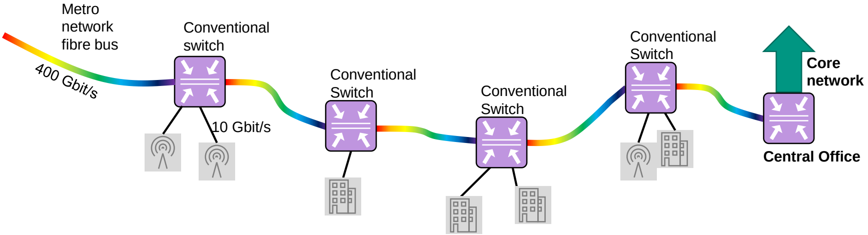
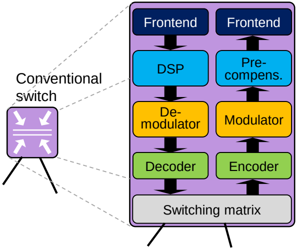
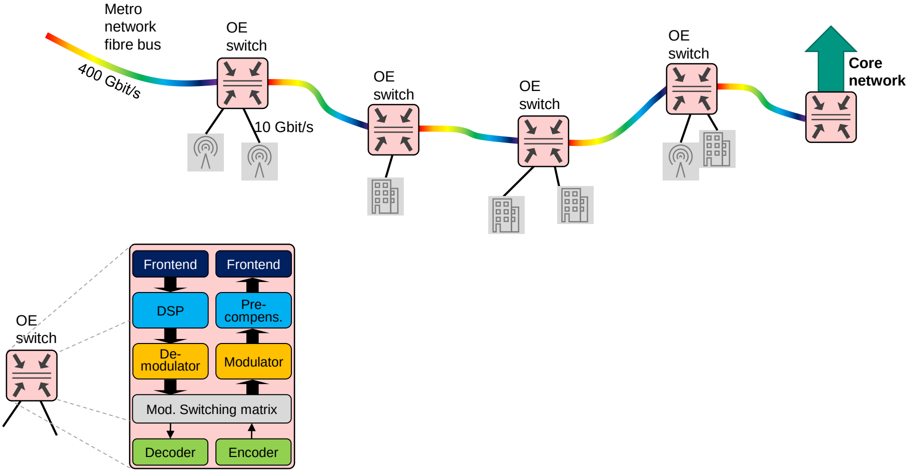




- Switch has an optical coherent front-end with full encoding and decoding in each hop
- Significant amount of data will not be dropped, but re-encoded for next hop
- Full decoding often not necessary in each hop, but only at destimation

Multi-Point-to-Point Aggregation Bus



[LBB+17] W. Lautenschläger, N. Benzaoui, F. Buchali, L. Dembeck, R. Dischler, B. Franz, U. Gebhard, J. Milbrandt, Y. Pointurier, D. Rösener, L. S. and A. Leven "Optical Ethernet – Flexible Optical Metro Networks," *IEEE/OSA J. Lightw. Technol.*, 2017

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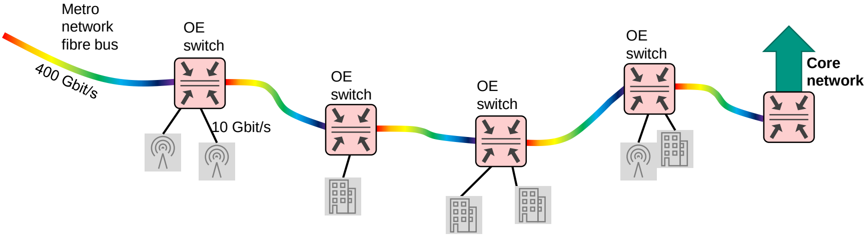

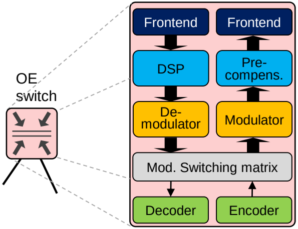

- Based on header, perform switching on noisy frames after hard-decision

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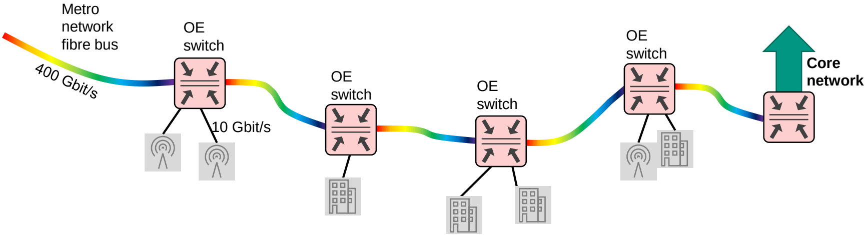

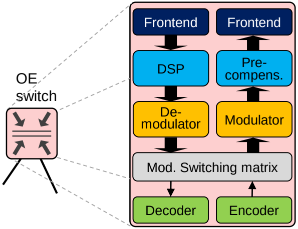

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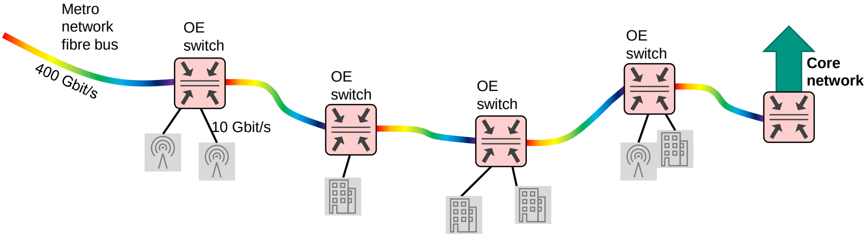

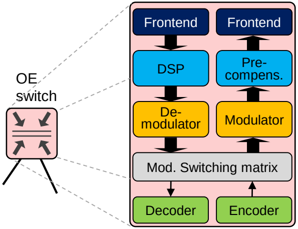

- Based on header, perform switching on noisy frames after hard-decision
- Carry out decoding only for data that are being switched
- Decision-and-forward relay

[LBB+17] W. Lautenschläger, N. Benzaoui, F. Buchali, L. Dembeck, R. Dischler, B. Franz, U. Gebhard, J. Milbrandt, Y. Pointurier, D. Rösener, L. S. and A. Leven "Optical Ethernet – Flexible Optical Metro Networks," *IEEE/OSA J. Lightw. Technol.*, 2017

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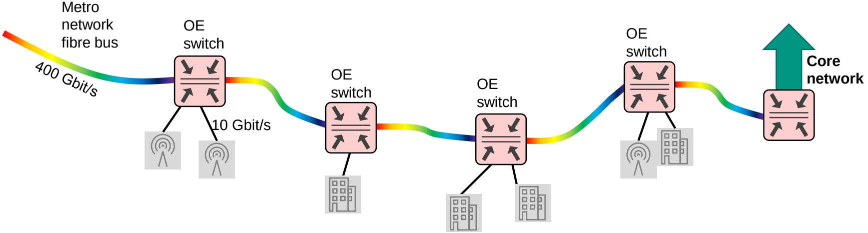

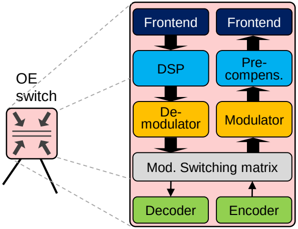

- Container-based transmission of codewords
- Adding data to empty parts of codeword

“0”

[LBB+17] W. Lautenschläger, N. Benzaoui, F. Buchali, L. Dembeck, R. Dischler, B. Franz, U. Gebhard, J. Milbrandt, Y. Pointurier, D. Rösener, L. S. and A. Leven “Optical Ethernet – Flexible Optical Metro Networks,” *IEEE/OSA J. Lightw. Technol.*, 2017

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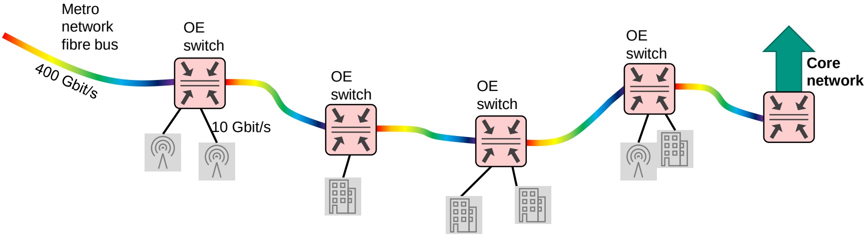

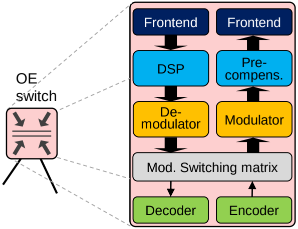

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Container	Data 1	"0"	Parity 1

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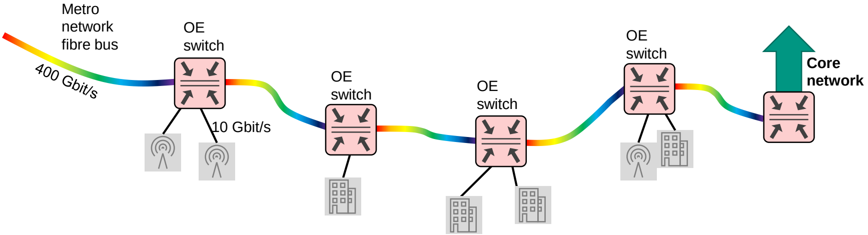

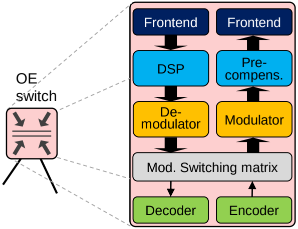

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Container	Data 1		"0"	Parity 1
+				
New data	"0"	Data 2	"0"	Parity 2

[LBB+17] W. Lautenschläger, N. Benzaoui, F. Buchali, L. Dembeck, R. Dischler, B. Franz, U. Gebhard, J. Milbrandt, Y. Pointurier, D. Rösener, L. S. and A. Leven "Optical Ethernet – Flexible Optical Metro Networks," *IEEE/OSA J. Lightw. Technol.*, 2017

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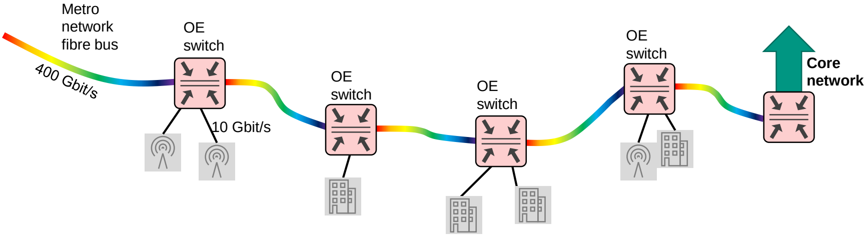

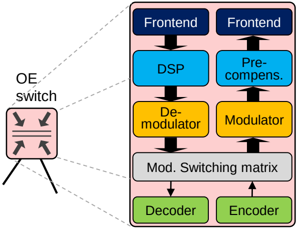

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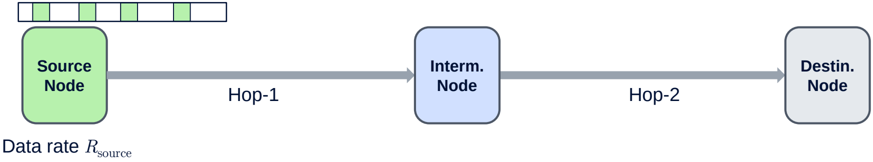
- Adding data to empty parts of codeword

Container	Data 1	"0"		Parity 1
+				
New data	"0"	Data 2	"0"	Parity 2
=				
New cont.	Data 1	Data 2	"0"	Parity

[LBB+17] W. Lautenschläger, N. Benzaoui, F. Buchali, L. Dembeck, R. Dischler, B. Franz, U. Gebhard, J. Milbrandt, Y. Pointurier, D. Rösener, L. S. and A. Leven "Optical Ethernet – Flexible Optical Metro Networks," *IEEE/OSA J. Lightw. Technol.*, 2017

End-to-end Coding in Low-Latency Networks

- Many applications require **ultra-low networking latency**, e.g., industrial control (Industrie 4.0), high-speed trading, sensor networks, etc…
- New networking architecture with end-to-end channel coding for such applications **[LBB+17], [SEB+17]**
- Toy Example:



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Karlsruhe Institute of Technology

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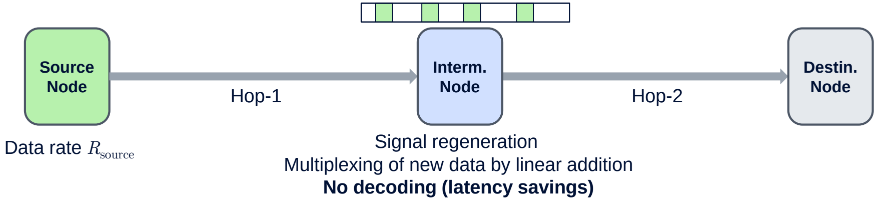

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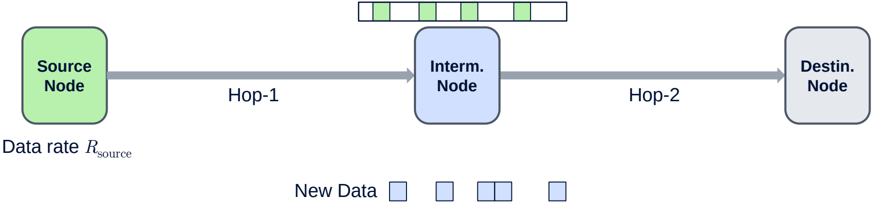

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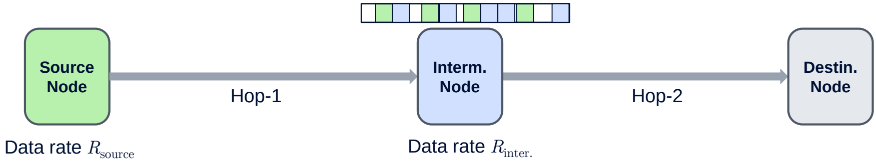

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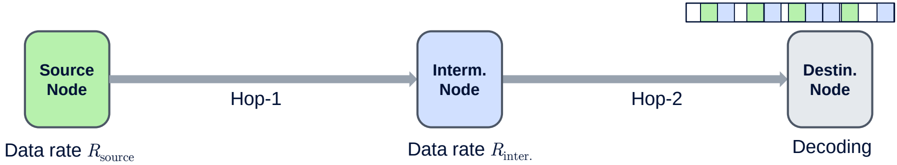

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KIT Karlsruhe Institute of Technology

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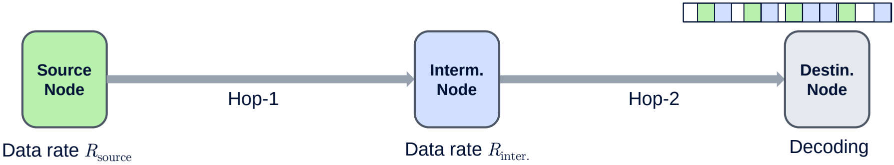

- Packets traveling from source to destination see worse channel than other packets (→ burst!)

[LBB+17] W. Lautenschläger, N. Benzaoui, F. Buchali, L. Dembeck, R. Dischler, B. Franz, U. Gebhard, J. Milbrandt, Y. Pointurier, D. Rösener, L. S. and A. Leven "Optical Ethernet – Flexible Optical Metro Networks," *IEEE/OSA J. Lightw. Technol.*, 2017

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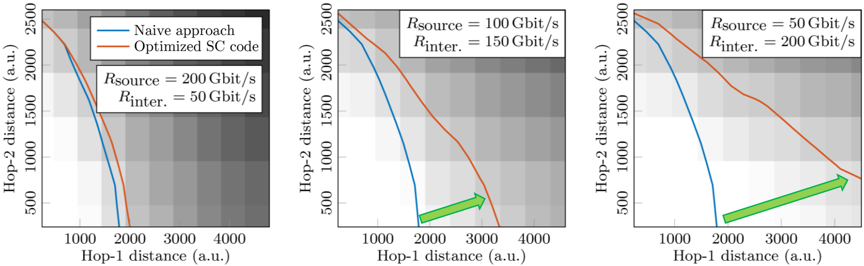
End-to-end Coding in Low-Latency Networks

- **Optimization of code parameters** using burst correction theory
- **Optimization of media access** (data placement) by burst theory
- Flexibility and adaptivity of effective rates to channel conditions
- **Hop-1 and Hop-2 achievable distance pairs significantly improved**



Overview

1. Introduction: Channel Coding and LDPC Codes
2. Spatially Coupled LDPC Codes
   - Motivation and Definition
   - Performance of SC-LDPC Codes
   - Practical Implementation of SC-LDPC Codes
   - Improvement of SC-LDPC Codes by Non-Uniform Coupling
3. Burst Correction Capabilities of Spatially Coupled LDPC Codes
   - Error Probability after Burst Erasures
   - Application Example
4. Conclusions

Conclusions

- Spatially Coupled LDPC Codes are a class of powerful codes that **extend** LDPC codes
- Record coding gains in optical communications
- Best approach to design parameters still unknown (various open research problems)
- General concept that applies to other areas of science as well (compressed sensing, statistical physics, ...)
- Applications in distributed storage, random access, fading channels, subscriber lines, etc...

- **Active area of research**

We Want You!

- **Multiple PhD positions available in**
  - Channel coding and reliable communications
  - Spatially coupled codes
  - High-speed data transmission
  - Machine learning applied to communications
  - Optical fiber communications
  - … and many more

- Do you want to join a young, dynamic group?
- Do you want to carry out innovative research in one of Germany largest research universities?

- Contact me for further details:

  `Laurent.Schmalen@kit.edu`

Overview

- Parity-Check Matrix of a Spatially Coupled LDPC Code
- ML Performance of LDPC Codes
- Analysis of SC-LDPC Codes over General Channels
- Simulation Example of Non-Uniform Coupling
- Burst erasure of complete spatial position
- Error Probability after Decoding with Burst Erasure
- The Block Erasure Channel

Parity-Check Matrix of a Spatially Coupled LDPC Code

- The parity-check matrix of a general SC-LDPC with coupling width w

$$
\boldsymbol{H}_{\text{SC}} = \begin{pmatrix}
\boldsymbol{H}_1(1) & & & & & & \\
\boldsymbol{H}_2(1) & \ddots & & & & & \\
\vdots & \ddots & \boldsymbol{H}_1(t) & & & & \\
\boldsymbol{H}_w(1) & \vdots & \boldsymbol{H}_2(t) & \boldsymbol{H}_1(t+1) & & & \\
 & \ddots & \vdots & \boldsymbol{H}_2(t+1) & \boldsymbol{H}_1(t+2) & & \\
 & & \boldsymbol{H}_w(t) & \vdots & \boldsymbol{H}_2(t+2) & \ddots & \\
 & & & \boldsymbol{H}_w(t+1) & \vdots & \ddots & \boldsymbol{H}_1(L) \\
 & & & & \boldsymbol{H}_w(t+2) & \vdots & \boldsymbol{H}_2(L) \\
 & & & & & \ddots & \vdots \\
 & & & & & & \boldsymbol{H}_w(L)
\end{pmatrix}
$$

is of size $\dim \boldsymbol{H}_{\text{SC}} = (L+w-1)m' \times Ln'$ where $\dim \boldsymbol{H}_i(t) = m' \times n'$.

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ML Dec. of Regular $[d_\mathsf{v}, d_\mathsf{c}]$ LDPC Codes

- The ML decoding threshold ϵ^{ML} (i.e., the ϵ up to which an ideal ML decoder successfully works) of a regular $[d_\mathsf{v}, d_\mathsf{c}]$ LDPC code on the BEC can be described as follows
- Let x_{ML} be the unique positive solution of the equation

$$x + \frac{1}{d_\mathsf{c}}(1-x)^{d_\mathsf{c}-1}(d_\mathsf{v} + d_\mathsf{v}(d_\mathsf{c}-1)x - d_\mathsf{c}x) - \frac{d_\mathsf{v}}{d_\mathsf{c}} = 0$$

- Then,

$$\epsilon^{\mathsf{ML}} = \frac{x_{\mathsf{ML}}}{\left(1-(1-x_{\mathsf{ML}})^{d_\mathsf{c}-1}\right)^{d_\mathsf{v}-1}}$$

- The proof of this fact is involved and beyond the scope of this lecture, however, we have

$$\lim_{d_\mathsf{v}\to\infty} \epsilon^{\mathsf{ML}} = \frac{d_\mathsf{v}}{d_\mathsf{c}} = 1 - r_d = C_{\mathsf{BEC}}$$

with exponentially fast convergence

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$$\lim_{d_\mathsf{v}\to\infty} \epsilon^{\mathsf{ML}} = \frac{d_\mathsf{v}}{d_\mathsf{c}} = 1 - r_d = C_{\mathsf{BEC}}$$

  with exponentially fast convergence
- We can use spatial coupling to construct capacity achieving codes (with $w\to\infty$, $L\to\infty$, $d_\mathsf{v}\to\infty$, $n\to\infty$, and $\ell\to\infty$)

Overview

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Analysis of SC-LDPC Codes for General Channels

- For more general channels, e.g., the BI-AWGN channel, we can use the Gaussian approximation to analyze the convergence behavior
- For every spatial position t we define mean values $\mu_{\xi,t}^{(\ell)}$ and $\mu_{\chi,t}^{(\ell)}$ which we initialize as

$$\mu_{\xi,t}^{(1)} = \begin{cases} \mu_c & \text{if } t \in \{1,2,\ldots,L\} \\ +\infty & \text{otherwise} \end{cases}$$

  where in practice, we replace $+\infty$ by an appropriately large number.
- The variable node outgoing message mean can be computed as

$$\mu_{\xi,t}^{(\ell)} = \mu_{c,t} + \frac{d_{\mathsf{v}} - 1}{w} \sum_{i=0}^{w-1} \mu_{\chi,t+i}^{(\ell)}$$

  where

$$\mu_{c,t} = \begin{cases} \mu_c & \text{if } t \in \{1,2,\ldots,L\} \\ +\infty & \text{otherwise} \end{cases}$$

Analysis of SC-LDPC Codes for General Channels (2)

- Note that the ***incoming*** message to the check node has a distribution that is a sum of Gaussian pdfs with different means (and variances) and with weight $1/w$.
- The mean of the check node outgoing message at position t can be computed as

$$1-\varphi(\mu_{\chi,t}) = \left[1-\frac{1}{w}\sum_{j=0}^{w-1}\varphi(\mu_{\xi,t-j}^{(\ell-1)})\right]^{d_{\mathsf{c}}-1}$$

- This leads to the overall update equation, combining variable and check node update

$$\mu_{\chi,t} = \varphi^{-1}\left(1-\left[1-\frac{1}{w}\sum_{j=0}^{w-1}\varphi\left(\mu_{c,t-j}+\frac{d_{\mathsf{v}}-1}{w}\sum_{i=0}^{w-1}\mu_{\chi,t+i-j}^{(\ell-1)}\right)\right]^{d_{\mathsf{c}}-1}\right)$$

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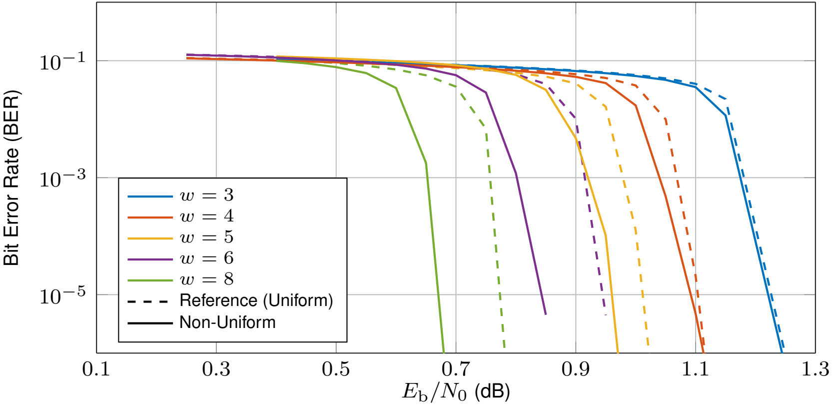
Simulation Example

- Simulation with windowed decoder ($W_D = 5w$ and $I = 1$ iteration)
- Code details and optimal values ν can be found in [SA19]



[SA19] L. Schmalen and V. Aref, "Spatially Coupled LDPC Codes with Non-uniform Coupling for Improved Decoding Speed," *Proc. ITW*, 2019, available online at `https://arxiv.org/abs/1904.07026`

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Burst Erasure of a Complete Spatial Position (2)

- We have the update equation

$$\xi_t^{(\ell)} = \epsilon_t \left(1 - \frac{1}{w}\sum_{i=0}^{w-1}\left(1 - \frac{1}{w}\sum_{j=0}^{w-1}\xi_{t+i-j}^{(\ell-1)}\right)^{d_{\mathsf{c}}-1}\right)^{d_{\mathsf{v}}-1} \qquad \forall t \in \{1,\ldots,L\}$$

with

$$\epsilon_t = \begin{cases} 1 & \text{if } t = t_b \\ 0 & \text{otherwise} \end{cases}$$

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with

$$\epsilon_t = \begin{cases} 1 & \text{if } t = t_b \\ 0 & \text{otherwise} \end{cases}$$

- This implies that $\xi_t^{(\ell)} = 0$, if $t \neq t_b$, and we only need to consider the case $t = t_b$ with

$$\xi_{t_b}^{(\ell)} = \left(1 - \frac{1}{w}\sum_{i=0}^{w-1}\left(1 - \frac{1}{w}\sum_{j=0}^{w-1}\xi_{t_b+i-j}^{(\ell-1)}\right)^{d_{\mathsf{c}}-1}\right)^{d_{\mathsf{v}}-1}$$

Burst Erasure of a Complete Spatial Position (3)

- In the inner sum, we only need to consider the case $j = i$, as the other contributions are zero, hence

$$\xi_{t_b}^{(\ell)} = \left(1 - \frac{1}{w}\sum_{i=0}^{w-1}\left(1 - \frac{1}{w}\xi_{t_b}^{(\ell-1)}\right)^{d_\mathsf{c}-1}\right)^{d_\mathsf{v}-1}$$
$$= \left(1 - \left(1 - \frac{1}{w}\xi_{t_b}^{(\ell-1)}\right)^{d_\mathsf{c}-1}\right)^{d_\mathsf{v}-1}$$

- We now make a change of variable $\kappa^{(\ell)} = \frac{1}{w}\xi_{t_b}^{(\ell)}$ leading to

$$w\kappa^{(\ell)} = \left(1 - \left(1 - \kappa^{(\ell-1)}\right)^{d_\mathsf{c}-1}\right)^{d_\mathsf{v}-1}$$
$$\Rightarrow \quad \kappa^{(\ell)} = \frac{1}{w}\left(1 - \left(1 - \kappa^{(\ell-1)}\right)^{d_\mathsf{c}-1}\right)^{d_\mathsf{v}-1}$$

Burst Erasure of a Complete Spatial Position (4)

- This is exactly the update equation of $[d_{\mathsf{v}}, d_{\mathsf{c}}]$ regular LDPC codes on the BEC (with ϵ replaced by $\frac{1}{w}$).
- We know that $\kappa^{(\ell)}$ converges to 0 iff $\frac{1}{w} < \epsilon^\star$ where $\epsilon^\star$ is the threshold of the $[d_{\mathsf{v}}, d_{\mathsf{c}}]$ LDPC code
- This leads to the following result

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Error Probability After Decoding

Theorem

Consider a code sampled uniformly from the $[d_\mathsf{v}, d_\mathsf{c}, w, L]$ *ensemble with* M *variable nodes per spatial position (SP) with w* $d_\mathsf{c} > 2$ *and* $wM \geq 2(d_\mathsf{v}+1)d_\mathsf{c}$*. If all variable nodes of a randomly chosen SP are erased, the (average) probability of BP decoding failure is lower-bounded by*

$$P_{\mathrm{B}}^{\mathrm{SPBC}} \geq \binom{M}{2}\left(1 - \frac{M^2}{\left(\frac{w}{d_\mathsf{c}}M - 3\right)^{d_\mathsf{v}}}\right)P_{\mathcal{R}}$$

where $P_{\mathcal{R}}$ *is the probability that two variable nodes from an SP of the code form a stopping set, given by* (1)*.*

Proof:

- Let N_2^{SP} denote the number of size-2 stopping sets formed by two VNs in an SP

Error Probability After Decoding

- Note that that

$$N_2^{\mathsf{SP}} = \sum_{0 \leq i < j \leq M} U_{ij} \quad \text{and} \quad \mathbb{E}\{N_2^{\mathsf{SP}}\} = \binom{M}{2} P(U_{ij} = 1) = \binom{M}{2} \mathbb{E}\{U_{ij}\}$$

  and

$$\mathbb{E}\{U_{ij}\} = P_{\mathcal{R}}$$

- We have

$$\begin{aligned} P_{\mathrm{B}}^{\mathrm{SPBC}} &= P(\text{ At least one stopping set in a SP }) \\ &\geq P\left(N_2^{\mathsf{SP}} \geq 1\right) \\ &\geq \frac{\left(\mathbb{E}\{N_2^{\mathsf{SP}}\}\right)^2}{\mathbb{E}\{(N_2^{\mathsf{SP}})^2\}} \end{aligned}$$

Error Probability After Decoding

- Furthermore

$$
\begin{aligned}
\mathbb{E}\left\{\left(N_2^{\mathsf{SP}}\right)^2\right\} &= \mathbb{E}\left\{\left(\sum_{1\leq i<j\leq M} U_{ij}\right)^2\right\} \\
&= \sum_{1\leq i<j\leq M} \mathbb{E}\left\{U_{ij}^2\right\} + \sum_{\substack{(i,j)\neq(k,l)\\ i<j,k<l}} \mathbb{E}\left\{U_{ij}U_{kl}\right\} \\
&\leq \binom{M}{2}\mathbb{E}\left\{U_{ij}\right\} + \frac{2\binom{M}{2}\left(\binom{M}{2}-1\right)}{\binom{wM\frac{d_{\mathsf{v}}}{d_{\mathsf{c}}}-2d_{\mathsf{v}}}{d_{\mathsf{v}}}}\mathbb{E}\left\{U_{ij}\right\}
\end{aligned}
$$

Error Probability After Decoding

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- Finally

$$P_{\mathrm{B}}^{\mathrm{SPBC}} \geq \frac{\binom{M}{2}^2 \mathbb{E}\{U_{ij}\}^2}{\binom{M}{2}\mathbb{E}\{U_{ij}\} + \frac{2\binom{M}{2}\left(\binom{M}{2}-1\right)}{\binom{wM\frac{d_v}{d_c}-2d_v}{d_v}}\mathbb{E}\{U_{ij}\}}$$

$$\geq \frac{\binom{M}{2}\mathbb{E}\{U_{ij}\}}{1 + \frac{2\binom{M}{2}}{\binom{wM\frac{d_v}{d_c}-2d_v}{d_v}}}$$

$$\overset{(a)}{\geq} \mathbb{E}\{N_2^{\mathsf{SP}}\}\left(1 - \frac{2\binom{M}{2}}{\binom{wM\frac{d_v}{d_c}-2d_v}{d_v}}\right)$$

$$\geq \mathbb{E}\{N_2^{\mathsf{SP}}\}\left(1 - \frac{M^2}{\left(\frac{wM}{d_c}-3\right)^{d_v}}\right)$$

where (a) is due to the fact that $\frac{1}{1+\tau} \geq 1-\tau$ for $\tau > -1$.

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The Block Erasure Channel (BLEC)

- Every SP is erased independently with probability p

$$\epsilon_t = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

- Erasures far apart (in spatial dimension) do not interfere

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- Erasures far apart (in spatial dimension) do not interfere
- Analysis consists of three parts
  (i) Partition SPs into segments of non-interfering erasures (at least $w-1$ non-erased positions separating segments). The expected number of SPs in such a segment is given by

$$\mathbb{E}\left[\tau_i\right] = \frac{1}{p}\left(\frac{1}{(1-p)^{w-1}} - 1\right) \approx (w-1)\left(1 + \frac{1}{2}(w-1)p + \frac{5}{6}(w-1)^2 p^2\right)$$

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  - (ii) There are N segments with $\sum_{i=1}^{N}\tau_i = L+w-1$. The average decoding failure is obtained as

$$P_{\text{B}}^{\text{BLEC}} = 1 - \mathbb{E}\left[\prod_{i=1}^{N}(1-P_e^{(i)})\right] \lessapprox \frac{L+w-1}{\mathbb{E}\left[\tau_i\right]}\mathbb{E}\left[P_e^{(1)}\right]$$

    where $P_e^{(1)}$ is the error probability of a segment

The Block Erasure Channel

- Analysis consists of three parts
  (iii) Finally, we compute $P_e^{(1)} = \sum_{i=1}^{\infty} Q_i$ where Q_i is the probability that a segment cannot be receovered if i SPs are erased. We consider decoding errors due to small stopping sets and decoding sets that cause the decoder to fail for any M (burst $> \beta_{\max}$). Putting all together

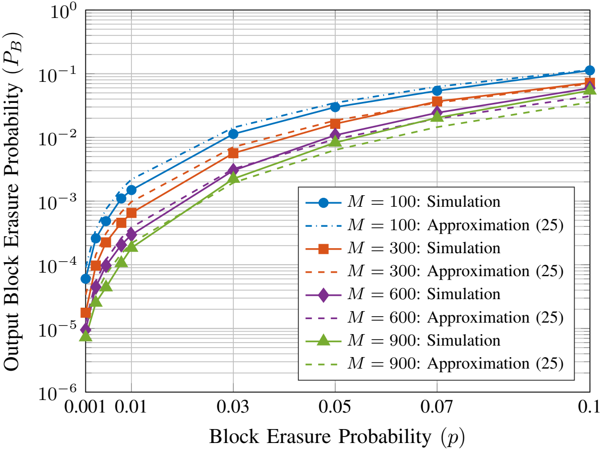
$$P_{\mathrm{B}}^{\mathrm{BLEC}} \approx (L+w-1)p(1-p)^{w-1}\left((1-p)^{w-1}\left[P_{\mathrm{B}}^{\mathrm{SPBC}} + pP_{\mathrm{B}}^{\mathrm{2PBC}}\right] + \frac{p^2}{(1-p)^{w-1}+p}\right)$$

with

$$P_{\mathrm{B}}^{\mathrm{2PBC}} \gtrapprox p(1-p)^{w-1}[1-(1-p)^{w-1}]\left(2\binom{M}{2}q_0 + M^2 q_1\right)$$

The Block Erasure Channel

- $[d_\mathsf{v} = 3, d_\mathsf{c} = 6, w = 4, L = 30]$ code ensemble



- Approximations track true behavior closely, even for p as large as 0.1 and M as small as 100

- With toolbox of approximations, we can approximate codes for the multi-point-to-point scheme